Proof of the DOZZ Formula

Antti Kupiainen

joint work with R. Rhodes, V. Vargas

MLI
October 2017

DOZZ formula

Dorn, Otto (1994) and Zamolodchikov, Zamolodchikov (1996):

$$C_\gamma(\alpha_1,\alpha_2,\alpha_3) = \left(\pi\,\mu\,\frac{\Gamma(\frac{\gamma^2}{4})}{\Gamma(1-\frac{\gamma^2}{4})}\left(\frac{\gamma}{2}\right)^{\frac{4-\gamma^2}{2}}\right)^{\frac{2Q-\bar\alpha}{\gamma}} \times \frac{\Upsilon'(0)\Upsilon(\alpha_1)\Upsilon(\alpha_2)\Upsilon(\alpha_3)}{\Upsilon(\frac{\bar\alpha-2Q}{2})\Upsilon(\frac{\bar\alpha-\alpha_1}{2})\Upsilon(\frac{\bar\alpha-\alpha_2}{2})\Upsilon(\frac{\bar\alpha-\alpha_3}{2})}$$

- $\alpha_i \in \mathbb{C}$, $\gamma \in \mathbb{R}$, $\mu > 0$
- $\bar\alpha = \alpha_1 + \alpha_2 + \alpha_3$, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$

Υ is an entire function on $\mathbb{C}$ with simple zeros defined by

$$\log \Upsilon(\alpha) = \int_0^\infty \left(\left(\frac{Q}{2}-\alpha\right)^2 e^{-t} - \frac{\sinh^2((\frac{Q}{2}-\alpha)\frac{t}{2})}{\sinh(\frac{t\gamma}{4})\sinh(\frac{t}{\gamma})}\right)\frac{dt}{t}$$

Structure Constant

$C_\gamma(\alpha_1, \alpha_2, \alpha_3)$ is the structure constant of **Liouville Conformal Field Theory**

Physics:

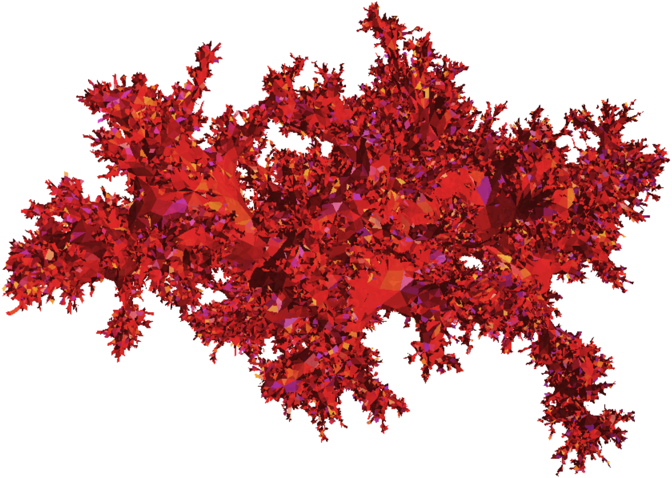
- String theory, 2d quantum gravity,
- 4d Yang-Mills; AGT correspondence

Mathematics:

- Fractal random surfaces
- Quantum cohomology

$\gamma = \sqrt{2}$, Quantum Sphere

Random fractal surfaces



F. David

Liouville Quantum Gravity

Random Riemannian metric g

On 2-sphere $S^2 = \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$:

$$g = e^{\gamma\phi(z)}(dx^2 + dy^2)$$

What is the probability law of $\phi(z)$?

Knizhnik, Polyakov, and Zamolodchikov '88:

$$\langle F(\phi) \rangle := \int \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} e^{-S(\phi)} D\phi$$

with $S(\phi)$ the **Liouville action functional**:

$$S(\phi) = \int (|\partial_z \phi(z)|^2 + \mu e^{\gamma\phi(z)}) dz$$

- μ is the cosmological constant
- γ depends on matter interacing with gravity

Random Surfaces

Random triangulation T of S^2:

$$\mathbb{P}(T) = e^{-\mu|T|} Z(T)$$

$|T|$ number of triangles, $Z(T)$ **partition function**

- Pure gravity (no matter): $Z(T) = 1$
- Ising model: $\sigma : \{\textit{triangles of } T\} \to \{-1, 1\}$

$$Z(T) = e^{\beta \sum_{t \sim t'} \sigma_t \sigma_{t'}}$$

Scaling limit: let $(\beta, \mu) \to (\beta_c, \mu_c)$ and rescale

Conjecture Obtain Liouville theory in the limit with $\gamma = \sqrt{8/3}$ (pure gravity), $\gamma = \sqrt{3}$ (Ising)

There are numerical checks

Random Surfaces

Example: Ising model $\gamma = \sqrt{3}$. Let

- σ be scaling limit of Ising spin
- $\tilde{\sigma}$ be scaling limit of Ising spin on a random planar map

Then

$$\tilde{\sigma}(z) = e^{\alpha\phi(z)}\sigma(z)$$

with ϕ the $\gamma = \sqrt{3}$ Liouville field and $\alpha = \frac{5}{2\sqrt{3}}$.

Hence we need to understand correlation functions of **vertex operators** $e^{\alpha\phi(z)}$in Liouville theory:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} \rangle = \int \prod_{i=1}^{n} e^{\alpha_i \phi(z_i)} e^{-S(\phi)} D\phi$$

Probabilistic Liouville Theory

What is the meaning of

$$e^{-\int(|\partial_z\phi(z)|^2+\mu e^{\gamma\phi(z)})dz}D\phi \quad ?$$

- $e^{-\int(|\partial_z\phi(z)|^2}D\phi$ in terms of **Gaussian Free Field (GFF)**
- GFF has covariance $-\Delta^{-1}$
- On S^2 $Ker(\Delta) = \{constants\}$
- $\phi = c + \psi$, $\psi \perp Ker(\Delta)$

Then define

$$\langle F(\phi)\rangle := \int_{\mathbb{R}} \mathbb{E}\left[F(c+\psi)e^{-\mu\int e^{\gamma(c+\psi(z))}dz}\right]dc$$

ψ is not a function:

$$\mathbb{E}\psi(z)\psi(z') = \log|z-z'|^{-1} + \text{smooth}$$

What does $e^{\gamma\psi(z)}$ mean?

Gaussian Multiplicative Chaos (GMC)

Regularize and renormalize: $\psi_\epsilon = \chi_\epsilon * \psi$

$$\lim_{\epsilon \to 0} e^{\gamma \psi_\epsilon(z) - \frac{\gamma^2}{2} \mathbb{E}\psi_\epsilon(z)^2} dz = M(dz) \quad \textit{almost surely}$$

M is a **random multifractal GMC measure**

We define

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(x_i)} \rangle := \lim_{\epsilon \to 0} \langle \prod_{i=1}^{n} e^{\alpha_i c} e^{\alpha_i \psi_\epsilon(x_i) - \frac{\alpha_i^2}{2} \mathbb{E}\psi_\epsilon(z_i)^2} \rangle_\epsilon \tag{1}$$

Existence of Liouville correlations

Theorem (DKRV 2014) *The limit (1) exists and is nontrivial if and only if:*

$$(A)\ \ \forall i : \alpha_i < Q \quad \text{and} \quad (B)\ \ \sum_i \alpha_i > 2Q$$

Remarks

- Recall that $Q := \frac{\gamma}{2} + \frac{2}{\gamma}$
- (A), (B) are called **Seiberg bounds**
- (A), (B) $\implies$ $n \geq 3$: **1- and 2-point functions are ∞**.

Proof is based on representation in terms of GMC

0-mode

Integrate first over the constant c:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(x_i)} \rangle = \mathbb{E}\left[\prod_i e^{\alpha_i \psi(x_i)} \int_{\mathbb{R}} e^{-2Qc} e^{\sum_i \alpha_i c - \mu e^{\gamma c} \int e^{\gamma \psi} dz} dc \right]$$

e^{-2Qc} for topological reasons.

The c-integral converges **if** $\sum_i \alpha_i > 2Q$:

$$\langle \prod_{i=1}^{n} e^{\alpha_i \phi(x_i)} \rangle = \frac{\Gamma(s)}{\mu^s \gamma} \mathbb{E}\left[\prod_i e^{\alpha_i \psi(x_i)} \left(\int e^{\gamma \psi} dz \right)^{-s} \right]$$

where $s := (\sum_i \alpha_i - 2Q)/\gamma$.

This explains Seiberg bound (B)

Modulus of Chaos

Cameron-Martin theorem: $\psi(z)$ under $\prod_i e^{\alpha_i \psi(x_i)} \mathbb{P}$ equals

$$\psi(z) + \sum_i \alpha_i \mathbb{E}\psi(z)\psi(z_i) \text{ under } \mathbb{P}$$

Result: Liouville correlations are given by

$$\langle \prod_{i=1}^n e^{\alpha_i \phi(x_i)} \rangle = \frac{C_g \Gamma(s)}{\mu^s} \prod_{i<j} |z_i - z_j|^{-\alpha_i \alpha_j}$$

$$\times \mathbb{E} \Big(\int \prod_i \frac{1}{|z - z_i|^{\gamma \alpha_i}} M(dz) \Big)^{-s}$$

Seiberg bound (A):

$\frac{1}{|z-z_i|^{\gamma\alpha_i}}$ is M-integrable almost surely **if and only if** $\alpha_i < Q$:

$$" e^{\alpha\phi} \equiv 0" \quad \alpha \geq Q$$

Conformal Field Theory

Liouville correlations are expectations of functions of GMC.

However they have interesting symmetries:

Theorem (DKRV 2014)

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *be a Möbius map. Then*

$$\langle \prod_i e^{\alpha_i \phi(f(z_i))} \rangle = \prod_i |f'(z_i)|^{-2\Delta_{\alpha_i}} \langle \prod_i e^{\alpha_i \phi(z_i)} \rangle$$

where $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$.

In particular 3-point function is determined by the **Structure constants**

$$C(\alpha_1, \alpha_2, \alpha_3) = \langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(1)} e^{\alpha_3 \phi(\infty)} \rangle$$

Structure constants

We obtain a probabilistic expression for structure constants

$$C(\alpha_1, \alpha_2, \alpha_3) \propto \mathbb{E}\left(\int \frac{(|z| \vee 1)^{\gamma\bar{\alpha}}}{|z|^{\gamma\alpha_1}|1-z|^{\gamma\alpha_2}} M(dz)\right)^{\frac{2Q-\bar{\alpha}}{\gamma}}$$

$\bar{\alpha} := \alpha_1 + \alpha_2 + \alpha_3$.

The DOZZ formula is an explicit expression for this expectation.

Its original derivation was somewhat mysterious:

"It should be stressed that the arguments of this section have nothing to do with a derivation. These are rather some motivations and we consider the expression proposed as a guess which we try to support in the subsequent sections"

Integrability

DOZZ is an **integrability** result for Liouville model and for multiplicative chaos.

It is believed that all N-point functions are recursively determined by the structure constants (**conformal bootstrap**).

Does the probabilistic expression satisfy DOZZ and bootstrap?

We prove it satisfies DOZZ

Dilemma

The DOZZ proposal $C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3)$ is a **meromorphic** function of $\alpha_i \in \mathbb{C}$. In particular for **real** α's

$$C_{DOZZ}(\alpha_1, \alpha_2, \alpha_3) \neq 0 \quad \text{if} \quad \alpha_i > Q$$

The probabilistic $C(\alpha_1, \alpha_2, \alpha_3)$ is **identically zero** in this region:

$$C(\alpha_1, \alpha_2, \alpha_3) \equiv 0 \quad \alpha_i \geq Q$$

What is going on? DOZZ is too beautiful to be wrong!

Remark. *One can renormalize $e^{\alpha\phi}$ for $\alpha \geq Q$ so that $C(\alpha_1, \alpha_2, \alpha_3) \neq 0$. However the result does not satisfy DOZZ.*

Analyticity

Theorem (KRV 2017)

(A) *The structure constants* $C(\alpha_1, \alpha_2, \alpha_3)$ *are analytic in* α_i *in a neighborhood of* $\alpha_i \in (0, Q)$.

(B) *The structure constants* $C(\alpha_1, \alpha_2, \alpha_3)$ *have a unique analytic continuation to meromorphic function* ***satisfying the DOZZ conjecture****.*

Note! Due to renormalization $|e^{i\beta\phi(x_i)}| = \infty$ so that even (A) is subtle!

Belavin-Polyakov-Zamolodchicov equation

Consider a **4-point function**

$$F(u) := \langle e^{\alpha_0\phi(u)} e^{\alpha_1\phi(0)} e^{\alpha_2\phi(1)} e^{\alpha_3\phi(\infty)} \rangle$$

Theorem (KRV2016) *For $\alpha_0 = -\frac{\gamma}{2}$ **or** $\alpha_0 = -\frac{2}{\gamma}$ F satisfies a hypergeometric equation*

$$\partial_u^2 F + \frac{a}{u(1-u)}\partial_u F - \frac{b}{u(1-u)}F = 0 \quad (*)$$

Proof: Gaussian integration by parts combined with regularity estimates.

Reason: $e^{\alpha_0\phi(u)}$ are **degenerate conformal fields**

Solution

Theorem (KRV2016) *The space of real solutions to* $(*)$ *is* ***one dimensional*** *so that*

$$F(u) = A|\mathcal{F}_-(u)|^2 + B|\mathcal{F}_+(u)|^2$$

where A/B *is an* ***explicit*** *function of* α_i, γ *and* $\mathcal{F}_\pm$ *are Gauss hypergeometric functions* $\mathcal{F}_-(0) = 1$*,* $\mathcal{F}_+(0) = 0$ *.*

On the other hand A, B determined by asymptotics of

$$F(u) \propto \mathbb{E}\left(\int \frac{(|z| \vee 1)^{\gamma\bar{\alpha}}}{|z-u|^{\gamma\alpha_0}|z|^{\gamma\alpha_1}|1-z|^{\gamma\alpha_2}} M(dz)\right)^{\frac{2Q-\bar{\alpha}}{\gamma}}$$

as $u \to 0$. Obviously

$$F(u) = C(\alpha_1 + \alpha_0, \alpha_2, \alpha_3) + o(1)$$

so that $A = 1$. The $o(1)$ results from a delicate analysis of GMC.

Reflection

Theorem *Let* $\alpha_0 = -\frac{\gamma}{2}$ *or* $\alpha_0 = -\frac{2}{\gamma}$. *Then*
(a) ***If*** $\alpha_1 - \alpha_0 < Q$ *(recall* $\alpha_0 < 0$*)*

$$B = D(\alpha_1)C(\alpha_1 - \alpha_0, \alpha_2, \alpha_3)$$

with $D(\alpha_1)$ *explicit (ratio of 14* Γ *functions)*
(b) If $\alpha_1 - \alpha_0 > Q$ *we get the same with the replacement*

$$C(\alpha_1 - \alpha_0, \alpha_2, \alpha_3) \to R(\alpha - \alpha_0)C(2Q - (\alpha_1 - \alpha_0), \alpha_2, \alpha_3)$$

where R is given by a ***probabilistic*** *expression in terms of* ***tail behavior*** *of multiplicative chaos*

Note! (b) says that if $\alpha > Q$ we have the replacement

$$e^{\alpha\phi} \to R(\alpha)e^{(2Q-\alpha)\phi}$$

Periodicity

Since A/B is explicit and $A = 1$ we get

$$C(\alpha_1 - \alpha_0, \alpha_2, \alpha_3) = D(\alpha_0, \alpha_1, \alpha_2, \alpha_3) C(\alpha_1 + \alpha_0, \alpha_2, \alpha_3)$$

with $\alpha_0 = -\frac{\gamma}{2}$ **or** $-\frac{2}{\gamma}$. and $D(\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ explicit.

$$D(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = -\frac{1}{\pi\mu} \frac{\Gamma(-\alpha_0^2)\Gamma(-\alpha_0\alpha_1)\Gamma(-\alpha_0\alpha_1 - \alpha_0^2)\Gamma(\frac{\alpha_0}{2}(2\alpha_1 - \bar{\alpha}))}{\Gamma(\frac{\alpha_0}{2}(2Q - \bar{\alpha}))\Gamma(\frac{\alpha_0}{2}(2\alpha_3 - \bar{\alpha}))\Gamma(\frac{\alpha_0}{2}(2\alpha_2 - \bar{\alpha}))}$$
$$\times \frac{\Gamma(1 + \frac{\alpha_0}{2}(\bar{\alpha} - 2Q))\Gamma(1 + \frac{\alpha_0}{2}(\bar{\alpha} - 2\alpha_3))\Gamma(1 + \frac{\alpha_0}{2}(\bar{\alpha} - 2\alpha_2))}{\Gamma(1 + \alpha_0^2)\Gamma(1 + \alpha_0\alpha_1)\Gamma(1 + \alpha_0\alpha_1 + \alpha_0^2)\Gamma(1 + \frac{\alpha_0}{2}(\bar{\alpha} - 2\alpha_1))}$$

Teschner '95: DOZZ is the **unique analytic** solution of these equations **if** we could extend $C(\alpha_1, \alpha_2, \alpha_3)$ beyond the region $\alpha_i < Q$.

Reflection coefficient

Idea: use

$$C(\alpha_1, \alpha_2, \alpha_3) = R(\alpha_1)C(2Q - \alpha_1, \alpha_2, \alpha_3)$$

to obtain **analytic continuation** of $C(\alpha_1, \alpha_2, \alpha_3)$ to $\alpha_1 > Q$.

What is R?

Let $\alpha < Q$, D any neighborhood of origin and

$$Z_D := \int_D \frac{1}{|z|^\alpha} M(dz)$$

Then

$$\mathbb{P}(Z > x) = R(\alpha)|x|^{-\frac{2(Q-\alpha)}{\gamma}}(1 + o(z))$$

$R(\alpha)$ has an explicit expression in terms of multiplicative chaos (**2-point Quantum Sphere** of Duplantier and Sheffield)

Integrability

DOZZ formula implies

$$R(\alpha) = -\left(\left(\frac{\gamma}{2}\right)^{\frac{\gamma^2}{2}-2}\tilde{\mu}\right)^{\frac{2(Q-\alpha)}{\gamma}} \frac{\Gamma(\frac{\gamma}{2}(\alpha - Q))\Gamma(\frac{2}{\gamma}(\alpha - Q))}{\Gamma(\frac{\gamma}{2}(Q-\alpha))\Gamma(\frac{2}{\gamma}(Q-\alpha))}.$$

To conclude the proof of DOZZ formula we need to show the probabilistic R equals R_{DOZZ}.

This is done by proving analyticity of R in the probabilistic region $\alpha < Q$ and deriving shift identities for R.

Summary

The mysteries of Liouville theory can be addressed with a rigorous probabilistic analysis of the functional integral.

Liouville theory is an example of **integrable probability**

Future work:

- Prove **conformal bootstrap**
- Probabilistic construction of **Virasoro algebra** representation theory
- **Analytic continuation in γ**: e.g. for γ purely imaginary, $c < 1$, C is conjectured to give 3-point probabilities for FK Potts models.