Applied Mathematics

Statistical solutions for hyperbolic conservation laws

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Consider the hyperbolic conservation law

$$\begin{aligned} \partial_t u + \nabla \cdot f(u) = 0 \\ u(x,0) = u_0(x). \end{aligned} \tag{1}$$

In practically all real-world applications, the initial data u_0 will not be known exactly due to measurement error, low measurement resolution, etc. However, we may assume that we know certain statistics about the data (mean, variance, etc.), allowing us to sample the initial data according to these statistics. We argue that the correct way of representing uncertain data is via one of two *equivalent* representations:

1. As a family μ_t of probability measures on a function space ($L^1(\mathbb{R})$ in our case).

2. As a family $\nu_t = (\nu_t^1, \nu_t^2, \ldots)$ of Young measures giving statistics and correlations of the solution at different spatial points.

We derive a *hierarchy* of evolutionary PDEs in terms of either μ_t or ν_t, obtaining so-called *statistical solutions* of (1). We show existence, uniqueness and stability for statistical solutions of scalar conservation laws and linear systems. We also outline numerical methods for the efficient computation of statistical solutions.

This is joint work with Siddhartha Mishra (ETH Zürich) and Samuel Lanthaler (EPFL).