Rigidity of quasisymmetric mappings between horizontal self-affine carpets

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Based on the collaboration "*Rigidity of quasisymmetric mappings on self-affine carpets*" with Antti Käenmäki and Tuomo Ojala, [KOR16].

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- They identified qs-maps of $\mathbb{R}$ as the boundary values of the quasiconformal self maps of the upper half plane.
- Tukia and Väisälä [TV80] took qs-mappings to metric spaces.

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Definition

Let (X, d) and (Y, ϱ) be metric spaces. A homeomorphism $f\colon X \to Y$ is called quasisymmetric if there exists an increasing homeomorphism $\eta\colon [0, \infty) \to [0, \infty)$ so that

$$\frac{\varrho(f(x), f(y))}{\varrho(f(x), f(z))} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right)$$

for all different $x, y, z \in X$.

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Composition of qs-maps is a qs-map.

Three directions or research

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1. Conformal dimension

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2. Rigidity

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1. Conformal dimension
2. Rigidity
3. Equivalence

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- Given X, with $\dim_H X = s$, what can be said of $\dim_H f(X)$ for a quasisymmetric f?
- How much can the dimension drop in a quasisymmetry.

Conformal dimension

Definition

$\mathcal{C}\dim_{\mathrm{H}} X = \inf\{\dim_{\mathrm{H}} f(X) : f \text{ is quasisymmetric}\}$
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- $\mathcal{C}\dim_{\mathrm{H}} \in \{0\} \cup [1, \infty]$ (Kovalev2006 [Kov06])
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- Products of the form $\mathbb{R} \times C$, where C is a Cantoset, are minimal for conformal dimension. (Pansu [Pan89]?, Mackay & Tyson [MT10])
- For a Bedford–MacMullen carpet E, $\mathcal{C}\dim_A E = 0$ if the vertical projection of E is not a line segment and otherwise $\mathcal{C}\dim_A E = \dim_A E$. (Mackay [Mac11])

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5. "Central carpets" are not qs-equivalent [BM13]

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2. $\varphi_i(x_1, x_2) = (\alpha_1(i)x_i, \alpha_2(i)x_2) + (b_1, b_2)$, where $0 < \alpha_2(i) < \alpha_1(i) < 1$ for all i, and

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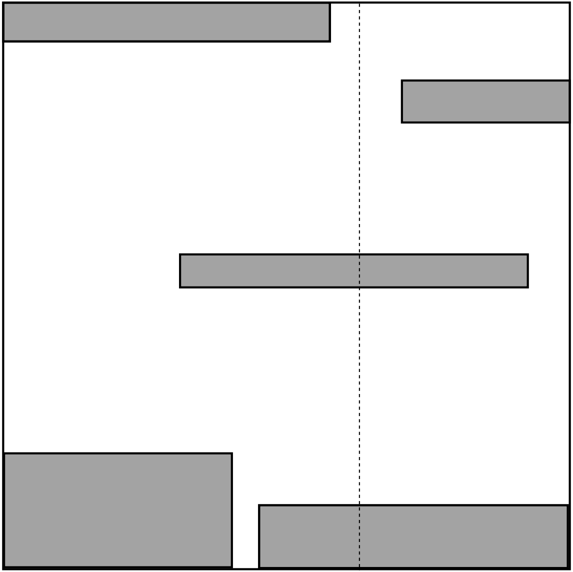
3. every vertical line in $[0,1]^2$ intersects $\varphi_i([0,1])$ for at least two different i.

Horizontal carpet



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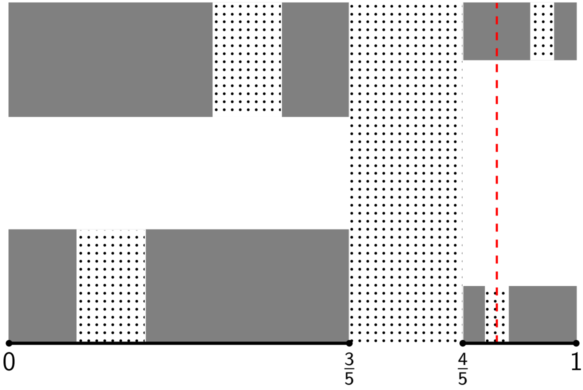
□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Definition

A mapping $f : X \to Y$ is *quasi-Lipschitz* if

$$\frac{\log \varrho(f(x), f(y))}{\log d(x, y)} \to 1$$

uniformly as $d(x, y) \to 0$.

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2. Of topological reasons, fibers are mapped to fibers.
3. Tangent mapping is bi-Lipschitz
4. Hard part: pull some information to back to original sets and mapping.

□

The hard part:

- Assuming that f is not quasi-Lipschitz, we find a sequence of pairs (x_i, y_i), with $|x_i - y_i| \to 0$ of points in which f obeys a true Hölder behavior.

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- Go to weak tangent along this sequence, and the true Hölder behaviour is there in the limit.
- Contradiction.
- This did not seem so hard?

Corollaries

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- Minimality result $\Rightarrow$ quasisymmetry between horizontal carpets preserves Assouad dimension.
- Rigidity result $\Rightarrow$ quasisymmetry between horizontal carpets preserves Hausdorff dimension (and upper/lower Minkowski dimension).

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We actually only need that this maximal weak tangent is minimal to conformal Hausdorff or Assouad dimensnion.

Thanks for listening!

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