Hausdorff dimensions and hitting probabilities of random covering sets

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Institut Mittag-Leffler, 28 September 2017

Random covering problem on the circle

Covering model

- Let $\{\xi_n\}$ be a sequence of i.i.d. random variables uniformly distributed on the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ ($\xi_n : \Omega \to \mathbb{T}$, $\mathbb{P} \circ \xi_n^{-1} = \mathcal{L}$)
- Let $\{l_n\}$ be a decreasing sequence of positive real numbers $(0 < l_n < 1, l_n \downarrow 0)$
- Random intervals : $I_n(\omega) := \xi_n(\omega) + (0, l_n)$
- Random covering set

$$\begin{aligned} E(\omega) &:= \{t \in \mathbb{T} : t \in I_n(\omega) \text{ infinitely often}\} \\ &= \limsup_{n\to\infty} I_n(\omega) \end{aligned}$$

- Alternatively,

$$E(\omega) = \left\{ t \in \mathbb{T} : \sum_{n=1}^{\infty} \chi_{(0,l_n)}(t - \xi_n(\omega)) = +\infty \right\}$$

Sizes of covering sets

- the roles of the two measures
  $\mathbb{P}$: measures the randomness of the initial points of the random intervals ($\mathbb{P} \circ \xi_n^{-1} = \mathcal{L}$)
  $\mathcal{L}$: measures the lengths of the random intervals
- Questions : How can we describe the covering set $E(\omega)$ and uncovered set $E(\omega)^c$? How large are they, for example, from the topological, measure, dimensional viewpoints etc ?

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- Questions : How can we describe the covering set $E(\omega)$ and uncovered set $E(\omega)^c$? How large are they, for example, from the topological, measure, dimensional viewpoints etc?
- Kahane (1985)
  E is almost surely dense on $\mathbb{T}$ and is of second category.
- Borel-Cantelli Lemma and Fubini's Theorem implies almost surely

$$\mathcal{L}(E(\omega)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} l_n < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} l_n = \infty. \end{cases}$$

Precise questions

- Case $\sum_{n=1}^{\infty} l_n = +\infty$,
  - (Dvoretzky's problem, 1956) ?$\Longleftrightarrow E = \mathbb{T}$ a.s.
    Find a sufficient and necessary condition such that
    $$\mathbb{P}(E(\omega) = \mathbb{T}) = 1.$$
  - (Kahane's problem, 1968) Let $K \subset \mathbb{T}$ be a compact set. Find a sufficient and necessary condition for
    $$\mathbb{P}(K \subset E(\omega)) = 1.$$
  - (Carleson's problem) How can we describe the infinity of the set of intervals covering a given point ?
  - If $E \neq \mathbb{T}$, how large is the uncovered set ? $\dim_{\mathrm{H}} E^c =?$
- Case $\sum_{n=1}^{\infty} l_n < +\infty$,
  - $\dim_{\mathrm{H}} E =?$
  - (hitting probability problem) $? \Longrightarrow \mathbb{P}(E \cap F \neq \emptyset) > 0$ ($F \subset \mathbb{T}$ is a determinstic set)
  - $\dim_{\mathrm{H}} E \cap F =?$

Dvoretzky covering problem

Case $\sum_{n=1}^{\infty} l_n = \infty$

- Dvoretzky's problem (1956)
  Find a sufficient and necessary condition such that

$$\mathbb{P}(E(\omega) = \mathbb{T}) = 1.$$

- $\sum_{n=1}^{\infty} l_n = \infty$ is necessary
- Dvoretzky(1956)
  There exist sequence $\{l_n\}$ such that almost surely

$$\mathcal{L}(E(\omega)) = 1, \quad \text{but} \quad E(\omega) \neq \mathbb{T}.$$

  That is, $\mathcal{L}$-almost every point is covered but not every point on the circle is covered.
- Dvoretzky(1956)

$$\limsup_{n\to\infty}(nl_n - 2\log\frac{1}{l_n}) > -\infty \Longrightarrow \mathbb{P}(E(\omega) = \mathbb{T}) = 1$$

Dvoretzky covering problem

- Billard (1965), Kahane (1968)

$$\limsup_{n\to\infty} \frac{1}{n} \exp(l_1 + l_2 + \cdots + l_n) = \infty$$

is a sufficient condition, and

$$\sum_{n=1}^{\infty} l_n^2 \exp(l_1 + l_2 + \cdots + l_n) = \infty$$

is a necessary one.

- $l_n = \frac{c}{n}$
From the above conditions, when $c > 1$ the whole unit circle is infinitely covered, while this is not the case for $0 < c < 1$. The condition is invalid for $c = 1$.

Dvoretzky covering problem

- Orey (unpublished), Mandelbrot (1972)
  improve the sufficient condition to

$$\limsup_{n\to\infty} \frac{1}{n} \exp(l_1 + l_2 + \cdots + l_n) > 0$$

which implies that $c = 1$ also gives an infinite covering.

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  which implies that $c = 1$ also gives an infinite covering.

Theorem (Shepp, 1972)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \exp(l_1 + l_2 + \cdots + l_n) = \infty \Longleftrightarrow \mathbb{P}(E(\omega) = \mathbb{T}) = 1$$

Covering a subset (Kahane's problem)

- Kahane's problem (1968) Let $K \subset \mathbb{T}$ be a compact set. Find a sufficient and necessary condition for

$$\mathbb{P}(K \subset E) = 1.$$

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- Remark : $K \subset E \iff E^c \cap K = \emptyset$
- (Kahane 1968)$(l_n = \frac{c}{n}, 0 < c < 1)$
$$\mathbb{P}\big(K \subset E\big) = \begin{cases} 0 & \text{if } \dim_{\mathrm{H}}(K) < c, \\ 1 & \text{if } \dim_{\mathrm{H}}(K) > c. \end{cases}$$
- (Hawkes 1973) Suppose $\dim_{\mathrm{H}}(K) = \dim_{\mathrm{P}}(K)$, then
$$\mathbb{P}\big(K \subset E\big) = \begin{cases} 0 & \text{if } \dim_{\mathrm{H}}(K) < \tau, \\ 1 & \text{if } \dim_{\mathrm{H}}(K) > \tau, \end{cases}$$
where $\tau = \limsup_{n\to\infty} \frac{\sum_{i=1}^{n} l_i}{\log n}$.

Covering a subset (Kahane's problem)

- Kahane (1987)
  Let $x^+ = \max\{x, 0\}$ and

$$k(t) := \exp\left(\sum_{n=1}^{\infty}(l_n - t)^+\right).$$

Then

$$\mathbb{P}(K \subset E) = 0$$

if and only if K carries a positive measure σ such that

$$\int_{\mathbb{T}}\int_{\mathbb{T}} k(t-s)\sigma(dt)\sigma(ds) < \infty,$$

that is, K has a positive capacity with respect to the potential $k(\cdot)$.

Carleson's problem

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  How can we describe the infinity of the set of intervals covering a given point ?

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  How can we describe the infinity of the set of intervals covering a given point ?
- Fan and Kahane (1993) ($l_n = \frac{c}{n}$)
  $c > 0$:

$$c < \limsup_{N\to\infty} \frac{\max_{t\in\mathbb{T}} \sum_{n=1}^{N} \chi_{I_n(\omega)}(t)}{\log N} < \infty \quad \text{a.s.}$$

  $c > 1$: (when $c = 1$, the below inequalities are not true)

$$0 < \liminf_{N\to\infty} \frac{\min_{t\in\mathbb{T}} \sum_{n=1}^{N} \chi_{I_n(\omega)}(t)}{\log N} < c \quad \text{a.s.}$$

Carleson's problem

- How many times a point is covered ? (Fan 2002) ($l_n = \frac{c}{n}, c > 0$)

$$E_\beta = \left\{ t \in \mathbb{T} : \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \chi_{I_n(\omega)}(t) = \beta \right\}$$

  For a certain interval of $\beta > 0$, the Hausdorff dimension of E_β is equal to $1 - [\beta \log(\beta/c) - (\beta - c)]$, which implies that points on the circle are differently covered.

- Barral and Fan (2005)
  They calculated the Hausdorff dimension of the sets E_β for all $\beta > 0$.

$$\lim_{n\to\infty} n l_n = \infty \Longrightarrow \text{a.s.} \lim_{N\to\infty} \frac{\sum_{n=1}^{N} \chi_{I_n(\omega)}(t)}{\sum_{n=1}^{N} l_n} = 1 \quad \forall t \in \mathbb{T}$$

- Example $l_n = \frac{\log n}{n}$

the size of uncovered set

- Kahane (1968) $l_n = \frac{c}{n}(0 < c < 1)$

$$\dim_{\mathrm{H}} E^c = 1 - c \quad a.s.$$

- Kahane (1985)

$$\dim_{\mathrm{H}} E^c = 1 - \limsup_{n\to\infty} \frac{l_1 + \cdots + l_n}{\log n} \quad a.s.$$

Hausdorff dimension of covering set

Case $\sum_{n=1}^{\infty} l_n < \infty$

- Fan and Wu (2004)($l_n = \frac{a}{n^\gamma}, a > 0, \gamma > 1$)

$$\dim_H(E) = \frac{1}{\gamma} \quad \text{a.s.}$$

- Durand (2010)

$$\dim_H(E) = \alpha = \alpha(l_n) := \inf\left\{ s > 0 : \sum_{n=1}^{\infty} l_n^s < \infty \right\} \quad a.s.,$$

  Moreover, the set E is a set with large intersection.

- Remark : The dimensional result above can be obtained directly by so-called mass transference principle (Beresnevich and Velani, 2006)

Hitting probability of covering set

Case $\sum_{n=1}^{\infty} l_n < \infty$

- Hitting probability question :
  Given a sequence $\{l_n\}$ with $\sum_{n=1}^{\infty} l_n < +\infty$, under what conditions on a measurable set F, we have

$$\mathbb{P}\big(E \cap F \neq \emptyset\big) > 0?$$

Furthermore, how large is the intersection set $E \cap F$?

Hitting probability of covering set

- It can be shown that

$$\limsup_{k\to\infty} \frac{\log_2 n_k}{k} = \alpha,$$

where

$$n_k = \#\left\{n \in \mathbb{N} : l_n \in [2^{-k+1}, 2^{-k+2})\right\} \qquad (k \geq 2).$$

- Condition (C) : There exists an increasing sequence of positive integers $\{k_i\}$ such that

$$\lim_{i\to\infty} \frac{k_{i+1}}{k_i} = 1 \quad \text{and} \quad \lim_{i\to\infty} \frac{\log_2 n_{k_i}}{k_i} = \alpha < 1.$$

Examples : $l_n = \frac{a}{n^\gamma}, a > 0, \gamma > 1$; $l_n = \frac{1}{\beta^n}, \beta > 1$.

Hitting probability of covering set

Theorem (Li,Shieh,Xiao,2013)

Let E be the random covering set associated with the sequence $\{l_n\}$. If the condition (C) holds, then for every analytic set $F \subset \mathbb{T}$, we have

$$\mathbb{P}\big(E \cap F \neq \emptyset\big) = \begin{cases} 0 & \textit{if } \dim_{\mathrm{P}}(F) < 1 - \alpha, \\ 1 & \textit{if } \dim_{\mathrm{P}}(F) > 1 - \alpha. \end{cases}$$

Remark

The conclusion $\dim_{\mathrm{P}}(F) < 1 - \alpha$ implies $\mathbb{P}(E \cap F \neq \emptyset) = 0$ holds even without the condition (C).

Hitting probability of covering set

Theorem (Li,Shieh,Xiao,2013)

Let E be the random covering set associated with the sequence $\{l_n\}$ which satisfies the condition (C). If $\dim_{\rm P}(F) > 1-\alpha$, then

$$\dim_{\rm P}(E \cap F) = \dim_{\rm P}(F) \qquad a.s.$$

and

$$\dim_{\rm H}(F) - (1-\alpha) \le \dim_{\rm H}(E \cap F) \le \dim_{\rm P}(F) - (1-\alpha) \quad a.s.$$

In particular, if $\dim_{\rm H}(F) = \dim_{\rm P}(F)$, then

$$\dim_{\rm H}(E \cap F) = \dim_{\rm H}(F) - (1-\alpha) \quad a.s.$$

Hitting probability of covering set

Theorem (Li, Suomala, 2014)

Let $F \subset \mathbb{T}$ be an analytic set. Then

$$\mathbb{P}(E \cap F \neq \emptyset) = \begin{cases} 0 & \textit{if } \dim_{\mathrm{P}}(F) < 1 - \alpha, \\ 1 & \textit{if } \dim_{\mathrm{H}}(F) > 1 - \alpha. \end{cases}$$

Furthermore,

$$\dim_{\mathrm{H}}(F) - (1 - \alpha) \leq \dim_{\mathrm{H}}(E \cap F) \leq \dim_{\mathrm{P}}(F) - (1 - \alpha) \quad a.s.$$

Proposition (Li, Suomala, 2014)

There are (l_n) such that $\alpha(l_n) = 1$ and a closed set $F \subset \mathbb{T}$ with $\dim_{\mathrm{P}}(F) = 1$ while $E \cap F = \emptyset$ almost surely.

Hitting probability of covering set

Proposition (Li, Suomala, 2014)

Given $0 \leq \alpha, t \leq 1$, there is a sequence (l_n) with $\alpha = \alpha(l_n)$ and a closed set $F \subset \mathbb{T}$ with $\dim_{\mathrm{H}}(F) = t, \dim_{\mathrm{P}}(F) = 1$ such that almost surely, $\dim_{\mathrm{H}}(E \cap F) = \min\{\alpha, t\}$. In particular, it is possible that a.s.

$$\dim_{\mathrm{H}}(F) + \alpha - 1 < \dim_{\mathrm{H}}(E \cap F) < \dim_{\mathrm{P}}(F) + \alpha - 1.$$

Dynamical covering

- doubling map $T : \mathbb{T} \to \mathbb{T}, x \mapsto 2x(\text{mod}1)$
- Let ν_ϕ, ν_ψ be two T-invariant probability Gibbs measures on $\mathbb{T}$ associated to normalized Hölder potentials ϕ and ψ (that is, the pressures of ϕ and ψ are equal to zero).
- $\mathbb{P} \to \nu_\phi$, $\mathcal{L} \to \nu_\psi$
- $l_n = 1/n^\gamma$
- Fan, Schmeling and Troubetzkoy (2013) For ν_ϕ-a.e. x, we have

$$\dim_{\mathrm{H}} E = \begin{cases} \gamma^{-1} & \text{if } \gamma^{-1} \le h_{\nu_\phi} (:= \text{measure-theoretic entropy of } \nu_\phi) \\ D_{\nu_\phi}(\gamma^{-1}) & \text{if } h_{\nu_\phi} < \gamma^{-1} < e_{\max} (:= \int (-\phi) d\mathcal{L}) \\ 1 & \text{if } \gamma^{-1} \ge e_{\max} \end{cases}$$

and

$$\dim_{\mathrm{H}} F = \begin{cases} 1 & \text{if } \gamma^{-1} \le e_{\max} \\ D_{\nu_\phi}(\gamma^{-1}) & \text{if } \gamma^{-1} > e_{\max} \end{cases}$$

where $D_{\mu_\phi}(t) := \dim_{\mathrm{H}}\{y : \lim_{r \to 0} \frac{\log \nu_\phi(B(x,r))}{\log r} = t\}$.

Dynamical covering

Remark

The previous result was generalized to expanding Markov map with a finite partition by Liao and Seuret (2013).

Randomized version of the Littlewood conjecture

Theorem (Haynes and Koivusalo, 2017)

Let $(\gamma_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ be sequences of independent random variables taking values which are uniformly distributed (with respect to Lebesgue measure) in $[0,1)$. Then almost surely we have that, for all $\alpha,\beta\in\mathbb{R}$,

$$\liminf_{n\to\infty} n(\log n)\|n\alpha-\gamma_n\|\|n\beta-\delta_n\|\le 1.$$

Littlewood conjecture

For every $\alpha,\beta\in\mathbb{R}$,

$$\liminf_{n\to\infty} n\|n\alpha\|\|n\beta\|=0.$$

stronger statement : for every $\alpha,\beta\in\mathbb{R}$,

$$\liminf_{n\to\infty} n(\log n)\|n\alpha\|\|n\beta\|<\infty.$$

Random covering problem in higher dimensional torus

Covering model

- Let $\{\xi_n\}$ be a sequence of i.i.d. random variables uniformly distributed on the torus $\mathbb{T}^d$ ($\xi_n : \Omega \to \mathbb{T}^d$, $\mathbb{P} \circ \xi_n^{-1} = \mathcal{L}^d$)
- Let $\{g_n\}$ be a sequence of subsets of the torus $\mathbb{T}^d$
- Random sets : $G_n(\omega) := \xi_n(\omega) + g_n$
- Random covering set

$$\begin{aligned} E(\omega) &:= \{t \in \mathbb{T}^d : t \in G_n(\omega) \text{ infinitely often}\} \\ &= \limsup_{n\to\infty} G_n(\omega) \end{aligned}$$

- Borel-Cantelli Lemma and Fubini's Theorem implies almost surely

$$\mathcal{L}^d(E(\omega)) = \begin{cases} 0 & \text{if } \sum_{n=1}^{\infty} \mathcal{L}^d(g_n) < \infty \\ 1 & \text{if } \sum_{n=1}^{\infty} \mathcal{L}^d(g_n) = \infty. \end{cases}$$

Dvoretzkoy covering problem on the torus

Case $\sum_{n=1}^{\infty} \mathcal{L}^d(g_n) = \infty$

- Assumption on g_n
  Let g be a convex set in $\mathbb{R}^d$ and g_n be homothetic to g with volume v_n decreasingly tending to zero as $n \to \infty$.
- El Hélou (1978)
  Let $v_n = \frac{a}{n}$. Then
  $a > 1 \Longrightarrow \mathbb{P}(\mathbb{T}^d = E) = 1$
  $0 < a < 1 \Longrightarrow \mathbb{P}(\mathbb{T}^d = E) = 0$
  $0 < a < 1, \ \dim_{\mathrm{H}} K < ad \Longrightarrow \mathbb{P}(K \subset E) = 1$
  $0 < a < 1, \ \dim_{\mathrm{H}} K > ad \Longrightarrow \mathbb{P}(K \subset E) = 0$
- Kahane (1990) Let g be a simplex. Then

$$\mathbb{P}(\mathbb{T}^d = E) = 1$$

if and only if

$$\int_0^1 \exp \sum_{n=1}^{\infty} v_n \left(1 - (\frac{s}{v_n})^{1/d}\right)^+ ds = \infty.$$

Dvoretzkoy covering problem on the torus

- Open problem
  When g is a ball or cube, the Dvoretzky's problem in $\mathbb{T}^d$ is still open.
- Shiu (2013)
  Let $d = 2$ and g_n be a sequence of squares with length v_n. Denote $K = \{0\} \times \mathbb{R}/\mathbb{Z}$. Then

$$\mathbb{P}(K \subset E(\omega)) = 1$$

if and only if

$$\sum_{n=1}^{\infty} \frac{v_n}{(\sum_{i=1}^{n} v_i)^2} \exp(\sum_{i=1}^{n} v_i^2) = \infty.$$

Hausdorff dimension of covering set

Case $\sum_{n=1}^{\infty} \mathcal{L}^d(g_n) < \infty$

Remark (Hausdorff dimension)

Since $\{\omega \in \Omega : \dim_{\mathrm{H}} E(\omega) = s\}$ is a tail event for all $0 \leq s \leq d$ and so, by Kolmogorov's zero-one law, there exists $\alpha \in [0, d]$ such that

$$\dim_{\mathrm{H}} E = \alpha \quad a.s.$$

Theorem (Koivusalo, Järvenpää, Järvenpää, Li, Suomala, Xiao, 2017)

Let $F \subset \mathbb{T}^d$ be an analytic set. Then

$$\dim_{\mathrm{P}} F < d - \alpha \Rightarrow \mathbb{P}(E \cap F \neq \emptyset) = 0$$

and

$$\dim_{\mathrm{P}} F \geq d - \alpha \Rightarrow \dim_{\mathrm{H}}(E \cap F) \leq \alpha + \dim_{\mathrm{P}} E - d.$$

Affine random covering sets

- Assume that (g_n) is a sequence of rectangles.

Definition

For a rectangle g and $0 < s \leq d$, define the singular value function of g as

$$\Phi^s(g) = a_1(g) \cdots a_{m-1}(g) a_m(g)^{s-(m-1)}$$

where $0 < a_d(g) \leq \cdots \leq a_1(g) < 1$ are the lengths of the edges of g in decreasing order and m is the integer such that $m - 1 < s \leq m$.

Affine random covering sets

Theorem (Järvenpää,Järvenpää,Koivusalo,Li,Suomala,2014)

Assume that (g_n) is a sequence of rectangles such that for all $i = 1, 2, \cdots, d$ the sequence of lengths $a_i(g_n)$ decreases to 0 as n tends to ∞. Then almost surely

$$\dim_{\mathrm{H}} E = \alpha(g_n),$$

where

$$\alpha(g_n) = \inf\{0 < s \leq d : \sum_{n=1}^{\infty} \Phi^s(g_n) < \infty\}.$$

Hausdorff dimension of covering set

Theorem (Persson, 2015)

Let (g_n) be a sequence of open sets in $\mathbb{T}^d$. Then the covering set E is almost surely in the class $\mathcal{G}^s(\mathbb{T}^d)$, where s is defined by

$$s = \inf\{t \geq 0 : \sum_{i=1}^{\infty} \frac{\mathcal{L}^d(g_n)^2}{I_t(g_n)} < \infty \text{ or } t = d\}$$

with the t-energy of g_n, $I_t(g_n) = \int_{g_n} \int_{g_n} \|x - y\|^{-s} dxdy$, and $\mathcal{G}^s(\mathbb{T}^d)$ is the largest collection of G_δ subsets of $\mathbb{T}^d$ with the property that it is closed under countable intersections and images of bi-Lipschitz maps, and any set in $\mathcal{G}^s(\mathbb{T}^d)$ has Hausdorff dimension at least s.

Remark

The author proved that the above s equals to $\alpha(g_n)$ when g_n are open rectangles, and here the decreasing condition on $a_i(g_n)$ was removed.

Hausdorff dimension of covering set

Definition

Given $F \subset \mathbb{R}^d$, we say that a point $x \in F$ has positive Lebesgue density with respect to F if

$$\liminf_{r \to 0} \frac{\mathcal{L}^d(F \cap B(x,r))}{\mathcal{L}^d(B(x,r))} > 0$$

and, moreover, the set F has positive Lebesgue density if all of its points have positive Lebesgue density with respect to F.

$$G_s(A) := \sup \left\{ \frac{\mathcal{L}^d(F)^2}{I_s(F)} : \ F \subset A \text{ is Lebesgue measurable and } \mathcal{L}^d(F) > 0 \right\}$$

Hausdorff dimension of covering set

Theorem (Feng, Järvenpää, Järvenpää, Suomala, arXiv : 1508.07881)

Assume that g_n is Lebesgue measurable and has positive Lebesgue density for all $n \geq 1$. Then a.s.

$$\dim_{\mathrm{H}} E = \sup\{0 \leq s \leq d : \sum_{n=1}^{\infty} G_s(g_n) = \infty\}.$$

Remark

1. They also showed that $\dim_{\mathrm{P}} E = d$ a.s. provided that g_n are Lebesgue measurable and $\mathcal{L}^d(g_n) > 0$ for infinitely many $n \in \mathbb{N}$.
2. The distribution of ξ_n can be given by any Radon probability measure on $\mathbb{T}^d$ which is not purely singular with respect to the Lebesgue measure.
3. The setting can be generalized to the smooth Riemann manifolds in this paper.

Hausdorff dimension of covering set

Theorem (Seuret, 2017)

Let μ be an invariant Gibbs measure on an irreducible topological Markov shift Σ associated with a Hölder continuous potential. Let (ξ_n) be a sequence of i.i.d. random variables with the distribution μ and $G_n = B(\xi_n, n^{-s})$. Then a.s.

- $$\dim_H E = \begin{cases} s^{-1} & \text{when } s \geq (\dim_H \mu)^{-1} \\ \underline{D}_\mu(s^{-1}) & \text{when } H_s^{-1} \leq s < (\dim_H \mu)^{-1} \\ \dim_H \Sigma & \text{when } s < H_s^{-1} \end{cases}$$

- $$\dim_H E^c = \begin{cases} \dim_H \Sigma & \text{when } s \geq H_s^{-1} \\ \underline{D}_\mu(s^{-1}) & \text{when } H_{\max}^{-1} < s < H_s^{-1} \\ 0 & \text{when } s < H_{\max}^{-1} \end{cases}$$

$\mathsf{D}_\mu(s^{-1}) = \dim_H\{x \in \Sigma : d_\mu(x) = s^{-1}\}$

Hausdorff dimension of covering set

Theorem (Ekström and Persson, arXiv : 1612.07110)

Let μ be a Borel probability measure on $\mathbb{R}^d$. Let (ξ_n) be a sequence of i.i.d. random variables with the distribution μ and $G_n = B(\xi_n, n^{-s})$. If $s > (\overline{\dim}_{\mathrm{H}}\mu)^{-1}$, then a.s.

$$\dim_{\mathrm{H}} E \geq \frac{1}{s} - \operatorname*{ess\,inf}_{\substack{x\sim\mu \\ \underline{d}_\mu(x) > s^{-1}}} (\overline{d}_\mu(x) - \underline{d}_\mu(x)).$$

$\underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$

$\overline{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$

$\overline{\dim}_{\mathrm{H}}\mu = \operatorname*{ess\,sup}_{x\sim\mu} \underline{d}_\mu(x)$

Hitting probability of affine random covering set

Hitting probability problem

Let E be a affine random covering set. Under what conditions on a measurable set F, we have

$$\mathbb{P}\big(E \cap F \neq \emptyset\big) > 0?$$

Hitting probability of affine random covering set

Hitting probability problem

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Counterexmple

Let (g_n) be a sequence of rectangles in $\mathbb{T}^2$ with the side length parallel to the x-axis $\alpha_1(g_n) = n^{-\varepsilon}$ and that to the y-axis $\alpha_2(g_n) = n^{-(1+\varepsilon)}$, where $0 < \varepsilon < 1$. Then the Hausdorff dimension $\alpha = 1 + \frac{1-\varepsilon}{1+\varepsilon}$. Let $F = [0, 1] \times \{b\}$ with $0 < b < 1$. Then

$$\dim_{\mathrm{H}} F = 1 > \frac{2\varepsilon}{1+\varepsilon} =: 2 - \alpha.$$

However,

$$\mathbb{P}(E \cap F \neq \emptyset) = 0.$$

Covering sets involving random rotations

Covering model (with rotation)

- Let $\{\xi_n\}$ be a sequence of i.i.d. random variables uniformly distributed on the torus $\mathbb{T}^d$ ($\xi_n : \Omega \to \mathbb{T}^d$, $\mathbb{P} \circ \xi_n^{-1} = \mathcal{L}^d$)
- Let $\{h_n\}$ be a sequence of i.i.d. random variables uniformly distributed on $\mathcal{O}(d)$ ($h_n : \Omega \to \mathcal{O}(d)$, $\mathbb{P} \circ h_n^{-1} = \theta$), where $\mathcal{O}(d)$ is the orthogonal group on $\mathbb{R}^d$ and θ is the Haar measure on $\mathcal{O}(d)$.
- ξ_n and h_n are independent
- Let $\{g_n\}$ be a sequence of subsets of the torus $\mathbb{T}^d$
- Random sets : $G_n(\omega) := \xi_n(\omega) + h_n(g_n)$
- Random covering set

$$\begin{aligned} E(\omega) &:= \{t \in \mathbb{T}^d : t \in G_n(\omega) \text{ infinitely often}\} \\ &= \limsup_{n\to\infty} G_n(\omega) \end{aligned}$$

Hitting probability of covering set

Remark (Hausdorff dimension)

$$\dim_{\mathrm{H}} E = \alpha^r \quad a.s.$$

Theorem (Koivusalo, Järvenpää, Järvenpää, Li, Suomala, Xiao, 2017)

Let $F \subset \mathbb{T}^d$ *be an analytic set with* $\dim_{\mathrm{H}} F > d - \alpha^r$. *Assume that* $\max\{\alpha^r, \dim_{\mathrm{H}} F\} > \frac{d+1}{2}$ *Then*

$$\alpha^r + \dim_{\mathrm{H}}(F) - d \le \dim_{\mathrm{H}}(E \cap F) \le \alpha^r + \dim_{\mathrm{P}}(F) - d \quad a.s.$$

Random covering problem in metric space

Random covering sets in metric space

- Assume that (X, dist) is a compact metric space endowed with a Borel probability measure μ which is Ahlfors d-regular for some $0 < d < \infty$. That is, there exists a constant $0 < C < \infty$ such that

$$C^{-1}r^d \leq \mu(B(x,r)) \leq Cr^d$$

  for all $x \in X$ and $0 < r < \text{diam}(X)$.
- Let $\{\xi_n\}$ be a sequence of i.i.d. random variables uniformly distributed on X ($\xi_n : \Omega \to X$, $\mathbb{P} \circ \xi_n^{-1} = \mu$)
- Let $\{r_n\}$ be a sequence of positive real numbers and $r_n \to 0$ as $n \to \infty$.
- Random sets : $G_n(\omega) := B(\xi_n(\omega), r_n)$
- Random covering set

$$\begin{aligned} E(\omega) &:= \{t \in X : t \in G_n(\omega) \text{ infinitely often}\} \\ &= \limsup_{n\to\infty} G_n(\omega) \end{aligned}$$

Hausdorff dimension of covering set

Remark (Hausdorff dimension)

The mass transference principle gives the Hausdorff dimension of E as the following

$$\dim_{\mathrm{H}}(E) = \alpha = \alpha(g_n) := \inf\left\{ s > 0 : \sum_{n=1}^{\infty} r_n^s < \infty \right\} = \limsup_{n\to\infty} \frac{\log n}{-\log r_n},$$

where $r_n = \operatorname{diam}(g_n)$.

- Condition (C) : There exists an increasing sequence of positive integers $\{k_i\}$ such that

$$\lim_{i\to\infty} \frac{k_{i+1}}{k_i} = 1 \quad \text{and} \quad \lim_{i\to\infty} \frac{\log_2 n_{k_i}}{k_i} = \alpha < d.$$

where $n_k = \#\left\{ n \in \mathbb{N} : r_n \in [2^{-k+1}, 2^{-k+2}) \right\} \qquad (k \geq 2).$

Hitting probability of covering set

- $\{g_n\}$ is a sequence of balls of X.

Theorem (Koivusalo, Järvenpää, Järvenpää, Li, Suomala, Xiao, 2017)

Let $F \subset X$ be an analytic set. Then

$$\mathbb{P}\big(E \cap F \neq \emptyset\big) = \begin{cases} 0 & \textit{if } \dim_{\mathrm{P}}(F) < d - \alpha, \\ 1 & \textit{if } \dim_{\mathrm{H}}(F) > d - \alpha. \\ 1 & \textit{if } \dim_{\mathrm{P}}(F) > d - \alpha \textit{ and Condition (C) holds.} \end{cases}$$

Hitting probability of covering set

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Theorem (Koivusalo, Järvenpää, Järvenpää, Li, Suomala, Xiao, 2017)

Let $F \subset X$ be an analytic set. Then

$$\max\{0, \alpha + \dim_{\mathrm{H}}(F) - d\} \leq \dim_{\mathrm{H}}(E \cap F) \leq \min\{d, \alpha + \dim_{\mathrm{P}}(F) - d\} \quad a.s.$$

Hausdorff dimension of random covering sets

- Let $X_1, \cdots, X_d$ be metric spaces and the product space $X_1^d = \times_{i=1}^d X_i$ with the metric
$$d(x,y) = \max_{1\le i\le d} d_i(x_i, y_i).$$
- The closed rectangle in X_1^d with centre $x = (x_1, \dots, x_d)$ and "side radii" $r = (r_1, \dots, r_d)$ is the set
$$\overline{R}(x,r) = \times_{i=1}^d \overline{B}(x_i, r_i).$$
- For$r = (r_1, \dots, r_d)$ and $s = (s_1, \dots, s_d)$ such that $r_i > 0$ and $s_i \ge 0$ for every i, let τ be a permutation of $\{1, \dots, d\}$ such that $r_{\tau(1)} \ge \cdots \ge r_{\tau_d}$ and define the singular value function Φ_r^s on $[0, s_1 + \cdots + s_d]$ by

$\Phi_r^s(t) = r_{\tau(i)}^{t - \sum_{j=1}^{i-1} s_{\tau(j)}} \cdot \Pi_{j=1}^{i-1} r_{\tau(j)}^{s_{\tau(j)}}$ for $t \in [s_{\tau(1)} + \cdots + s_{\tau(i-1)}, s_{\tau(1)} + \cdots + s_{\tau(i)}]$

Hausdorff dimension of random covering sets

- Ekström, Järvenpää, Järvenpää, Suomala (arXiv : 1705.02616v1)
  For $i = 1, \ldots, d$, let X_i be a compact metric space and let μ_i be an s_i-regular Borel probability measure on X_i. Let $\mu = \times_{i=1}^{d} \mu_i$. Then a.s.

$$\dim_{\mathrm{H}} E = \inf\{t : \sum_n \Phi^s_{r_n}(t) < \infty\} \wedge (s_1 + \cdots + s_d).$$

Thanks for your attention !