Regularity of aperiodic sequences and their associated subshifts

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29.09.2017

Joint work with F. Dreher, M Gröger, M. Kesseböhmer, M. Steffens and T. Samuel.
Based on 'Regularity of aperiodic minimal subshifts' by T. Samuel.

Work partially supported by the DFG sponsored grant

Sturmain subshifts

[KS12] – J. Kellendonk and J. Savinien.
Spectral triples and characterization of aperiodic order.
Proc. Lond. Math. Soc. (1) 104 (2012), 123–157.

[GKM+16] – M. Gröger, M. Keßeböhmer, A. Mosbach, T. Samuel and M. Steffens.
A classification of aperiodic order via spectral metrics and Jarník sets.
Submitted (2016) 25 pages – Pre-print: `arxiv.org/abs/1601.06435`.

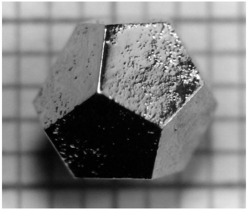
l-Grigorchuk subshift

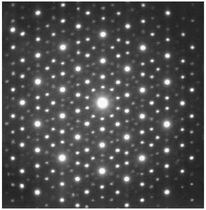
[GLN17] – R. Grigorchuk, D. Lenz and T. Nagnibeda.
Spectra of Schreier graphs of Grigorchuk's group and Schroedinger operators with aperiodic order.
To appear in: Math. Ann. (2017) 29 pages.

[DKM+17] – F. Dreher, M. Keßeböhmer, A. Mosbach, T. Samuel and M. Steffens.
Regularity of aperiodic minimal subshifts.
Bull. Math. Sci. (2017) 1–22.

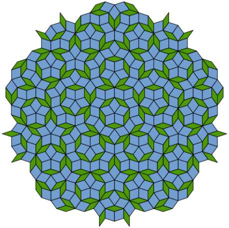
A physical motivation

Quasicrystals - Ho-Mg-Zn icosahedral quasicrystal





Aperiodic order - Penrose tiling



Aperiodic minimal subshifts

- For $n \in \mathbb{N}$ and for a finite collection of symbols $\mathcal{A}$, which we refer to as an alphabet, we define $\mathcal{A}^n$ to be the set of all finite words of length n, and set

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- A subshift is called aperiodic if it admits no periodic elements.
- A subshift is called minimal if every point has a dense orbit.

Regularity of subshifts

Definition (Repulsive)

A subshift Y is called repulsive if

$$\inf\left\{\frac{|W|-|w|}{|w|} : w, W \in \mathcal{L}(Y), w \text{ is a prefix and suffix of } W, \text{ and } W \neq w \neq \emptyset\right\} > 0.$$

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For a subshift Y and for $n \in \mathbb{N}$ set

$$Q(n) := \sup\{p \in \mathbb{N}: \text{there exists } W \in \mathcal{L}(Y) \text{ with } |W| = n \text{ and } W^p \in \mathcal{L}(Y)\}$$

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Definition (Linearly repetitive)

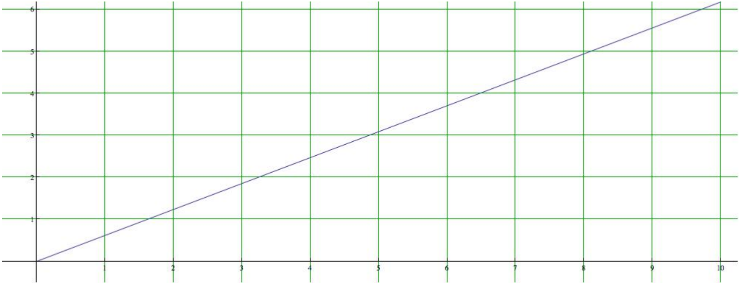
A subshift Y is called linearly repetitive if

$$\limsup_{n\to\infty} \frac{R(n)}{n} < \infty,$$

where the repetitive function $R: \mathbb{N} \to \mathbb{N}$ of a subshift Y assigns to n the smallest n' such that any element of $\mathcal{L}(Y)$ with length n' contains (as factors) all elements of $\mathcal{L}(Y)$ with length n.

Example: Sturmian subshifts

Sturmian word with irrational slope $\theta \in (0, 1)$.



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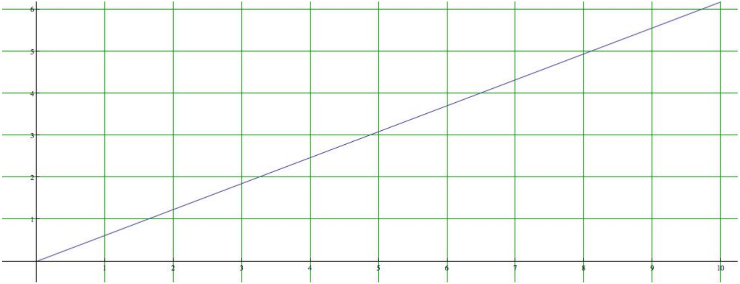

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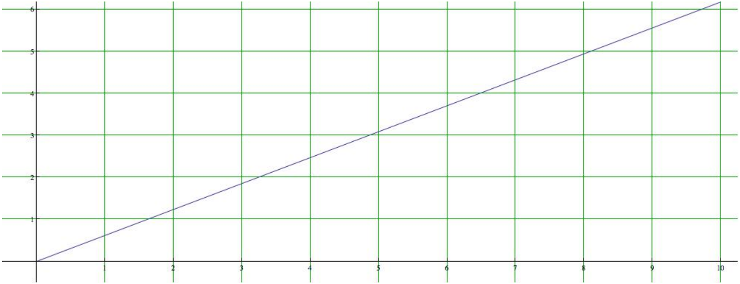

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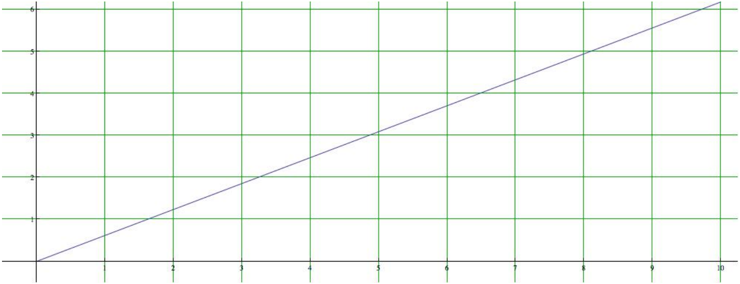

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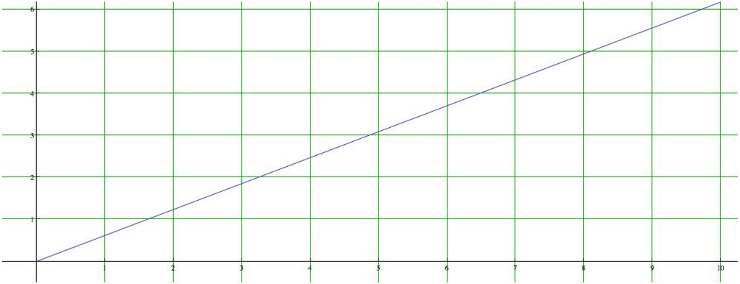

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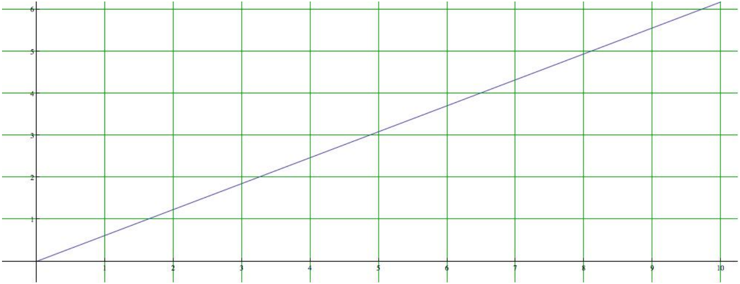

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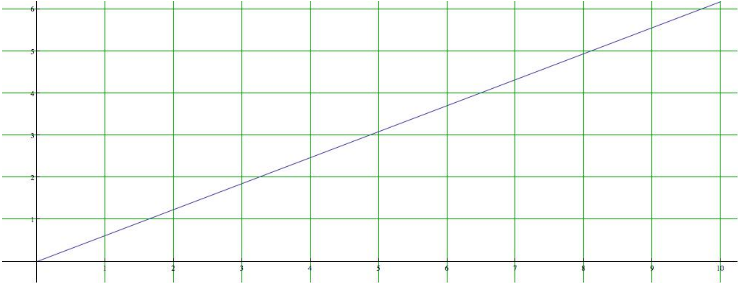

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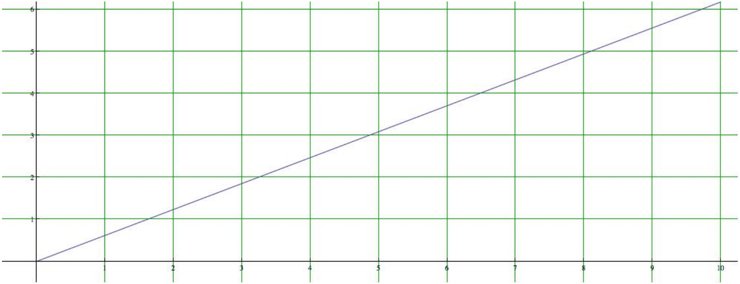

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Definition (Sturmian word)

Let $\theta \in [0, 1]$ be irrational. Define the Sturmian word $x := (x_n)_{n \in \mathbb{N}}$ of slope θ by

$$x_n := \lceil \theta(n+1) \rceil - \lceil \theta n \rceil.$$

Definition (Sturmian subshift)

Let $x := (x_n)_{n\in\mathbb{N}}$ denote a Sturmian word of slope θ. The set

$$X = \Omega(x) := \overline{\{\sigma^k(x_1, x_2, \dots) \colon k \in \mathbb{N}_0\}}$$

is called the Sturmian subshift of slope θ.

Here the closure is taken with respect to the product topology.

Jacques Charles François Sturm
(29.09.1803–15.12.1855)

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Properties of Sturmian subshifts

- A Sturmian subshift is aperiodic.
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Theorem ([HM40, FBF$^+$02, KS12])

The following are equivalent.

- *The continued fraction entries of θ are bounded.*
- *A Sturmian subshift X of slope θ is linearly repetitive.*
- *A Sturmian subshift X of slope θ is repulsive.*
- *A Sturmian subshift X of slope θ is power free.*

Definition (α-repulsive)

Let $\alpha \geq 1$ be given. For a subshift Y set

$$\ell_\alpha := \liminf_{n\to\infty} A_{\alpha,n},$$

where for a given natural number $n \geq 2$,

$$A_{\alpha,n} := \inf\left\{\frac{|W| - |w|}{|w|^{1/\alpha}} : w, W \in \mathcal{L}(X), w \text{ is a prefix and suffix of } W, |W| = n \text{ and } W \neq w \neq \emptyset\right\}.$$

If ℓ_α is finite and non-zero, then we say that Y is α-repulsive.

Definition (α-finite)

Recall:

$$Q(n) := \sup\{p \in \mathbb{N} \colon \text{there exists } W \in \mathcal{L}(Y) \text{ with } |W| = n \text{ and } W^p \in \mathcal{L}(Y)\}.$$

For $\alpha \geq 1$ we say that a subshift is α-finite if the value

$$\limsup_{n\to\infty} \frac{Q(n)}{n^{\alpha-1}}$$

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Definition (α-repetitive)

Recall: The repetitive function $R\colon \mathbb{N} \to \mathbb{N}$ of a subshift Y assigns to r the smallest r' such that any element of $\mathcal{L}(Y)$ with length r' contains (as factors) all elements of $\mathcal{L}(Y)$ with length r.

Let $\alpha \geq 1$ be given and set

$$R_\alpha := \limsup_{n\to\infty} \frac{R(n)}{n^\alpha}.$$

A subshift Y is called α-repetitive if R_α is finite and non-zero, where R denotes the repetitive function of Y.

These results also holds for subshifts, which are not Sturmian.

Proposition ([GKM+16])

1-repulsive implies repulsive.

1-finite implies power free.

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Theorem ([DKM+17])

A subshift that is α-repulsive or α-finite is aperiodic.

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$$\langle a, x, y, z \colon 1 = a^2 = x^2 = y^2 = z^2 = \kappa^k((az)^4) = \kappa^k((axayay)^4) \text{ for all } k \in \mathbb{N}_0 \rangle$$

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- Alternatively, this subshift can be generated by the three semi-group homomorphisms τ_β, where $\beta \in \{x, y, z\}$ is defined by

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- The word η is the unique infinite word such that, for all $n \in \mathbb{N}$, η has the prefix

$$(\tau_x \circ \tau_y \circ \tau_z)^n(a) = \underbrace{\underbrace{\tau_x \circ \tau_y \circ \tau_z} \circ \underbrace{\tau_x \circ \tau_y \circ \tau_z} \circ \cdots \circ \underbrace{\tau_x \circ \tau_y \circ \tau_z}}_{n\text{–times}}(a).$$

Generating the sequence

First 15 letters

$$\kappa^3(a) = \tau_x \circ \tau_y \circ \tau_z(a) = (a, x, a, y, a, x, a, z, a, x, a, y, a, x, a).$$

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One can show that the η is the word obtained from the following procedure.

$$(a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_\dots)$$

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Generating the sequence

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$$\kappa^3(a) = \tau_x \circ \tau_y \circ \tau_z(a) = (a, x, a, y, a, x, a, z, a, x, a, y, a, x, a).$$

One can show that the η is the word obtained from the following procedure.

$$(a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_a\_\ldots)$$
$$(axa\_axa\_axa\_axa\_axa\_axa\_axa\_axa\_axa\_axa\_axa\_axa\_axa\_ax\ldots)$$
$$(axayaxa\_axayaxa\_axayaxa\_axayaxa\_axayaxa\_axayaxa\_axayax\ldots)$$
$$(axayaxazaxayaxa\_axayaxazaxayaxa\_axayaxazaxayaxa\_axayax\ldots)$$

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What happens if we fill in the same letter multiple times in a row?

We now introduce a more general class of subshifts based on this latter construction, which we call l-Grigorchuk subshifts, where each $l = (l_k)_{k\in\mathbb{N}}$ is a sequence of natural numbers.

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For $j \in \mathbb{N}$, define $\tau^{(j)}$ by

$$\tau^{(j)}(a) := \begin{cases} \tau_x^{l_1} \circ \tau_y^{l_2} \circ \tau_z^{l_3} \circ \cdots \circ \tau_z^{l_j}(a) & \text{if } j \equiv 0 \pmod 3, \\ \tau_x^{l_1} \circ \tau_y^{l_2} \circ \tau_z^{l_3} \circ \cdots \circ \tau_x^{l_j}(a) & \text{if } j \equiv 1 \pmod 3, \\ \tau_x^{l_1} \circ \tau_y^{l_2} \circ \tau_z^{l_3} \circ \cdots \circ \tau_y^{l_j}(a) & \text{if } j \equiv 2 \pmod 3, \end{cases}$$

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Proposition ([DKM+17])

For $l = (l_k)_{k \in \mathbb{N}}$, there exists a unique $\eta_l \in \{a, x, y, z\}^{\infty}$ with prefix $\tau^{(j)}(a)$, for all $j \in \mathbb{N}_0$. Moreover,

$$\Omega(\eta) := \overline{\{\sigma^k(\eta_1, \eta_2, \dots) \colon k \in \mathbb{N}_0\}}$$

is an aperiodic minimal subshift.

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Theorem ([DKM+17])

For $\alpha \geq 1$ an l-Grigorchuk subshift is α-repulsive, and hence α-finite if and only if

$$\limsup_{n\to\infty} \left| l_{n+1} + (1-\alpha) \sum_{i=1}^{n} l_i \right| < \infty.$$

What can be deduced from the sequence *l* about the subshift?

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Theorem ([DKM+17])

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- If I is a bounded sequence, then the associated I-Grigorchuk subshift is 1-repetitive and 1-repulsive, and hence, 1-finite.

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- Let $b \geq 2$ denote a fixed integer. If $I = (b^n)_{n \in \mathbb{N}}$, then the associated I-Grigorchuk subshift is b-repulsive, and hence b-finite, and b^2-repetitive.

Example ([DKM$^+$17])

- If l is a bounded sequence, then the associated l-Grigorchuk subshift is 1-repetitive and 1-repulsive, and hence, 1-finite.
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- Let $(b_n)_{n\in\mathbb{N}}$ denote a bounded sequence, and set $l_n = 2^{n/2} - b_{n/2}$ if n is even, and set $l_n = b_{(n+1)/2}$ otherwise. The associated l-Grigorchuk subshift is 2-repetitive, however, it is not α-repulsive nor α-finite, for any value of $\alpha \geq 1$.

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- Let $(b_n)_{n \in \mathbb{N}}$ denote a bounded sequence, and set $l_n = 2^{n/2} - b_{n/2}$ if n is even, and set $l_n = b_{(n+1)/2}$ otherwise. The associated l-Grigorchuk subshift is 2-repetitive, however, it is not α-repulsive nor α-finite, for any value of $\alpha \geq 1$.
- If $l = (l_n)_{n \in \mathbb{N}}$ is a sequence of natural number such that there exists a non-constant polynomial P with $l_n = P(n)$, then the l-Grigorchuk subshift is neither α-repulsive, α-finite nor α-repetitive, for any value of $\alpha \geq 1$.

The second example actually holds in general.

Proposition ([DKM+17])

Let I be a sequence of natural numbers. If the I-Grigorchuk subshift is α-repulsive, and hence α-finite, then it is α^2-repetitive.

Proof

We set $c := \limsup_{n\to\infty} |l_{n+1} + (1-\alpha)\sum_{i=1}^{n} l_i|$, which is a finite real number. For all $\epsilon > 0$, there exists an $N \in \mathbb{N}$, such that, for all $n \geq N$,

$$\alpha - \frac{c+\epsilon}{\sum_{i=1}^{n} l_i} \leq 1 + \frac{l_{n+1}}{\sum_{i=1}^{n} l_i} \leq \alpha + \frac{c+\epsilon}{\sum_{i=1}^{n} l_i},$$

Observe that, for all $\delta \geq 1$ and $n \in \mathbb{N}$,

$$l_{n+2} + (1-\delta)\sum_{i=1}^{n+1} l_i = l_{n+2} + l_{n+1} + \left(1 - \delta\left(1 + \frac{l_{n+1}}{\sum_{i=1}^{n} l_i}\right)\right)\sum_{i=1}^{n} l_i.$$

Hence applying the previous inequalites gives

$$\left| l_{n+2} + l_{n+1} + (1 - \delta\alpha) \sum_{i=1}^{n} l_i \right| \leq \left| \delta(c + \epsilon) + l_{n+2} + (1 - \delta) \sum_{i=1}^{n+1} l_i \right|,$$

for all $n \geq N$. As $\delta \to \alpha$, the result follows.

Recall

For $\beta \geq 1$ an l-Grigorchuk subshift is β-repetitive if and only if

$$\limsup_{n\to\infty} \left| l_{n+2} + l_{n+1} + (1 - \beta) \sum_{i=1}^{n} l_i \right| < \infty.$$

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Thank you for your attention.