Fractals and Patterns

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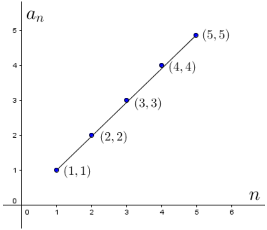
Arithmetic progressions

Definition

An arithmetic progression of length n (with $n \geq 3$) is a sequence of distinct vectors in $\mathbb{R}^d$ $(\vec{a_1}, \cdots, \vec{a_n})$ such that the diferences $\overrightarrow{a_{j+1}} - \vec{a_j}$ are the same vector (or gap) $\vec{v} \neq \vec{0}$ for all j.



Observation

An arithmetic progression of length n is $\{\vec{a_1}, \cdots, \vec{a_n}\}$ where $\frac{\vec{a_i} + \overrightarrow{a_{i+2}}}{2} = \overrightarrow{a_{i+1}}$ for all $1 \leq i \leq n-2$.

Motivation: Discrete context

Definition

The upper density of a set $E \subseteq \mathbb{N}$ is defined by
$\overline{d}(E) := \limsup_{n\to\infty} \frac{\#(E\cap\{1,\cdots,n\})}{n}$.

Theorem (Roth 1954)

If a set $E \subseteq \mathbb{N}$ has positive upper density, then E contains arithmetic progressions of length 3.

Theorem (Szemerédi 1975)

If a set $E \subseteq \mathbb{N}$ has positive upper density, then E contains arbitrarily long arithmetic progressions.

Motivation: Discrete context

- Green and Tao (2008) proved that the set of prime numbers contains arbitrarily long arithmetic progressions.

There have been many results for more general "geometrical configurations", for example:

- Bergelson and Leibman (1996) studied polynomial configurations with rational coefficients. They proved that any *subset of integers of upper positive density contains certain polynomial configurations.*
- Tao and Ziegler (2016) extended the previous results to polynomial configurations in the set of *prime numbers.*

Motivation: Continuous context

We denote $H := \{\phi : [0,1] \to [0,1] \text{ homeomorphism}\}$. Boshernitzan and Chaika proved a dichotomy for Borel subsets:

Theorem (Boshernitzan, Chaika 2016)

Given a Borel subset $A \subseteq [0,1]$, we have two possibilities:

$\exists \phi \in H$ *such that* $\phi(A)$
does not contains any arithmetic progression of length 3

or

$\forall \phi \in H, \phi(A)$*contains arithmetic progressions of arbitrary finite length.*

Motivation: Continuous context

From the Lebesgue density theorem, one can deduce that any set $E \subseteq \mathbb{R}^N$ of positive Lebesgue measure contains homothetic copies of every finite set.

Conjecture (Erdős)

For any infinite set $A \subseteq \mathbb{R}$ there exists a set $E \subseteq \mathbb{R}$ of positive Lebesgue measure which does not contain any homothetic copy of A.

A partial result in this direction is

Theorem (Falconer 1984)

Let $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ a non-increasing sequence, such that $x_n \to_{n\to\infty} 0$, and $\underline{\lim}_{n\to\infty} \frac{x_{n+1}}{x_n} = 1$. Then, there exist a closed set $E \subseteq \mathbb{R}$ with $\mathcal{L}(E) > 0$ such that $\forall b \in \mathbb{R}\ \forall c \in \mathbb{R}_{\neq 0}$ we have that $cx_n - b \notin E$ for infinite numbers $n \in \mathbb{N}$.

It is not known what happens with the set $A := \{2^{-n} : \ n \in \mathbb{N}\}$.

Motivation: Sets and patterns

We saw that any set $E \subseteq \mathbb{R}$ of positive Lebesgue measure contains a homothetic copy of every finite set. So, the following question is natural:

Question (Iosevich)

If $A \subset \mathbb{R}$ is a finite set and $E \subseteq [0, 1]$ is a set of Hausdorff dimension 1, must E contain a homothetic copy of A?

Theorem (Keleti 2008)

Given a countable family of triplets, there exists a 1-dimensional set that does not contain a homothetic copy of any of those triplets.

Maga (2010/11) extended Keleti's constructions to the plane.

When does a set contain arithmetic progressions?

A set $E \subseteq \mathbb{R}^d$ contains an arithmetic progression $(\overrightarrow{x_1}, \cdots, \overrightarrow{x_n})$

$$\iff \frac{\overrightarrow{x_i} + \overrightarrow{x_{i+2}}}{2} = \overrightarrow{x_{i+1}} \text{ for all } 1 \leq i \leq n-2$$

$$\iff \psi(\overrightarrow{x_1}, \cdots, \overrightarrow{x_n}) = \overrightarrow{0},$$

where we define $\psi := (\psi_1, \cdots, \psi_{n-2})$ with
$\psi_i(\overrightarrow{x_1}, \cdots, \overrightarrow{x_n}) := \frac{\overrightarrow{x_i}+\overrightarrow{x_{i+2}}}{2} - \overrightarrow{x_{i+1}}$ for all $1 \leq i \leq n-2$.

What about other relationships among points?

When does a set E contain four (distinct) points forming a parallelogram?

A set $E \subseteq \mathbb{R}^d$ contains four points $\vec{x_1}, \vec{x_2}, \vec{x_3}, \vec{x_4}$ forming a parallelogram
$\iff \exists \vec{x_1}, \vec{x_2}, \vec{x_3}, \vec{x_4} \in E$ such that $\vec{x_2} - \vec{x_1} = \vec{x_4} - \vec{x_3}$
$\iff \exists \vec{x_1}, \vec{x_2}, \vec{x_3}, \vec{x_4} \in E$ such that $\psi(\vec{x_1}, \vec{x_2}, \vec{x_3}, \vec{x_4}) = 0$,

where $\psi(\vec{x_1}, \vec{x_2}, \vec{x_3}, \vec{x_4}) := \vec{x_2} - \vec{x_1} + \vec{x_3} - \vec{x_4}$.

What is a pattern?

A pattern is a relationship among points, given by the zero set of a (possibly multi-valued) function. Formally:

Definition

Given $E \subseteq \mathbb{R}^d$ we say that $\psi : \mathbb{R}^{dk} \to \mathbb{R}^n$ with $k \geq 2$ is a pattern in E, if there exist distinct $\vec{x_1}, \cdots, \vec{x_k} \in E$ such that $\psi(\vec{x_1}, \cdots, \vec{x_k}) = \vec{0}$.
In the particular case that ψ is a linear function, we say that it is a linear pattern.

Dimension functions

Definition

We say that an increasing and right-continuous function $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a dimension function if $h(0) = 0$, $h(t) > 0$ if $t > 0$.

The set of dimension functions is partially ordered, considering the order defined by

$$h_2 \prec h_1 \text{ if } \lim_{x \to 0^+} \frac{h_1(x)}{h_2(x)} = 0.$$

h-Hausdorff measure

We will use the notation $|U|$ for the diameter of the set U.

Definition

If $E \subseteq \bigcup_{i\in\mathbb{N}} U_i$ with $0 < |U_i| \leq \delta$ for all i, we say that $\{U_i\}_{i\in\mathbb{N}}$ is a δ-covering of E.

Definition

The Hausdorff measure associated with h is

$$\mathcal{H}^h(E) := \lim_{\delta\to 0} \inf \left\{ \sum_i h(|U_i|) : \{U_i\}_i \text{ a } \delta\text{-covering of } E \right\}.$$

This definition generalises the α-dimensional Hausdorff measure, which is the particular case $h(x) := x^\alpha$.
The relation of order says that $x^s \prec x^t$ if and only if $s < t$.

Sets avoiding patterns

- One can deduce from a theorem and a remark given by Máthé (2012), the following result:

Given a countable set of linear functions, there exists a compact set $E \subseteq \mathbb{R}^d$ of full Hausdorff dimension, such that E does not contain any of those linear patterns.

- This generalises the results of Keleti and Maga mentioned above.
- They considered only Hausdorff dimension (and not Hausdorff measures for general dimension functions).

Sets avoiding patterns

- Máthé constructed large sets avoiding polynomial patterns. The sets are not of full dimension if the polynomials are not linear.
- At least in some cases, there is no set of full dimension without the given non-linear patterns. For example, the function to avoid an angle of 90° is the polynomial function of degree 2 given by $\psi(\vec{x}, \vec{y}, \vec{z}) := \langle \vec{z} - \vec{x}, \vec{y} - \vec{x} \rangle$; and it was proved by Harangi, Keleti, Kiss, Maga, Máthé, Mattila and Strenner that every Borel subset of $\mathbb{R}^2$ of Hausdorff dimension largen than 1 contains three points forming an angle of 90° (in particular, no full dimensional Borel set avoiding 90° angles exists in the plane).
- Since I wanted to study large sets for an arbitrary dimension function h with $h \prec x^d$, I focus on the case of linear patterns.

Main Theorem

Theorem (Y. 2017)

Let h be a dimension function with $h \prec x^d$, and let $(\psi_k)_{k\in\mathbb{N}}$ a sequence of non-zero linear functions with $\psi_k : (\mathbb{R}^d)^{m_k} \to \mathbb{R}$ and $m_k \geq 2$.

Then there exists a compact set $E \subseteq \mathbb{R}^d$ such that $\mathcal{H}^h(E) > 0$, and $\psi_k(\overrightarrow{x_1}, \cdots, \overrightarrow{x_{m_k}}) \neq 0$ for all $k \in \mathbb{N}$ and all distinct vectors $\overrightarrow{x_1}, \cdots, \overrightarrow{x_{m_k}} \in E$.

In particular, if we choose $h(x) := -\log(x)x^d$, we obtain a set of Hausdorff dimension d with the same properties.
This improves the results of Keleti, Maga and the linear case of Máthé mentioned above.

Some ideas about the proof of the main theorem

- I use Keleti's idea of defining the cubes to kill the patterns at later stages of the construction, but there are several differences.
- I also have to modify the location of the cubes to fit any linear pattern.
- I do not need to have separation between the cubes of the same level, but I need a uniform bound for the amount of offspring of each cube.
- I have two types of construction levels: in one type everything is chosen (most levels are of this type, to ensure $\mathcal{H}^h(E) > 0$), and in the other type the patterns are "killed".

Sharpness of the main theorem

In the main theorem it is impossible to have a set E that works for every $h \prec x^d$.

Theorem (Besicovitch 1956)

Given $E \subseteq \mathbb{R}^d$ and a dimension function h such that $\mathcal{H}^h(E) = 0$, then there exists $g \prec h$ such that $\mathcal{H}^g(E) = 0$.

So, if $\mathcal{H}^h(E) > 0$ for all $h \prec x^d$, then $\mathcal{L}(E) = c_d \mathcal{H}^{x^d}(E) > 0$. Hence, since a set of positive Lebesgue measure contains an arithmetic progression of length 3 (which corresponds on the zero set of the linear function $\psi(x, y, z) := \frac{x+z}{2} - y$), in the main theorem it is not possible to have a set E that works for every $h \prec x^d$.

Some applications 1: Set of quotients

Corollary

Given a dimension function h with $h \prec x$ and a countable set $A \subseteq \mathbb{R}_{\neq 1}$, there exists a compact set $E \subseteq [1,2]$ such that $\mathcal{H}^h(E) > 0$ and the set $\frac{E}{E} := \{\frac{y}{x} \ : \ x, y \in E\}$ does not contain any element of A.

Take $\psi_a(x,y) := ax - y$ for all $a \in A$, and apply the main theorem.

Some applications 2: Set of differences

Corollary

Given a dimension function $h \prec x$ and a countable set $B \subseteq \mathbb{R}_{\neq 0}$, there exists a compact set $F \subseteq [0, \log(2)]$ such that $\mathcal{H}^h(F) > 0$ and the set $F - F := \{y - x \ : \ x, y \in F\}$ does not contain any element of B.

We define $A := e^B := \{e^b : \ b \in B\} \subseteq \mathbb{R} \setminus \{1\}$. By the previous corollary we have a compact set $E \subseteq [1, 2]$ such that $\mathcal{H}^h(E) > 0$ and $\frac{y}{x} \neq a$ for all $x, y \in E$ and all $a \in A$. We choose $F := \log(E)$.

In particular, if B is a countable and dense set, we obtain a set F of positive $\mathcal{H}^h$-measure whose set of differences has empty interior. This contrasts with Steinhaus' theorem, asserting that the difference set of a set of positive Lebesgue measure contains an interval.

Some applications 3: Planes

Corollary

Given a dimension function h with $h \prec x$ and given a countable set of planes $\{\pi_k\}_k$ in $\mathbb{R}^3$ containing the origin, there exists a compact set $E \subseteq \mathbb{R}$ with $\mathcal{H}^h(E) > 0$ such that

$$(x, y, z) \notin \pi_k \ \forall k \text{ for all distinct } x, y, z \in E.$$

Each of those planes π_k is given by an equation $\psi_k(x, y, z) := a_k x + b_k y + c_k z = 0$. And applying the main theorem, the result follows.

Some applications 4: Proportions

Corollary

Given a dimension function h with $h \prec x$ and a countable set $A \subseteq (1, +\infty)$, there exists a compact set $E \subseteq \mathbb{R}$ with $\mathcal{H}^h(E) > 0$ such that

$$\frac{z-x}{z-y} \notin A \quad \forall x < y < z \text{ in } E.$$

Let $A := \{\alpha_k\}_k$. If we choose $\pi_k : \ x - \alpha_k y + (\alpha_k - 1)z = 0$ in the previous corollary, the result is follows.

Choosing $h(x) := -\log(x)x$ we recover a result of Keleti. Also, if we choose $A = \{2\}$, E does not contain any arithmetic progression.

Some applications 5: Trapezoids

Corollary

Let $d \in \mathbb{N}$ and let h a dimension function such that $h \prec x^d$. Let $(\beta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}_{\neq 0}$.

We get a compact set $E \subseteq \mathbb{R}^d$ such that $\mathcal{H}^h(E) > 0$, and for all $n \in \mathbb{N}$, E does not contain the vertices of any trapezoid with the lengths of the parallel sides in proportion β_n.

Taking $\psi_n(\vec{x_1}, \vec{x_2}, \vec{x_3}, \vec{x_4}) := \vec{x_1} - \vec{x_2} - \beta_n(\vec{x_3} - \vec{x_4})$, the result follows by applying the main theorem.
In particular, when $\beta_n = 1$ for all $n \in N$, the set E does not contain the vertices of any parallelogram. This is an improvement over a result given by Maga.

Some applications 6: in $\mathbb{C}$

We have a complex version of the main theorem taking $d = 2s$ and identifying $\mathbb{C}$ with $\mathbb{R}^2$.

Corollary

Let h be a dimension function with $h \prec x^2$, and let $(P_n)_{n\in\mathbb{N}} = (x_n, y_n, z_n)_{n\in\mathbb{N}}$ a sequence of triplets of different complex numbers. Then there exists a compact set $E \subseteq \mathbb{C}$, with $\mathcal{H}^h(E) > 0$, that does not contain a similar copy of any of the given triplets.

Define $\alpha_n := \frac{z_n - x_n}{z_n - y_n}$, $\psi_n(x, y, z) = (\alpha_n - 1)z - \alpha_n y + x$ and appliyng the complex version of the main theorem, the result follows.
Taking $h(x) := -x^2 \log(x)$, we recover the result of Maga.

We studied **large sets avoiding patterns.**

On the other hand, it is interesting to study **small sets containing a lot of geometrical configurations.**

What is a configuration?

Definition

Let $E \subseteq \mathbb{R}^N$ and $\mathcal{F} := \{f_i : \mathbb{R}^N \to \mathbb{R}^N, i \in \Lambda\}$ a set of functions, we say that *E contains the configuration $(f_i)_{i\in\Lambda}$* if

$$\text{there exist } t \in \mathbb{R}^N \text{ such that } f_i(t) \in E \ \forall i \in \Lambda,$$

or equivalently, if

$$\bigcap_{i\in\Lambda} f_i^{-1}(E) \neq \emptyset.$$

When Λ is finite (or countable), we say that the configuration is finite (or countable).
If the functions f_i are non-constant polynomials (or linear functions), we say that the configuration is a polynomial configuration (or linear configuration).

When does a set contain arithmetic progressions?

Given a vector $\vec{v} \in \mathbb{R}^d_{\neq\{\vec{0}\}}$,

$$\begin{aligned} &\text{a set } E \subseteq \mathbb{R}^d \text{ contains an arithmetic progression with gap } \vec{v} \\ &\iff \exists \vec{a} \text{ such that } \vec{a}, \vec{a}+\vec{v}, \cdots, \vec{a}+(n-1)\vec{v} \in E \\ &\iff \exists \vec{a} \in E \cap (E-\vec{v}) \cap \cdots \cap (E-(n-1)\vec{v}) \\ &\iff \bigcap_{0\leq i\leq n-1} f_i^{-1}(E) \neq \emptyset; \end{aligned}$$

where we define $f_i(\vec{x}) := \vec{x} + i\vec{v}$.

This is a difference between patterns and configurations: in the case of configurations we need to know the gap $\vec{v}$, in the case of patterns we can characterize every arithmetic progression of length n at the same time.

Small sets containing configurations

Davies, Marstrand and Taylor constructed very small sets containing prescribed linear configurations.

Theorem (Davies, Marstrand, Taylor 1959/60)

Given a dimension function h, there exists a closed set $E \subseteq \mathbb{R}$ *such that* $\mathcal{H}^h(E) = 0$ *and* $\bigcap_{i=1}^n (a_i E + b_i) \neq \emptyset$ *for any finite subset of linear (real) functions of non-zero slope.*
They also proved that there exists an $\mathcal{F}_\sigma$ *set* $E \subseteq \mathbb{R}$ *such that* $\bigcap_{i \in \Lambda} (a_i E + b_i) \neq \emptyset$ *(for* $a_i \neq 0$*) for any countable set* Λ *and* $\mathcal{H}^h(E) = 0$.

Small sets containing configurations

Theorem (Molter, Y. 2016)

Given a dimension function h and a family of continuous functions $\mathcal{F}$ *on* $\mathbb{R}^N$ *satisfying certain (technical) conditions, then there exists a perfect set* $E \subseteq \mathbb{R}^N$*, such that* $\mathcal{H}^h(E) = 0$ *and* $\bigcap_{1 \leq i \leq n} f_i^{-1}(E) \neq \emptyset$ *for each finite subset* $\{f_1, \cdots, f_n\}$ *of* $\mathcal{F}$*.*

- In particular the family of bilipschitz functions on $\mathbb{R}^N$, and the family of non-constant polynomials with real coefficients satisfy the assumptions of the previous theorem.
- We also prove an analogous result for intersection of images, and for countably many intersections obtaining an $\mathcal{F}_\sigma$ set without isolated points.

Thank you for your attention!!!

The most important thing is...

Mathematicians are so curious!

... And we brought chocolates to share with you :)