

The Combinatorics of Break Divisors

Chi Ho Yuen

Joint Work with Spencer Backman (Einstein Institute for Mathematics) and Matt Baker (Georgia Tech)

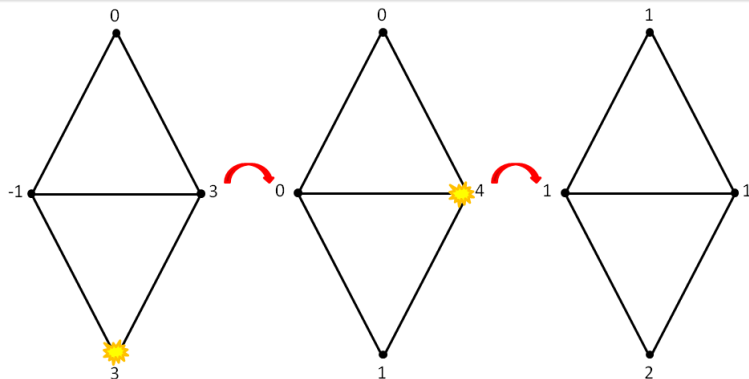
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Divisors and Chip-Firing on Finite Graphs

Definition

- *Divisor*: A configuration to put “chips” on vertices.
- $\text{Div}^d(G)$: The set of degree d divisors.
- *Chip-firing*: Remove $\deg(v)$ chips from v , and send one chip along each incident edge.



Definition

The *Jacobian* $\text{Jac}(G) = \text{Pic}^0(G)$ is $\text{Div}^0(G)/\sim$.

$D \sim D'$ if D, D' differ by a series of chip-firing moves.

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- Related to MANY areas: statistical physics/probability, combinatorics, graph theory, commutative algebra, representation theory...
- Tropical geometry: $\text{Jac}(G)$ keeps track of the information of degenerated line bundles over semistable models.

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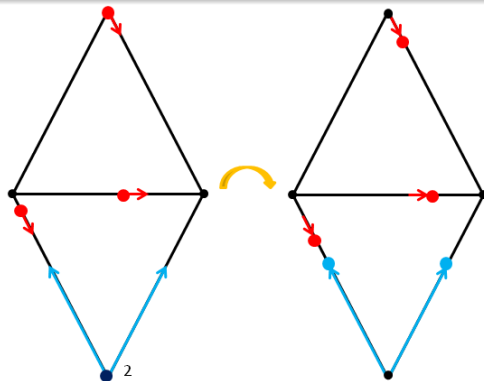
Theorem (Kirchhoff's Matrix-Tree Theorem (Version N+1))

$|\text{Jac}(G)|$ equals the number of spanning trees of G .

Metric Graphs and Divisors

Definition

- *Metric graph* Γ : Metric version of a graph, each edge is an interval (as metric space).
- *Divisor*: We can now add chips inside edges.
- *Chip-firing*: Fire chips across cuts.



$\text{Pic}^0(\Gamma)$ has the canonical structure of a g -dimensional real torus ($g := |E| - |V| + 1$ is the *genus* of Γ).

- 1 Fix a model G for Γ , and work with $C_1(G; \mathbb{R}) \cong \mathbb{R}^E$.
- 2 The subspace $H_1(G; \mathbb{R})$ and lattice $H_1(G; \mathbb{Z})$ are spanned by cycles of G (over \mathbb{R} and \mathbb{Z} , respectively).
- 3 The *tropical Jacobian* $\text{Jac}(\Gamma)$ is the torus $H_1(G; \mathbb{R})/H_1(G; \mathbb{Z})$.

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- $\text{Pic}^0(G) \cong \text{Jac}(\Gamma)$: Given a divisor $D = p_1 + \dots + p_k - q_1 - \dots - q_k$, pick a path γ_i from p_i to q_i , map D to $\gamma_1 + \dots + \gamma_k \in C_1(G; \mathbb{R})$, project to $H_1(G; \mathbb{R})$ and then to $\text{Jac}(\Gamma)$.

The Geometry of $\text{Div}_+^g(\Gamma) \rightarrow \text{Jac}(\Gamma)$

$\text{Div}_+^g(\Gamma) \cong \Gamma^g / \mathfrak{S}_g$: The space of degree g *effective* divisors.

Theorem (Mikhalkin–Zharkov 2008, An–Baker–Kuperberg–Shokrieh 2014)

There exists a unique continuous section to the map $\text{Div}_+^g(\Gamma) \rightarrow \text{Jac}(\Gamma)$.

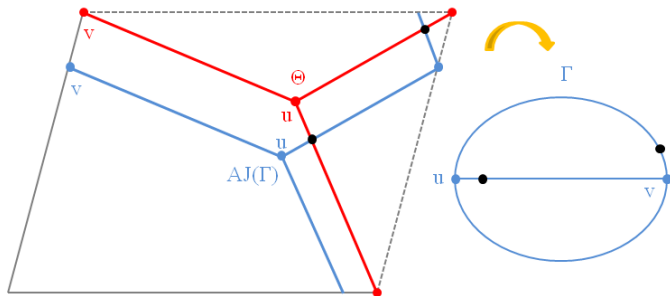
Remark

Not true in the classical world, e.g., when C is a genus 2 hyperelliptic curve, $\text{Div}_+^2(C) = \text{Sym}^2(C) \rightarrow \text{Jac}(C)$ is a blow-down map.

Proof of Existence (Mikhalkin–Zharkov)

Idea: Construct the section via tropical intersection theory.

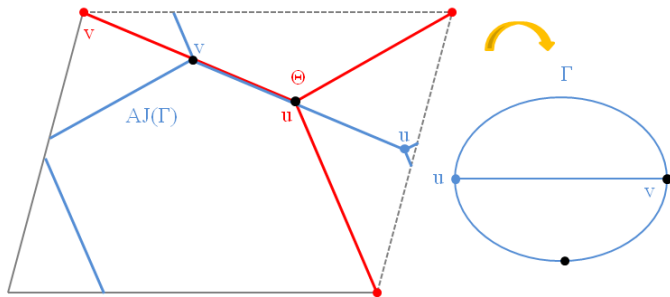
- 1 Map Γ to $\text{Jac}(\Gamma)$ by the *Abel-Jacobi map* AJ .
- 2 Take the intersection of $AJ(\Gamma)$ and a shifted copy of $\Theta = \sum_{i=1}^{g-1} AJ(\Gamma) \subset \text{Jac}(\Gamma)$ (the *tropical theta divisor*). (Θ can also be defined using tropical theta function.)
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- 3 Tropical magic: even when the intersection is not transverse, one can still take stable intersection.

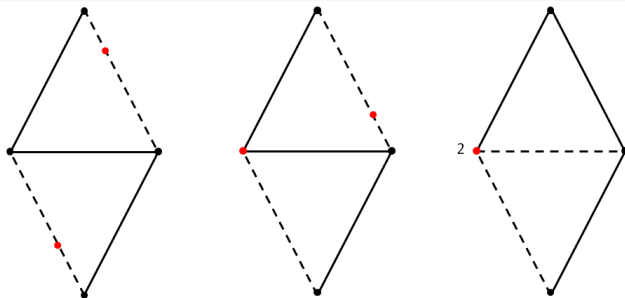


Proof of Existence (ABKS)

Idea: Write down the section combinatorially.

Definition

A *break divisor* of Γ is a divisor obtained by first choosing a spanning tree T of G and then put a chip at every edge not in T .

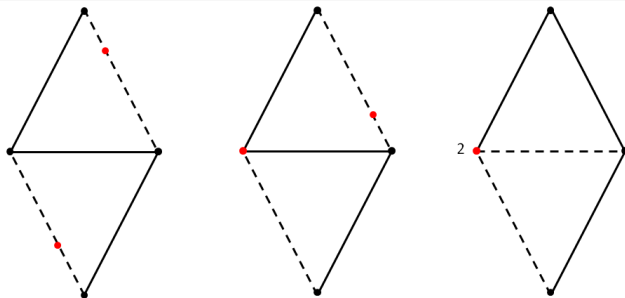


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Theorem (ABKS 2014)

Break divisors form a set of representatives for $\text{Pic}^g(\Gamma)$ (and $\text{Pic}^g(G)$).

Observation

- For a spanning tree T with $e_1, \dots, e_g \notin T$, the break divisors associated to T (essentially) form a parallelotope $\mathcal{C}_T := e_1 \times \dots \times e_g$.
- Vertices of the cells are the elements of $\text{Jac}(G)$.

The ABKS Decomposition

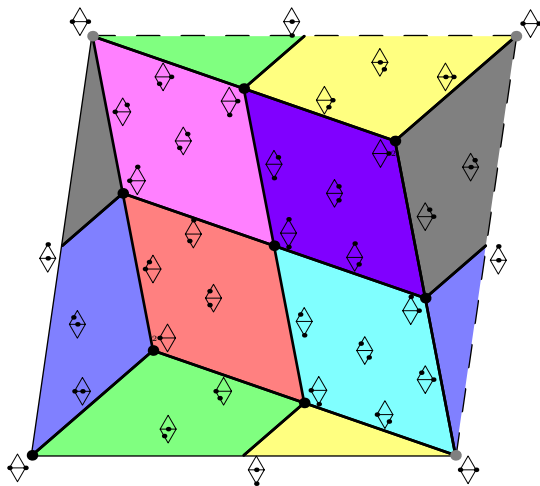
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Corollary (ABKS 2014)

$\text{Jac}(\Gamma) = \bigcup_T \mathcal{C}_T$, and the interior of two distinct cells are disjoint.

ABKS Decomposition: The Picture



Theorem (Kirchhoff's Matrix-Tree Theorem (Version $N+2$))

The volume of $\text{Jac}(\Gamma)$ is the determinant of the reduced Laplacian.

Proposition (Baker 2017+)

The Zhang measure (tropical canonical measure) of Γ is the distribution of chips of a random break divisor.

Analogous discretized statement.

Theorem (Shen 2017+)

For a nodal curve X with dual graph Γ , any Simpson compactification of $\text{Pic}^g(X)$ has a stratification “anti-isomorphic” to the face poset of the ABKS decomposition of $\text{Jac}(\Gamma)$. Specifically,

- 1 $\widehat{\text{Pic}}^g(X) = \coprod_F J^F$, where F runs over the faces of the ABKS decomposition.
- 2 $\text{codim}(J^F) = \dim(F), \forall F$.
- 3 J^F is “glued” to $J^{F'}$ iff F' is a face of F .

Geometric Bijections

Choose some generic vector \mathbf{v} . Shift each vertex along $-\epsilon\mathbf{v}$ so it lands in a cell, now we have a bijection $\Phi_{\mathbf{v}}$ between spanning trees and $\text{Jac}(G)$.

- Setup: \mathbf{v} picks an orientation for every cycle C of G , by choosing the one with positive inner product with \mathbf{v} .

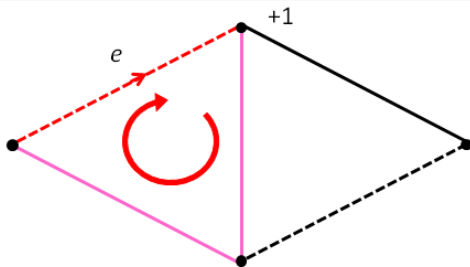
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Theorem (Y. 2017 (Matrix-Tree Theorem Version N+3))

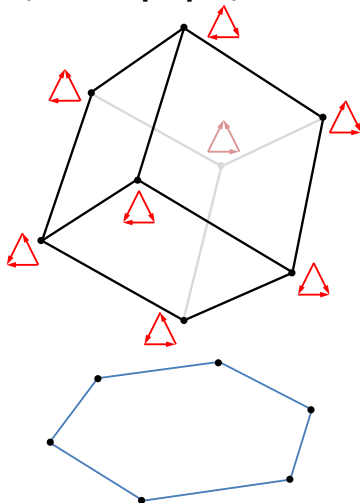
$\Phi_{\mathbf{v}}$ maps a spanning tree T to a break divisor as follows: orient each $e \notin T$ according to its fundamental cycle, put a chip at the head of e .



Zonotopes

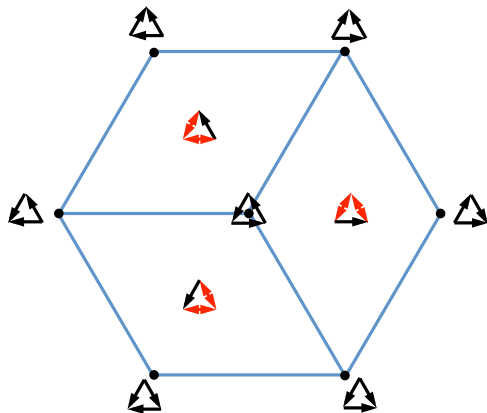
A : The incidence matrix of a graph G (or any matrix).

The zonotope Z_A is the projection of $[0, 1]^E$ by A .



Zonotopal Tilings

Using a variation of regular subdivision, Z_A can be partitioned into parallelotopes (with respect to some vector \mathbf{w}).

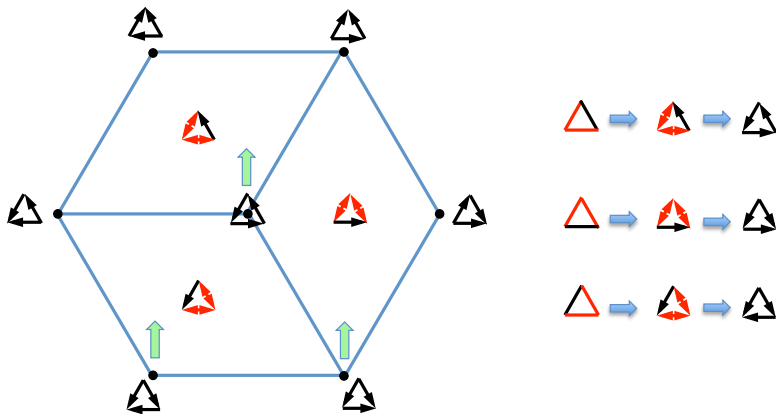


To experts: By Bohne–Dress, Zonotopal tilings \Leftrightarrow Single-element liftings.

Shifting (Again)

Shift the vertices along some generic \mathbf{v} as last time. Again we have a bijection $\Phi_{\mathbf{w}, \mathbf{v}}$ between spanning trees and (some) vertices.

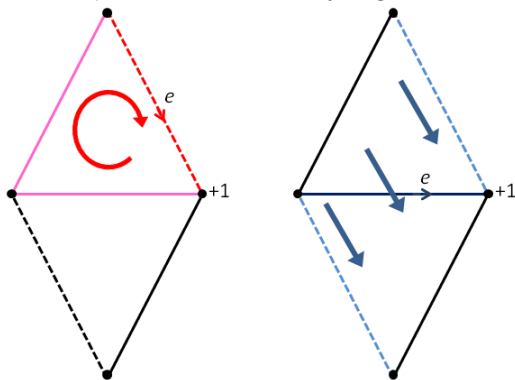
Vertices $\approx \text{Pic}^m(G) \approx \text{Jac}(G)$ by taking indegree sequence.



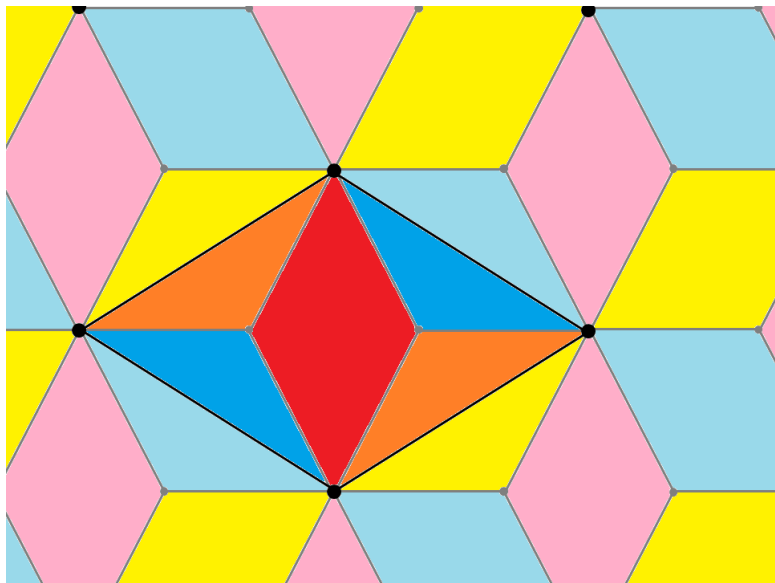
Generalized Geometric Bijections [Backman-Baker-Y. 2016+]

The vectors \mathbf{w} (for tiling) and \mathbf{v} (for shifting) pick an orientation for every cycle and cut of G . The bijection $\Phi_{\mathbf{w}, \mathbf{v}}$ can be described as follows:

- Given a spanning tree T .
- Orient each $e \notin T$ according to its fundamental cycle.
- Orient each $e \in T$ according to its fundamental cut.
- (Optional) Put a chip at the head of every edge.



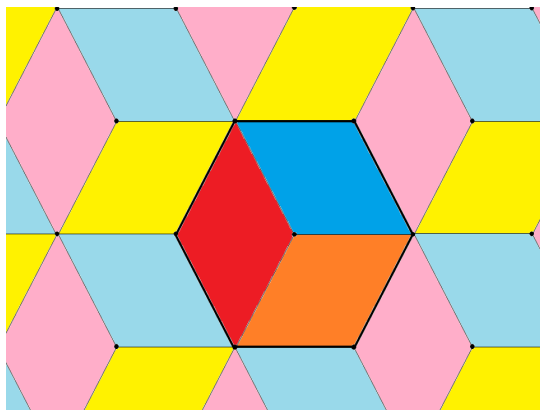
Tiling of $\text{Jac}(\Gamma)$



Tiling of Zonotopes

Proposition (Y. 2017+)

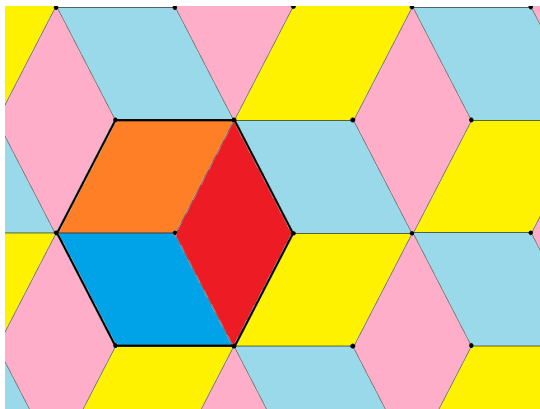
The ABKS decomposition of G (lifted to the universal cover), coincides with certain zonotopal tiling of $Z_{M(G)}^$ (tiling the whole space).*



Tiling of Zonotopes

Observation

Many zonotopal tilings lead to the same ABKS decomposition.



Observation

Any bijection ϕ between $\text{Jac}(G)$ and the collection of spanning trees induces a $\text{Jac}(G)$ -action (torsor) by $[D] \cdot T := \phi([D] + \phi^{-1}(T))$.

Group Action–Tiling Duality

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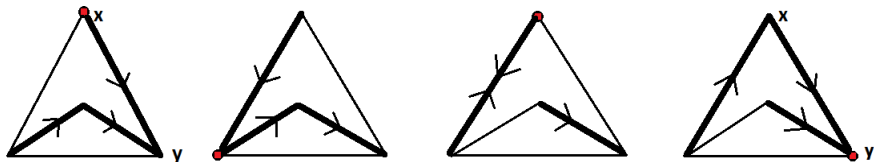
- Technical Statement: $\Phi_{\mathbf{w},\mathbf{v}}$ and $\Phi_{\mathbf{w},\mathbf{v}'}$ induce isomorphic group actions iff the extension tilings of G^* induced by \mathbf{v} and \mathbf{v}' differ by a translation.
- Punchline: The group action “dual” to ABKS decomposition was studied by combinatorists before.

Special Group Actions for Plane Graphs

Olivier Bernardi's process (08): Fix a starting edge (v, f) in a plane graph.

- For every spanning tree T , starting with (v, f) , walk along edges in T .
- Cut every $e \notin T$ twice, put a chip at the end that was being cut first.

There is an analogous *rotor-routing* that also produces $\text{Jac}(G)$ -actions.

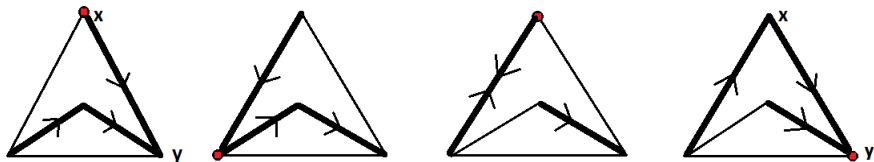


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Theorem (Baker–Wang 2017, Chan–Church–Grochow 2015)

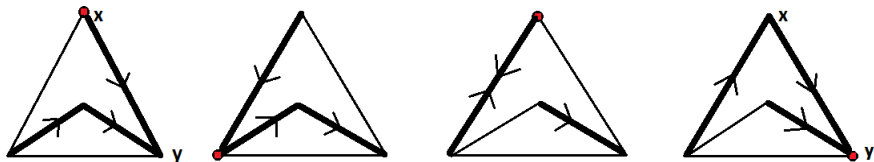
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Proposition (Y. 2017+)

The ABKS decomposition of G is the “dual” of such action for G^ .*

The canonical group action for a plane graph

“ \Leftrightarrow ”

The canonical tropical structure on its dual

- (8) Jordan Ellenberg: “By the Matrix-Tree theorem, the number of spanning trees of a graph G equals the size of the critical group of G . When are the spanning trees a torsor for the critical group? Making G into a ribbon graph, and given choice of sink, we do get a torsor structure from the rotor-router model [21]. But under what conditions can we get a natural torsor action without this additional data?”
- (c) Matt Baker: “A recent paper of mine and several coauthors [2] almost achieves this through a polyhedral decomposition of the real g -dimensional torus (where g is the genus of the graph). The cells of this decomposition are canonically in bijection with the spanning trees, the vertices are canonically in bijection with $\text{Pic}^g(G)$, and $\text{Pic}^g(G)$ is canonically a torsor for the critical group. The only thing that is not canonical here is the bijection between vertices and cells of the decomposition.”
- (f) Melody Chan: “In my recent paper with Church and Grochow [14], we look at when the choice of sink is needed in the rotor-router model. We show that the resulting torsor is independent of the choice of sink if and only if the graph G is planar.”

From the problem session of the 2013 AIM workshop on chip-firing (scribed by Sam Hopkins).

On going work (Backman–Santos–Y. 2018+): Extension to general oriented matroids.

Key Ingredient: *Oriented Matroid Programming*.

Problem

- *Extensions to other notions of matroids?*
- *Extensions to other combinatorial objects?*
- *Applications in tropical Brill-Noether theory?*

Thank you!