Nevanlinna–Herglotz Functions and Some Applications to Spectral Theory

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Herglotz-Nevanlinna Functions and Their Applications
Institut Mittag-Leffler, Djursholm, Sweden
May 8–12, 2017
1 Topics Discussed

2 Scalar Nevanlinna–Herglotz Functions

3 Matrix-Valued Nevanlinna–Herglotz Functions

4 Operator-Valued Nevanlinna–Herglotz Functions

5 Epilogue
Topics Discussed:

- **Scalar Nevanlinna–Herglotz Functions**
  - Basic facts
  - Plenty of Examples (from very elementary to more sophisticated ones)
  - More basic facts
  - Aronszajn–Donoghue and Simon–Wolff theory
  - A model approach to rank-one perturbations
  - A model approach to self-adjoint extensions of symmetric operators with deficiency indices $(1, 1)$
  - The model approach applies to continuous and discrete half-line Hamiltonian systems such as, Sturm–Liouville operators (up to 3 coefficients), Dirac-type operators, Jacobi operators (tri-diagonal), CMV (Cantero–Moral–Velázquez) operators (five-diagonal, but with lots of zeros strategically placed make it effectively a 2nd order operator)
  - Literature hints
Topics Discussed (contd.):

- **Matrix-Valued Nevanlinna–Herglotz Functions**
  - Basic facts
  - Matrix analogs of Aronszajn–Donoghue and Simon–Wolff theory
  - Applies to continuous and discrete half-line and full-line Hamiltonian systems
  - Literature hints

- **Operator-Valued Nevanlinna–Herglotz Functions**
  - Basic facts
  - Applies to Schrödinger, Dirac-type, and Jacobi operators with operator-valued coefficients
  - Applies to Dirichlet-to-Neumann (resp., Robin-to-Robin) operators for elliptic PDEs
  - Literature hints
• **Epilogue**
  
  − Hints at some of the questions asked during the talk and subsequently during the conference are provided together with a few more references.
Let $\mathbb{C}_\pm = \{z \in \mathbb{C} \mid \pm \text{Im}(z) > 0\}$.

**Definition 1**

$m : \mathbb{C}_+ \to \mathbb{C}$ is called a **Nevanlinna–Herglotz function** (in short, a **N-H fct.**) if $m$ is analytic on $\mathbb{C}_+$ and $m(\mathbb{C}_+) \subseteq \mathbb{C}_+$.

That is, we’re looking at analytic self-maps on $\mathbb{C}_+$.

But actually, it suffices to assume $m(\mathbb{C}_+) \subseteq \overline{\mathbb{C}_+}$.

Unless explicitly stated otherwise, we always extend $m$ to $\mathbb{C}_-$ by **reflection**, i.e.,

$$m(z) = \overline{m(\overline{z})}, \quad z \in \mathbb{C}_-.$$
There is considerable disagreement concerning the proper name of functions satisfying the conditions in Definition 1: One finds the names Nevanlinna, Pick, Nevanlinna-Pick, Herglotz, and $R$-functions. Sometimes this depends on the geographical origin of authors and at times whether the open upper half-plane $\mathbb{C}_+$ or the conformally equivalent open unit disk $D$ is involved).

However, Herglotz definitely studied what’s also called Caratheodory functions (functions analytic in the open unit disk with nonnegative real part), and Nevanlinna studied the open upper half-plane, $\mathbb{C}_+$. So pure mathematicians got this right by calling them Nevanlinna functions, and mathematical physicists, who settled on Herglotz functions, definitely got this wrong.

Anway, an object with so many names must be really important!
Gustav Herglotz (2 February 1881 – 22 March 1953):

According to Wikipedia, a German Bohemian mathematician, best known for his works on the theory of relativity and seismology, also worked in many areas of applied and pure mathematics. (E.g., celestial mechanics, theory of electrons, special and general relativity, hydrodynamics, differential geometry, number theory.) Habilitation under Felix Klein in Göttingen, 1904. After detours via Vienna and Leipzig he became the successor of Carl Runge in Göttingen in 1925. One of his students was Emil Artin.
Rolf Herman Nevanlinna (22 October 1895 – 28 May 1980):

According to Wikipedia, one of the most famous Finnish mathematicians, particularly appreciated for his work in complex analysis. His most important mathematical achievement is the value distribution theory of meromorphic functions. Rolf Nevanlinna’s article “Zur Theorie der meromorphen Funktionen”, which contains the Main Theorems was published in Acta Mathematica in 1925. Hermann Weyl called it “one of the few great mathematical events of the century”.

![Rolf Herman Nevanlinna](image)
**Addition and Composition:** If \( m(z) \) and \( n(z) \) are Nevanlinna–Herglotz functions, then so are
\[
m(z) + n(z) \quad \text{and} \quad m(n(z)).
\]

**Elementary examples** of Nevanlinna–Herglotz functions are, e.g.,
\[
c + id, \quad c + dz, \quad c \in \mathbb{R}, \quad d \geq 0,
\]
\[
z^r, \quad 0 < r < 1,
\]
\[
\ln(z),
\]
choosing the obvious branches, and
\[
\tan(z), \quad -\cot(z),
\]
\[
\frac{a_{2,1} + a_{2,2}z}{a_{1,1} + a_{1,2}z}, \quad a = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \in \mathbb{C}^{2 \times 2}, \quad a^* j_2 a = j_2, \quad j_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},
\]
i.e., certain **linear fractional transformations** (the group of automorphisms of \( \mathbb{C}_+ \)), hence the special case
\[
-1/z.
\]
As a consequence,

\[-1/m(z), \quad m(-1/z), \quad \ln(m(z)),\]

and

\[m_a(z) = \frac{a_{2,1} + a_{2,2}m(z)}{a_{1,1} + a_{1,2}m(z)},\]

with \(a \in \mathbb{C}^{2 \times 2}\) as above, are all N-H functions whenever \(m(z)\) is N-H.

Most importantly, let \(H\) be a \textbf{self-adjoint} operator in a separable complex Hilbert space \(\mathcal{H}\) with \((\cdot, \cdot)_\mathcal{H}\) the scalar product on \(\mathcal{H} \times \mathcal{H}\) linear in the second factor. Consider the \textbf{resolvent} of \(H\), \((H - z)^{-1}, z \in \mathbb{C}\setminus \mathbb{R}\). Then for all \(f \in \mathcal{H}\),

\[(f, (H - z)^{-1}f)_\mathcal{H}, \quad z \in \mathbb{C}\setminus \mathbb{R},\]

is the \textbf{prime example} of a \textbf{scalar N-H function} (use the spectral theorem). Actually,

\[(H - z)^{-1}, \quad z \in \mathbb{C}\setminus \mathbb{R},\]

is the \textbf{prime example} of an \textbf{operator-valued N-H function}. 
The fundamental result on N-H functions, in part due to Fatou, Herglotz, Luzin, Nevanlinna, Plessner, Privalov, de la Vallée Poussin, Riesz, and others:

**Theorem 2**

Let \( m(z) \) be a Nevanlinna–Herglotz function. Then

1. \( m(z) \) has finite normal limits \( m(\lambda \pm i0) = \lim_{\varepsilon \downarrow 0} m(\lambda \pm i\varepsilon) \) for a.e. \( \lambda \in \mathbb{R} \).

2. Suppose \( m(z) \) has a zero normal limit on a subset of \( \mathbb{R} \) having positive Lebesgue measure. Then \( m \equiv 0 \).

3. The Nevanlinna (resp., Riesz–Herglotz) representation holds:

\[
m(z) = c + d z + \int_{\mathbb{R}} d\omega(\lambda) \left( \frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right), \quad z \in \mathbb{C}_+,
\]

\[
c = \text{Re}(m(i)), \quad d = \lim_{\eta \uparrow \infty} m(i\eta)/(i\eta) \geq 0,
\]

\[
\int_{\mathbb{R}} \frac{d\omega(\lambda)}{1 + \lambda^2} < \infty. \quad \left( \text{I.e., “Superpositions” of } \frac{1}{\lambda - z} \leftrightarrow \text{prime example ....} \right)
\]
Theorem 2 (contd.)

(iv) Let \( (\lambda_1, \lambda_2) \subset \mathbb{R} \), then the **Stieltjes inversion formula** for \( \omega \) reads

\[
\frac{1}{2} \omega(\{\lambda_1\}) + \frac{1}{2} \omega(\{\lambda_2\}) + \omega((\lambda_1, \lambda_2)) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \int_{\lambda_1}^{\lambda_2} d\lambda \text{Im}(m(\lambda + i\varepsilon)).
\]

(v) The absolutely continuous (ac) part \( \omega_{ac} \) of \( \omega \) with respect to Lebesgue measure \( d\lambda \) on \( \mathbb{R} \) is given by

\[
d\omega_{ac}(\lambda) = \pi^{-1} \text{Im}(m(\lambda + i0))d\lambda.
\]

(vi) Any **poles** and **isolated zeros** of \( m \) are **simple** and located on the real axis, the **residues at poles** being **negative**.

Actually, much more is true: Denote by \( \omega = \omega_{ac} + \omega_s = \omega_{ac} + \omega_{sc} + \omega_{pp} \) the decomposition of \( \omega \) into its absolutely continuous (ac), singularly continuous (sc), pure point (pp), and singular (s) parts with respect to Lebesgue measure on \( \mathbb{R} \).
Theorem 3

Let $m(z)$ be a Nevanlinna–Herglotz function with representation as above. Then

(i) $d = 0$ and $\int_{\mathbb{R}} d\omega(\lambda)(1 + |\lambda|^s)^{-1} < \infty$ for some $s \in (0, 2)$

if and only if $\int_{1}^{\infty} d\eta \eta^{-s} \text{Im}(m(i\eta)) < \infty$.

(ii) Let $(\lambda_1, \lambda_2) \subset \mathbb{R}$, $\eta_1 > 0$. Then there is a constant $C(\lambda_1, \lambda_2, \eta_1) > 0$ such that

$$\varepsilon |m(\lambda + i\varepsilon)| \leq C(\lambda_1, \lambda_2, \eta_1), \quad (\lambda, \varepsilon) \in [\lambda_1, \lambda_2] \times (0, \varepsilon_0).$$

(iii) $\sup_{\eta > 0} \eta |m(i\eta)| < \infty$ if and only if $m(z) = \int_{\mathbb{R}} d\omega(\lambda)(\lambda - z)^{-1}$

and $\int_{\mathbb{R}} d\omega(\lambda) < \infty$.

In this case,

$$\int_{\mathbb{R}} d\omega(\lambda) = \sup_{\eta > 0} \eta |m(i\eta)| = -i \lim_{\eta \uparrow \infty} \eta m(i\eta).$$

(iv) For all $\lambda \in \mathbb{R}$, $\lim_{\varepsilon \downarrow 0} \varepsilon \text{Re}(m(\lambda + i\varepsilon)) = 0$,

$$\omega(\{\lambda\}) = \lim_{\varepsilon \downarrow 0} \varepsilon \text{Im}(m(\lambda + i\varepsilon)) = -i \lim_{\varepsilon \downarrow 0} \varepsilon m(\lambda + i\varepsilon).$$
Theorem 3 (contd.)

(v) Let $L > 0$ and suppose $0 \leq \text{Im}(m(z)) \leq L$ for all $z \in \mathbb{C}_+$. Then $d = 0$, $\omega$ is purely absolutely continuous, $\omega = \omega_{ac}$, and

$$0 \leq \frac{d\omega(\lambda)}{d\lambda} = \pi^{-1} \lim_{\epsilon \downarrow 0} \text{Im}(m(\lambda + i\epsilon)) \leq \pi^{-1} L \text{ for a.e. } \lambda \in \mathbb{R}.$$ 

(vi) Let $p \in (1, \infty)$, $[\lambda_3, \lambda_4] \subset (\lambda_1, \lambda_2)$, $[\lambda_1, \lambda_2] \subset (\lambda_5, \lambda_6)$. If

$$\sup_{0<\epsilon<1} \int_{\lambda_1}^{\lambda_2} d\lambda \left| \text{Im}(m(\lambda + i\epsilon)) \right|^p < \infty, \tag{*}$$

then $\omega = \omega_{ac}$ is purely absolutely continuous on $(\lambda_1, \lambda_2)$,

$$\frac{d\omega_{ac}}{d\lambda} \in L^p((\lambda_1, \lambda_2); d\lambda), \text{ and } \lim_{\epsilon \downarrow 0} \| \pi^{-1} \text{Im}(m(\cdot + i\epsilon)) - \frac{d\omega_{ac}}{d\lambda} \|_{L^p((\lambda_3, \lambda_4);d\lambda)} = 0.$$

Conversely, if $\omega$ is purely absolutely continuous on $(\lambda_5, \lambda_6)$, and if $\frac{d\omega_{ac}}{d\lambda} \in L^p((\lambda_5, \lambda_6); d\lambda)$, then (*) holds.
Theorem 3 (contd.)

(vii) Let \((\lambda_1, \lambda_2) \subset \mathbb{R}\). Then a local version of Wiener’s theorem reads for \(p \in (1, \infty)\),

\[
\lim_{\varepsilon \downarrow 0} \varepsilon^{p-1} \int_{\lambda_1}^{\lambda_2} d\lambda |\text{Im}(m(\lambda + i\varepsilon))|^p \\
= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p - \frac{1}{2})}{\Gamma(p)} \left(\frac{1}{2} \omega(\{\lambda_1\})^p + \frac{1}{2} \omega(\{\lambda_2\})^p + \sum_{\lambda \in (\lambda_1, \lambda_2)} \omega(\{\lambda\})^p\right).
\]

Moreover, for \(0 < p < 1\),

\[
\lim_{\varepsilon \downarrow 0} \int_{\lambda_1}^{\lambda_2} d\lambda |\pi^{-1}\text{Im}(m(\lambda + i\varepsilon))|^p = \int_{\lambda_1}^{\lambda_2} d\lambda \left| \frac{d\omega_{ac}(\lambda)}{d\lambda} \right|^p.
\]

Well, that’s a load of theoretical results, so let’s turn to many more concrete examples and their explicit N-H representations for a change!
We denote **Lebesgue measure on** \(\mathbb{R}\) by \(d\lambda\) and a **pure point measure** supported at \(x \in \mathbb{R}\) with mass one by \(\mu\{x\}\), \(\text{supp}(\mu\{x\}) = \{x\}\), \(\mu\{x\}(\{x\}) = 1\).

\[
c + id = c + d\pi^{-1} \int_{\mathbb{R}} d\lambda ((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}), \quad c \in \mathbb{R}, \ d \geq 0,
\]

\[
\ln(id) = \ln(d) + (i\pi/2) = \ln(d) + 2^{-1} \int_{\mathbb{R}} d\lambda ((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}), \quad d > 0,
\]

\[
c + dz, \quad c \in \mathbb{R}, \ d \geq 0,
\]

\[
-z^{-1} = \int_{\mathbb{R}} d\mu_{\{0\}}(\lambda) (\lambda - z)^{-1},
\]

\[
\ln(z) = \int_{-\infty}^{0} d\lambda ((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}),
\]

\[
\ln(-z^{-1}) = \int_{0}^{\infty} d\lambda ((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}),
\]

where \(\ln(\cdot)\) denotes the principal value of the logarithm (i.e., with cut along \((-\infty, 0]\) and \(\ln(\lambda) > 0\) for \(\lambda > 1\)).
Scalar Nevanlinna–Herglotz Functions

Scalar N-H Fcts: More Examples (contd.)

\[ z^r = \exp(r \ln(z)) \]

\[ = \cos(r \pi/2) + \pi^{-1} \sin(r \pi) \int_{-\infty}^{0} d\lambda \, |\lambda|^r ((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}), \]

\[ 0 < r < 1, \]

\[-z^{-r} = -\exp(-r \ln(z)) \]

\[ = -\cos(r \pi/2) \]

\[ + \pi^{-1} \sin(r \pi) \int_{-\infty}^{0} d\lambda \, |\lambda|^{-r} ((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}), \quad 0 < r < 1, \]

\[ \tan(z) = \sum_{n \in \mathbb{Z}}[((n + \frac{1}{2})\pi - z)^{-1} - (n + \frac{1}{2})\pi(1 + (n + \frac{1}{2})^2\pi^2)^{-1}) \]

\[ = \int_{\mathbb{R}} d\omega(\lambda) ((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}), \]

\[ \omega = \sum_{n \in \mathbb{Z}} \mu_{\{(n + \frac{1}{2})\pi\}}, \]
\[-\cot(z) = \sum_{n \in \mathbb{Z}} ((n\pi - z)^{-1} - n\pi (1 + n^2\pi^2)^{-1}) \]

\[= \int_{\mathbb{R}} d\omega(\lambda) ((\lambda - z)^{-1} - \lambda (1 + \lambda^2)^{-1}), \]

\[\omega = \sum_{n \in \mathbb{Z}} \mu\{n\pi\}. \]

The psi or digamma function,

\[\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)} = C + \sum_{n \in \mathbb{N}_0} ((-n - z)^{-1} + n(1 + n^2)^{-1}) \]

\[= \int_{\mathbb{R}} d\omega(\lambda) ((\lambda - z)^{-1} - \lambda (1 + \lambda^2)^{-1}), \]

\[\omega = \sum_{n \in \mathbb{N}_0} \mu\{-n\}, \quad C = -\gamma + \sum_{n \in \mathbb{N}_0} ((n + 1)^{-1} - n(1 + n^2)^{-1}). \]

Here $\Gamma(z)$ denotes the gamma function, $\gamma = -\psi(1) = .5772\ldots$ Euler’s constant, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. 
Scalar N-H Fcts: More Examples (contd.)

\[
\frac{z - \lambda_2}{z - \lambda_1} = 1 + (\lambda_2 - \lambda_1) \int_{\mathbb{R}} d\mu_{\lambda_1}(\lambda) (\lambda - z)^{-1}, \quad \lambda_1 < \lambda_2, \\
\ln\left(\frac{z - \lambda_2}{z - \lambda_1}\right) = \int_{\lambda_1}^{\lambda_2} d\lambda (\lambda - z)^{-1}, \quad \lambda_1 < \lambda_2.
\]

Next we turn to **Weyl–Titchmarsh m-functions** \(m_{\alpha}^{WT}(z)\) associated with the **half-line Laplacian**, \(\mathcal{H}_{\alpha}^{(0)}\) in \(L^2([0, \infty); dx)\), defined by

\[
(\mathcal{H}_{\alpha}^{(0)}g)(x) = -g''(x), \quad x > 0, \\
\text{dom } (\mathcal{H}_{\alpha}^{(0)}) = \{ g \in L^2([0, \infty); dx) \mid g, g' \in AC([0, R]) \text{ for all } R > 0 \}; \\
- g'' \in L^2([0, \infty); dx); \quad \sin(\alpha)g'(0+) + \cos(\alpha)g(0+) = 0, \quad \alpha \in (0, \pi).
\]

These are Nevanlinna–Herglotz functions of the type

\[
m_{\alpha}^{(0),WT}(z) = \frac{-\sin(\alpha) + \cos(\alpha)iz^{1/2}}{\cos(\alpha) + \sin(\alpha)iz^{1/2}} = \cot(\alpha) + \int_{\mathbb{R}} d\omega_{\alpha}^{(0),WT}(\lambda) (\lambda - z)^{-1},
\]
where the **spectral function**, \( \omega^{(0),WT}_{\alpha}(\lambda) \), of \( H^{(0)}_{\alpha} \) is explicitly given by

\[
\begin{cases}
0, & \lambda < 0, \: (\pi/2) \leq \alpha < \pi, \: \text{(no negative eigenvalue)} \\
-2 \frac{\cot(\alpha)}{\sin^2(\alpha)}, & -\infty < \lambda < -\cot^2(\alpha), \: 0 < \alpha < (\pi/2), \\
0, & -\cot^2(\alpha) < \lambda < 0, \: 0 < \alpha < (\pi/2), \: \text{(eigenvalue at } -\cot^2(\alpha)) \\
\frac{2}{\pi} \lambda^{1/2}, & \lambda \geq 0, \: \alpha = (\pi/2), \\
\frac{2}{\pi \sin^2(\alpha)} \left( \lambda^{1/2} - \cot(\alpha) \arctan \left( \frac{\lambda^{1/2}}{\cot(\alpha)} \right) \right), & \lambda \geq 0, \\
\end{cases}
\]

\[\alpha \in (0, \pi) \setminus \{\pi/2\}.\]

Similarly,

\[
m^{(0),WT}_{0} = iz^{1/2} = -2^{-1/2} + \pi^{-1} \int_{0}^{\infty} d\lambda \lambda^{1/2} [(\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}],
\]

with \( \omega^{(0),WT}_{0}(\lambda) = [2/(3\pi)] \chi_{[0, \infty)}(\lambda) \lambda^{3/2} \), corresponds to the remaining self-adjoint (Friedrichs or Dirichlet) boundary condition \( \alpha = 0 \), i.e., to \( g(0_+) = 0 \).
Finally we describe a class of N-H functions fundamental in Floquet theory of periodic Schrödinger operators with period π on \( \mathbb{R} \). Consider a sequence \( \{\lambda_n\}_{n \in \mathbb{N}_0} \subset \mathbb{R} \),

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \cdots
\]
such that asymptotically

\[
\lambda_{2n}, \lambda_{2n-1} \xrightarrow{n \to \infty} (n\pi)^2 + O(1).
\]

Define an entire function \( \Delta(z) \) (the Floquet discriminant) such that

\[
\Delta(z) - 1 = \frac{(\lambda_0 - z)}{2} \prod_{n \in \mathbb{N}} \frac{(\lambda_{4n-1} - z)(\lambda_{4n} - z)}{(2n\pi)^4},
\]

\[
\Delta(z) + 1 = 2 \prod_{n \in \mathbb{N}_0} \frac{(\lambda_{4n+1} - z)(\lambda_{4n+2} - z)}{((2n + 1)\pi)^4},
\]

and hence (note, we chose \( \lambda_0 = 0 \)),

\[
\Delta(z)^2 - 1 = (\lambda_0 - z) \prod_{n \in \mathbb{N}} \frac{(\lambda_{2n-1} - z)(\lambda_{2n} - z)}{(n\pi)^4}.
\]
Moreover, define (differential of the 2nd kind ......)

\[ \theta(z) = -\int_0^z d\zeta \Delta'(\zeta) [1 - \Delta(\zeta)^2]^{-1/2}, \quad z \in \mathbb{C}_+ \]

(choose the square root branch positive on (0, \lambda_1)). Then

\[ \cos(\theta(z)) = \Delta(z) \]

and, \( \theta \) is a Nevanlinna–Herglotz function with a representation of the type

\[ \theta(\zeta) = \zeta^{1/2} + c + \pi^{-1} \int_{\mathbb{R}} \text{Im}(\theta(\eta))d\eta \left[ (\eta - \zeta)^{-1} - \eta(1 + \eta^2)^{-1} \right], \quad c \in \mathbb{R}, \]

where \( \theta(0) = 0 \) determines \( c \). If \( \{\lambda_n\}_{n \in \mathbb{N}_0} \) represents the periodic and antiperiodic eigenvalues associated with a Schrödinger operator \( H = -\frac{d^2}{dx^2} + q \), with \( q \in L^1_{\text{loc}}(\mathbb{R}) \) real-valued and of period 1, \( \Delta(z) \) represents the corresponding Floquet discriminant and \( \theta(z) \) the Floquet (Bloch) momentum associated with \( H \).
Moreover,

\[ \theta(z) = \frac{i}{2} \int_{0}^{1} dx \ G(z, x, x)^{-1}, \quad z \in \mathbb{C}_+, \]

with \( G(z, x, y) = (H - z)^{-1}(x, y) \) the Green’s function of \( H \) in \( L^2(\mathbb{R}; dx) \).

In connection with the last formula for \( \theta \) one should keep in mind that \( G(\cdot, x, x) \), \( x \in \mathbb{R} \), is a N-H fct., so is \( -G(\cdot, x, x)^{-1} \), \( x \in \mathbb{R} \), and hence so is \( -\int_{0}^{1} dx \ G(\cdot, x, x)^{-1} \). However, in this context also multiplying the latter by the factor \(-i\) remains a N-H fct., i.e., \( \frac{i}{2} \int_{0}^{1} dx \ G(\cdot, x, x)^{-1} \), and hence \( \theta \), is indeed a N-H fct.
A WARNING (added after discussions with Y. Ivanenko, P. Kurasov, and A. Luger): In these lectures (and typically, in spectral theory) one always extends a N-H function \( m \) to \( \mathbb{C}_- \) by reflection, i.e., one uses

\[
m(z) = \overline{m(\overline{z})}, \quad z \in \mathbb{C}_-.
\]

This is quite different from analytic continuation as the following elementary examples show:

\[
m(z) = z + i = z + \frac{1}{\pi} \int_{\mathbb{R}} d\lambda \left[ \frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right], \quad z \in \mathbb{C}_+,
\]

\[
\overline{m(\overline{z})} = z - i, \quad z \in \mathbb{C}_-,
\]

\[
n(z) = -m(z)^{-1} = -\frac{1}{z + i} = \frac{1}{\pi} \int_{\mathbb{R}} \frac{d\lambda}{\lambda^2 + 1} \frac{1}{\lambda - z}, \quad z \in \mathbb{C}_+,
\]

\[
\overline{n(\overline{z})} = \frac{-1}{z - i}, \quad z \in \mathbb{C}_-.
\]

In particular, analytically continued N-H functions can have zeros and poles in the open lower complex half-plane \( \mathbb{C}_- \), in sharp contrast to N-E fcts. continued to \( \mathbb{C}_- \) by reflection (the latter can have zeros and singularities exclusively on \( \mathbb{R} \)).
As just discussed, the definition of \( m\big|_{\mathbb{C}_-} \) by means of reflection (i.e., employing \( m(z) = \overline{m(\overline{z})}, \ z \in \mathbb{C}_- \)), in general, does not represent the analytic continuation of \( m\big|_{\mathbb{C}_+} \). Greenstein '60 clarified the circumstances under which \( m \) can be analytically continued from \( \mathbb{C}_+ \) into a subset of \( \mathbb{C}_- \) through an interval \((\lambda_1, \lambda_2) \subseteq \mathbb{R}\).

**Lemma 4**

Let \( m \) be a N-H function and \((\lambda_1, \lambda_2) \subseteq \mathbb{R}, \lambda_1 < \lambda_2\). Then \( m \) can be analytically continued from \( \mathbb{C}_+ \) into a subset of \( \mathbb{C}_- \) through the interval \((\lambda_1, \lambda_2)\) if and only if the associated measure \( \omega \) is purely absolutely continuous on \((\lambda_1, \lambda_2)\), \( \omega\big|_{(\lambda_1, \lambda_2)} = \omega\big|_{(\lambda_1, \lambda_2), ac} \), and the density \( \omega' \) of \( \omega \) is real-analytic on \((\lambda_1, \lambda_2)\). In this case, the analytic continuation of \( m \) into some domain \( \mathcal{D}_- \subseteq \mathbb{C}_- \) is given by

\[
m(z) = \overline{m(\overline{z})} + 2\pi i \omega'(z), \quad z \in \mathcal{D}_-,
\]

where \( \omega'(z) \) denotes the complex-analytic extension of \( \omega'(\lambda) \) for \( \lambda \in (\lambda_1, \lambda_2) \). In particular, \( m \) can be analytically continued through \((\lambda_1, \lambda_2)\) by reflection, i.e., \( m(z) = \overline{m(\overline{z})} \) for all \( z \in \mathbb{C}_- \) if and only if \( \omega \) has no support in \((\lambda_1, \lambda_2)\).
Kotani ’84, ’87 has an interesting result in connection with applications of Nevanlinna–Herglotz functions to reflectionless Schrödinger and Dirac-type operators on $\mathbb{R}$ (i.e., solitonic, periodic, and certain classes of quasi-periodic and almost-periodic operators):

**Lemma 5**

Let $m$ be a N-H function and $(\lambda_1, \lambda_2) \subseteq \mathbb{R}$, $\lambda_1 < \lambda_2$. Suppose $\lim_{\varepsilon \to 0} \text{Re}(m(\lambda + i\varepsilon)) = 0$ for a.e. $\lambda \in (\lambda_1, \lambda_2)$. Then $m$ can be analytically continued from $\mathbb{C}_+$ into $\mathbb{C}_-$ through the interval $(\lambda_1, \lambda_2)$ and

$$m(z) = -\overline{m(\overline{z})}.$$ 

In addition, $\text{Im}(m(\lambda + i0)) > 0$, $\text{Re}(m(\lambda + i0)) = 0$ for all $\lambda \in (\lambda_1, \lambda_2)$. 
The Exponential N-H Representation:

Together with \( m(z), \ln(m(z)) \) is a N-H function, and since

\[
0 \leq \text{Im}(\ln(m(z))) = \arg(m(z)) \leq \pi, \quad z \in \mathbb{C}_+,
\]

the measure \( \hat{\omega} \) in the representation of \( \ln(m(z)) \), i.e., in the exponential representation of \( m(z) \), is purely absolutely continuous by Theorem 3 (\( \nu \)),

\[
d\hat{\omega}(\lambda) = \xi(\lambda) d\lambda \quad \text{for some } 0 \leq \xi \leq 1 \text{ a.e.}
\]

These exponential representations have been studied in great detail by Aronszajn and Donoghue '56, '64 and by Kac and M. Krein '74 (see also Markoff 1896):

**Theorem 6**

Suppose \( m(z) \) is a Nevanlinna–Herglotz function. Then

(i) There exists a \( \xi \in L^\infty(\mathbb{R}), 0 \leq \xi \leq 1 \text{ a.e.}, \) such that

\[
\ln(m(z)) = k + \int_\mathbb{R} \xi(\lambda) d\lambda \left[ (\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1} \right], \quad z \in \mathbb{C}_+,
\]

\[
k = \text{Re}(\ln(m(i))),
\]

where

\[
\xi(\lambda) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \text{Im}(\ln(m(\lambda + i\varepsilon))), \quad \text{for a.e. } \lambda \in \mathbb{R}.
\]
Theorem 6 (cont.)

(ii) Let $\ell_1, \ell_2 \in \mathbb{N}$ and $d = 0$. Then

$$\int_{-\infty}^{0} d\lambda \, \xi(\lambda) |\lambda|^{\ell_1} (1 + \lambda^2)^{-1} + \int_{0}^{\infty} d\lambda \, \xi(\lambda) |\lambda|^{\ell_2} (1 + \lambda^2)^{-1} < \infty$$

if and only if

$$\int_{-\infty}^{0} d\omega(\lambda) |\lambda|^{\ell_1} (1 + \lambda^2)^{-1} + \int_{0}^{\infty} d\omega(\lambda) |\lambda|^{\ell_2} (1 + \lambda^2)^{-1} < \infty$$

and

$$\lim_{z \to i\infty} m(z) = c - \int_{\mathbb{R}} d\omega(\lambda) \lambda (1 + \lambda^2)^{-1} > 0.$$
Theorem 6 (cont.)

(iii)

\[ \xi(\lambda) = 0 \text{ for } \lambda < 0 \text{ if and only if} \]
\[ d = 0, \quad [0, \infty) \text{ is a support for } \omega \text{ (i.e., } \omega((-\infty, 0)) = 0), \]
\[ \int_0^\infty d\omega(1 + \lambda)^{-1} < \infty, \text{ and } c \geq \int_0^\infty d\omega(\lambda)\lambda(1 + \lambda^2)^{-1}. \]

In this case
\[ \lim_{\lambda \downarrow -\infty} m(\lambda) = c - \int_0^\infty d\omega(\lambda')\lambda'(1 + \lambda'^2)^{-1} \]
and
\[ c > \int_0^\infty d\omega(\lambda)\lambda(1 + \lambda^2)^{-1} \text{ if and only if } \int_0^\infty d\lambda \xi(\lambda)(1 + \lambda)^{-1} < \infty. \]
Theorem 6 (cont.)

(iv) Let $(\lambda_1, \lambda_2) \subset \mathbb{R}$ and suppose $0 \leq A \leq \xi(\lambda) \leq B \leq 1$ for a.e. $\lambda \in (\lambda_1, \lambda_2)$ with $(B - A) < 1$. Then $\omega$ is purely absolutely continuous in $(\lambda_1, \lambda_2)$ and 
$$\frac{d\omega}{d\lambda} \in L^p((\lambda_3, \lambda_4); d\lambda)$$
for $[\lambda_3, \lambda_4] \subset (\lambda_1, \lambda_2)$ and all $p < (B - A)^{-1}$.

(v) The measure $\omega$ is purely singular, $\omega = \omega_s$, $\omega_{ac} = 0$ if and only if $\xi$ equals the characteristic function of a measurable subset $A \subset \mathbb{R}$, i.e., $\xi = \chi_A$. 
Support Theorems for Scalar N-H Fcts.:

A bit of **Aronszajn-Donoghue** and **Simon–Wolff** theory and underlying support theorems for $\omega$, $\omega_{ac}$, $\omega_s$:

Let $\mu, \nu$ be Borel measures on $\mathbb{R}$. Then $S_{\mu}$ is called a support of $\mu$ if $\mu(\mathbb{R}\setminus S_{\mu}) = 0$. The topological support $S_{\mu}^{cl}$ of $\mu$ is then the smallest closed support of $\mu$.

A support $S_{\mu}$ of $\mu$ is called minimal (or essential) relative to $\nu$ if for any smaller support $T_{\mu} \subseteq S_{\mu}$ of $\mu$, $\nu(S_{\mu} \setminus T_{\mu}) = 0$ (or equivalently, $\tilde{T} \subseteq S_{\mu}$ with $\mu(\tilde{T}) = 0$ implies $\nu(\tilde{T}) = 0$).

Minimal supports are unique up to sets of $\mu$ and $\nu$ measure zero and

$$S \sim T \quad \text{if and only if} \quad \mu(S \Delta T) = 0 = \nu(S \Delta T)$$

defines an equivalence class $\mathcal{E}_\nu(\mu)$ of minimal supports of $\mu$ relative to $\nu$ (with $S \Delta T = (S \setminus T) \cup (T \setminus S)$ the symmetric difference of $S$ and $T$).

Two measures, $\mu$ and $\nu$, are called orthogonal, $\mu \perp \nu$, if some of their supports are disjoint.

From now on the reference measure $\nu$ will be chosen to be Lebesgue measure on $\mathbb{R}$. 

Theorem 7

Let \( m \) be a Nevanlinna–Herglotz function. Then

(i) \( S_{\omega_{ac}} = \{ \lambda \in \mathbb{R} | \lim_{\varepsilon \downarrow 0} \Im(m(\lambda + i\varepsilon)) \text{ exists finitely and} \}
0 < \Im(m(\lambda + i0)) < \infty \} \) is a minimal support of \( \omega_{ac} \).

(ii) \( S_{\omega_s} = \{ \lambda \in \mathbb{R} | \lim_{\varepsilon \downarrow 0} \Im(m(\lambda + i\varepsilon)) = +\infty \} \)

and

\( S_{\omega_{sc}} = \{ \lambda \in S_{\omega_s} | \lim_{\varepsilon \downarrow 0} \varepsilon \Im(m(\lambda + i\varepsilon)) = 0 \} \) are minimal supports of \( \omega_s \) and \( \omega_{sc} \).

(iii) \( S_{\omega_{pp}} = \{ \lambda \in \mathbb{R} | \lim_{\varepsilon \downarrow 0} \varepsilon \Im(m(\lambda + i\varepsilon)) = -i \lim_{\varepsilon \downarrow 0} \varepsilon m(\lambda + i\varepsilon) > 0 \} \) is the

smallest support of \( \omega_{pp} \).

(iv) \( S_{\omega_{ac}}, S_{\omega_{sc}}, \) and \( S_{\omega_{pp}} \) are mutually disjoint minimal supports and

\( S_{\omega} = \{ \lambda \in \mathbb{R} | \lim_{\varepsilon \downarrow 0} \Im(m(\lambda + i\varepsilon)) \leq +\infty \text{ exists and} 0 < \Im(m(\lambda + i0)) \leq +\infty \} \)

\( = S_{\omega_{ac}} \cup S_{\omega_s} \) is a minimal support for \( \omega \).

(v) The set,

\( \tilde{S}_{\omega_{ac}} = \{ \lambda \in \mathbb{R} | 0 < \xi(\lambda) < 1 \}, \)

is a minimal support for \( \omega_{ac} \).
Of course
\[ \hat{S}_{\omega_{ac}} = \{ \lambda \in \mathbb{R} \mid \lim_{\varepsilon \downarrow 0} m(\lambda + i\varepsilon) \text{ exists finitely and } 0 < \text{Im}(m(\lambda + i0)) < \infty \} \]
is also a minimal support of \( \omega_{ac} \).

The equivalence relation for minimal supports motivates the introduction of equivalence classes associated with \( \omega \) and its decompositions \( \omega_{ac}, \omega_s \), etc. We will, in particular, use
\[ \mathcal{E}(\omega_{ac}) := \text{the equivalence class of minimal supports of } \omega_{ac}. \]

Next, we abbreviate the identity matrix in \( \mathbb{C}^n \) by \( I_n \), the unit circle in \( \mathbb{C} \) by \( S^1 = \partial D \), and introduce the set,
\[ A_2 = \{ a \in M_2(\mathbb{C}) \mid a^* j_2 a = j_2 \}. \]

Then \( |\det(a)| = 1 \), \( a \in A_2 \), and
\[ (a_{1,1}/a_{1,2}), (a_{1,1}/a_{2,1}), (a_{2,2}/a_{1,2}), (a_{2,2}/a_{2,1}) \in \mathbb{R}, \ a \in A_2, \ \text{as long as } a_{1,2} \neq 0, \]
respectively, \( a_{2,1} \neq 0 \). Moreover, we recall
\[ m_a(z) = \frac{a_{2,1} + a_{2,2}m(z)}{a_{1,1} + a_{1,2}m(z)}, \quad z \in \mathbb{C}_+, \quad (*) \]
and its general version
\[
m_a(z) = \frac{(ab^{-1})_{2,1} + (ab^{-1})_{2,2}m_b(z)}{(ab^{-1})_{1,1} + (ab^{-1})_{1,2}m_b(z)}, \quad a, b \in \mathcal{A}_2, \ z \in \mathbb{C}_+.
\]

The corresponding equivalence classes of minimal supports of \(\omega_{ac}\) and \(\omega_{a,ac}\) are then denoted by \(\mathcal{E}(\omega_{ac})\) and \(\mathcal{E}(\omega_{a,ac})\).

Here’s a part of the celebrated Aronszajn–Donoghue theory:

**Theorem 8**

Let \(m(z)\) and \(m_a(z), a \in \mathcal{A}_2\) be N-H functions related by the fractional transformation \((\ast)\) with corresponding measures \(\omega\) and \(\omega_a\). Then

(i) For all \(a \in \mathcal{A}_2\),
\[
\mathcal{E}(\omega_{a,ac}) = \mathcal{E}(\omega_{ac}),
\]
i.e., \(\mathcal{E}(\omega_{a,ac})\) is independent of \(a \in \mathcal{A}_2\) (and hence denoted by \(\mathcal{E}_{ac}\) below) and \(\omega_{a,ac} \sim \omega_{ac}\) for all \(a \in \mathcal{A}_2\).

(ii) Suppose \(\omega_b\) is a discrete point measure, \(\omega_b = \omega_{b,d}\), for some \(b \in \mathcal{A}_2\). Then \(\omega_a = \omega_{a,d}\) is a discrete point measure for all \(a \in \mathcal{A}_2\).
(iii) Define
\[ S = \{ \lambda \in \mathbb{R} | \text{there is no } a \in A_2 \text{ for which } \text{Im}(m_a(\lambda + i0)) \text{ exists and equals } 0 \} \].
Then \( S \in \mathcal{E}_{ac} \).

(iv) Suppose \( a_{1,2} \neq 0 \) (i.e., \( a \in A_2 \setminus \{ \gamma l_2 \}, \gamma \in S^1 \)). If \( \omega_{a,s}(\mathbb{R}) > 0 \) or \( \omega_s(\mathbb{R}) > 0 \), then \( \mathcal{E}(\omega_{a,s}) \neq \mathcal{E}(\omega_s) \) and there exist \( S_{a,s} \in \mathcal{E}(\omega_{a,s}) \), \( S_s \in \mathcal{E}(\omega_s) \) such that \( S_{a,s} \cap S_s = \emptyset \) (i.e., \( \omega_{a,s} \perp \omega_s \)). In particular,
\[ \tilde{S}_{\omega_{a,s}} = \{ \lambda \in \mathbb{R} | \lim_{\varepsilon \downarrow 0} m(\lambda + i \varepsilon) = -a_{1,1}/a_{1,2} \} \]
is a minimal support for \( \omega_{a,s} \) and the smallest support of \( \omega_{a,pp} \) equals
\[ S_{\omega_{a,pp}} = \{ \lambda \in \mathbb{R} | \lim_{\varepsilon \downarrow 0} m(\lambda + i \varepsilon) = -a_{1,1}/a_{1,2}, \int_{\mathbb{R}} d\omega(\lambda')(\lambda' - \lambda)^{-2} < \infty \} \].
Moreover,
\[ \omega_a(\{\lambda\}) = |a_{1,2}|^{-2} \left( d + \int_{\mathbb{R}} d\omega(\lambda')(\lambda' - \lambda)^{-2} \right)^{-1}, \quad \lambda \in \mathbb{R}. \]
(v) (An abstract Borg-type uniqueness theorem) Suppose $\omega_b$ is a discrete point measure for some (and hence for all) $b \in A_2$. Assume that $\text{supp}(\omega) = \{\lambda_{l_2,n}\}_{n \in I}$ and $\text{supp}(\omega_a) = \{\lambda_{a,n}\}_{n \in I}$ for some $a \in A_2$ with $a_{1,2} \neq 0$ are given, where $I$ is either $\mathbb{N}$, $\mathbb{Z}$, or a finite non-empty index set. Suppose in addition that one of the following conditions hold:

(α) $\omega(\mathbb{R})$ is known,

or

(β) $m(z_0)$ is known for some $z_0 \in \mathbb{C}_+$,

or

(γ) $\lim_{z \to i\infty}(m(z) - m^0(z)) = 0$, where $m^0(z)$ is a known N-H function.

Then the system of measures $\{\omega_b\}_{b \in A_2}$ and hence the system of Nevanlinna–Herglotz functions $\{m_b(z)\}_{b \in A_2}$ is uniquely determined.
Support Theorems for Scalar N-H Fcts. (contd.):

Sketch of Proof of Theorem 8 (v). Define

\[ F(z) = m(z) + \frac{a_{1,1}}{a_{1,2}}, \quad z \in \mathbb{C}_+. \]

Then \( F \) is a meromorphic N-H function which has simple zeros at \( \{\lambda_{a,n}\}_{n \in I} \) and simple poles at \( \{\lambda_{b,n}\}_{n \in I} \). In particular, its zeros and poles necessarily interlace and the exponential N-H representation for \( F \) then yields

\[ F(z) = \exp \left( k + \int_{\mathbb{R}} d\lambda \, \xi(\lambda)((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}) \right), \]

with \( \xi \) a piecewise constant function. Analyzing the formula

\[ \xi(\lambda) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \text{Im}(\ln(m(\lambda + i\varepsilon))) \text{ a.e.}, \]

shows that

\[ \xi(\lambda) = \chi_{\{\lambda \in \mathbb{R} \mid F(\lambda) < 0\}}(\lambda), \]

where \( \chi_{\mathcal{M}} \) denotes the characteristic function of a set \( \mathcal{M} \subseteq \mathbb{R} \) and hence \( \xi \) is uniquely determined by \( \text{supp}(\omega) \) and \( \text{supp}(\omega_a) \). Thus \( F(z) \) is uniquely determined except for the constant \( k \in \mathbb{R} \) (which cannot be determined from \( \text{supp}(\omega) \) and \( \text{supp}(\omega_a) \)). Either one of the conditions \((\alpha)-(\gamma)\) then will determine \( k \) and hence \( F(z) \), \( z \in \mathbb{C}_+ \). Thus \( m(z) \), and hence by \((*)\) \( m_b(z) \) for all \( b \in \mathcal{A}_2 \), are uniquely determined, which in turn determine \( \omega_b \) for all \( b \in \mathcal{A}_2 \). \( \square \)
Applications to Rank-One Perturbations:

The following model approach is due to Donoghue ’65:

Let \( \mathcal{H} \) be a separable complex Hilbert space with scalar product \( (\cdot, \cdot)_{\mathcal{H}} \), \( H_0 \) a self-adjoint operator in \( \mathcal{H} \) (which may or may not be bounded) with simple spectrum. Suppose \( f_1 \in \mathcal{H}, \| f_1 \|_{\mathcal{H}} = 1 \) is a cyclic vector for \( H_0 \) (i.e., \( \mathcal{H} = \text{lin.span}\{(H_0 - z)^{-1}f_1 \in \mathcal{H} | z \in \mathbb{C} \setminus \mathbb{R}\} \)), or equivalently, \( \mathcal{H} = \text{lin.span}\{E_0(\lambda)f_1 \in \mathcal{H} | \lambda \in \mathbb{R}\}, \) \( E_0(\cdot) \) the family of orthogonal spectral projections of \( H_0 \) and define

\[
H_\alpha = H_0 + \alpha P_1, \quad P_1 = (f_1, \cdot)_{\mathcal{H}} f_1, \quad \alpha \in \mathbb{R},
\]

(with \( \text{dom}(H_\alpha) = \text{dom}(H_0), \alpha \in \mathbb{R} \)). Denote by \( E_{H_\alpha}(\cdot) \) the family of orthogonal spectral projections of \( H_\alpha \) and define

\[
d\omega_\alpha(\lambda) = d\|E_{H_\alpha}(\lambda)f_1\|_{\mathcal{H}}^2, \quad \int_{\mathbb{R}} d\omega_\alpha(\lambda) = \|f_1\|_{\mathcal{H}}^2 = 1.
\]

By the spectral theorem for self-adjoint operators, \( H_\alpha \) in \( \mathcal{H} \) is unitarily equivalent to \( \hat{H}_\alpha \) in \( \hat{H}_\alpha = L^2(\mathbb{R}; d\omega_\alpha) \), where

\[
(\hat{H}_\alpha \hat{g})(\lambda) = \lambda \hat{g}(\lambda), \quad \hat{g} \in \text{dom}(\hat{H}_\alpha) = L^2(\mathbb{R}; (1 + \lambda^2)d\omega_\alpha),
\]

\[
H_\alpha = U_\alpha \hat{H}_\alpha U_\alpha^{-1}, \quad \mathcal{H} = U_\alpha L^2(\mathbb{R}; d\omega_\alpha),
\]
with $U_\alpha$ unitary,

$$U_\alpha : \hat{\mathcal{H}}_\alpha = L^2(\mathbb{R}; d\omega_\alpha) \to \mathcal{H}, \quad \hat{g} \to (U_\alpha \hat{g}) = \text{s-lim}_{N \to \infty} \int_{-N}^{N} d(E_\alpha(\lambda)f_1)\hat{g}(\lambda).$$

Moreover, $f_1 = U_\alpha \hat{f}_1$, $\hat{f}_1(\lambda) = 1$, $\lambda \in \mathbb{R}$.

The family of spectral projections $\hat{E}_\alpha(\lambda)$, $\lambda \in \mathbb{R}$ of $\hat{\mathcal{H}}_\alpha$ is then given by

$$(\hat{E}_\alpha(\lambda)\hat{g})(\mu) = \theta(\lambda - \mu)\hat{g}(\mu) \text{ for } \omega_\alpha\text{-a.e. } \mu \in \mathbb{R}, \hat{g} \in L^2(\mathbb{R}; d\omega_\alpha),$$

and $\theta(x) = 1$, $x \geq 0$, $\theta(x) = 0$, $x < 0$.

Introducing the Nevanlinna–Herglotz function

$$m_\alpha(z) = (f_1, (H_\alpha - z)^{-1}f_1)_\mathcal{H} = \int_{\mathbb{R}} \frac{d\omega_\alpha}{\lambda - z}, \quad z \in \mathbb{C}_+,$$

one verifies

$$m_\beta(z) = \frac{m_\alpha(z)}{1 + (\beta - \alpha)m_\alpha(z)}, \quad \alpha, \beta \in \mathbb{R}.$$ 

A comparison of these fractional transformations suggests the introduction of

$$a(\alpha, \beta) = \begin{pmatrix} 1 & \beta - \alpha \\ 0 & 1 \end{pmatrix} \in \mathcal{A}_2, \quad \alpha, \beta \in \mathbb{R}.$$
Moreover, since $\omega_\alpha(\mathbb{R}) = 1$, Theorem 8 applies (with $a(\alpha, \beta)_{1,1} = a(\alpha, \beta)_{2,2} = 1$, $a(\alpha, \beta)_{1,2} = \beta - \alpha$, $a(\alpha, \beta)_{2,1} = 0$).

If $f_1$ is not a cyclic vector for $H_0$ one introduces appropriate reducing subspaces and gets analogous results ....

Introducing the following set of N-H functions

$$\mathcal{N}_1 = \{ m : \mathbb{C}_+ \to \mathbb{C}_+ \text{ analytic} \mid m(z) = \int_\mathbb{R} d\omega(\lambda)(\lambda - z)^{-1}, \int_\mathbb{R} d\omega(\lambda) < \infty \},$$

we now turn to a realization theorem for Herglotz functions of the type $m_\alpha(z)$:

**Theorem 9**

(i) Any $m \in \mathcal{N}_1$ with associated measure $\omega$ can be realized in the form

$$m(z) = (f_1, (H - z)^{-1}f_1)_\mathcal{H}, \quad z \in \mathbb{C}_+,$$

where $H$ denotes a self-adjoint operator in some separable complex Hilbert space $\mathcal{H}$, $f_1 \in \mathcal{H}$, and

$$\int_\mathbb{R} d\omega(\lambda) = \|f_1\|^2_\mathcal{H}.$$
Theorem 10

(ii) Suppose \( m_\ell \in \mathcal{N}_1 \) with corresponding measures \( \omega_\ell, \ell = 1, 2, \) and \( m_1 \neq m_2. \) Then \( m_1 \) and \( m_2 \) can be realized as

\[
m_\ell(z) = (f_1, (H_\ell - z)^{-1}f_1)_\mathcal{H}, \quad \ell = 1, 2, \ z \in \mathbb{C}_+,
\]

where \( H_\ell, \ell = 1, 2 \) are self-adjoint rank-one perturbations of one and the same self-adjoint operator \( H_0 \) in some complex Hilbert space \( \mathcal{H} \) (which may be chosen separable) with \( f_1 \in \mathcal{H}, \) i.e.,

\[
H_\ell = H_0 + \alpha_\ell P_1, \quad P_1 = (f_1, \cdot)_\mathcal{H} f_1
\]

for some \( \alpha_\ell \in \mathbb{R}, \ell = 1, 2, \) if and only if the following conditions hold:

\[
\int_\mathbb{R} d\omega_1(\lambda) = \int_\mathbb{R} d\omega_2(\lambda) = \|f_1\|_{\mathcal{H}}^2,
\]

and for all \( z \in \mathbb{C}_+, \)

\[
m_2(z) = \frac{m_1(z)}{1 + (\alpha_2 - \alpha_1)m_1(z)}.
\]
Applications to Self-Adjoint Extensions:

We turn to **extensions** of a **symmetric** operator with **deficiency indices** \((1, 1)\), following **Donoghue’s** model approach:

Let \( \mathcal{H} \) be a separable complex Hilbert space, \( H \) a **closed, densely defined symmetric** operator with domain \( \text{dom}(H) \) and **deficiency indices** \((1, 1)\). Choose \( u_{\pm} \in \ker(H^* \mp i) \) with \( \|u_{\pm}\|_\mathcal{H} = 1 \) and denote by \( H_\alpha, \alpha \in [0, \pi) \) all **self-adjoint extensions** of \( H \), i.e.,

\[
H_\alpha(g + c(u_+ + e^{2i\alpha}u_-)) = Hg + c(iu_+ - ie^{2i\alpha}u_-),
\]

\[
\text{dom}(H_\alpha) = \{(g + c(u_+ + e^{2i\alpha}u_-)) \in \text{dom}(H^*) | g \in \text{dom}(H), c \in \mathbb{C}, \ u_{\pm} \in \ker(H^* \mp i)\}
\]

by von Neumann’s formula for self-adjoint extensions of \( H \). Let \( E_{H_\alpha}(\cdot) \) be the family of spectral projections of \( H_\alpha \) and suppose \( H_\alpha \) has simple spectrum for some (and hence for all) \( \alpha \in [0, \pi) \) (i.e., \( u_+ \) is a **cyclic vector** for \( H_\alpha \) for all \( \alpha \in [0, \pi) \)). Define

\[
d_\alpha(\lambda) = d\|E_{H_\alpha}(\lambda)u_+\|_{\mathcal{H}}^2, \quad \int_{\mathbb{R}} d_\alpha(\lambda) = \|u_+\|_{\mathcal{H}}^2 = 1, \ \alpha \in [0, \pi),
\]

then \( H_\alpha \) is **unitarily equivalent to multiplication by** \( \lambda \) in \( L^2(\mathbb{R}; d_\alpha) \) and \( u_+ \) can be mapped into the function identically \( 1 \).
However, it is more convenient to normalize $d\omega_\alpha(\lambda) = (1 + \lambda^2) d\nu_\alpha(\lambda)$, such that $\int_{\mathbb{R}} \frac{d\omega_\alpha(\lambda)}{1+\lambda^2} = 1$, $\int_{\mathbb{R}} d\omega_\alpha(\lambda) = \infty$, $\alpha \in [0, \pi)$.

Thus, $H_\alpha$ is unitarily equivalent to $\hat{H}_\alpha$ in $\hat{H}_\alpha = L^2(\mathbb{R}; d\omega_\alpha)$, where

$$(\hat{H}_\alpha \hat{g})(\lambda) = \lambda \hat{g}(\lambda), \quad \hat{g} \in \text{dom}(\hat{H}_\alpha) = L^2(\mathbb{R}; (1 + \lambda^2) d\omega_\alpha),$$

$H_\alpha = U_\alpha \hat{H}_\alpha U_\alpha^{-1}$, $\mathcal{H} = U_\alpha L^2(\mathbb{R}; d\omega_\alpha)$,

with $U_\alpha$ unitary,

$$U_\alpha : \hat{H}_\alpha = L^2(\mathbb{R}; d\omega_\alpha) \to \mathcal{H}, \quad \hat{g} \to U_\alpha \hat{g} = \text{s-lim}_{N \to \infty} \int_{-N}^{N} d(E_{H_\alpha}(\lambda) u_+)(\lambda - i) \hat{g}(\lambda).$$

Moreover, $u_+ = U_\alpha \hat{u}_+$, $\hat{u}_+(\lambda) = (\lambda - i)^{-1}$, and

$$(\hat{H}(\alpha) \hat{g})(\lambda) = \lambda \hat{g}(\lambda),$$

$\hat{g} \in \text{dom}(\hat{H}(\alpha)) = \{ \hat{h} \in \text{dom}(\hat{H}_\alpha) \mid \int_{\mathbb{R}} d\omega_\alpha(\lambda) \hat{h}(\lambda) = 0 \}$,

where

$$H = U_\alpha \hat{H}(\alpha) U_\alpha^{-1}.$$
Thus $\hat{H}(\alpha)$ in $L^2(\mathbb{R}; d\omega_\alpha)$ is a canonical representation for a densely defined, closed, symmetric operator $H$ with deficiency indices $(1, 1)$ in a separable complex Hilbert space $\mathcal{H}$ with cyclic deficiency vector $u_+ \in \ker(H^* - i)$.

Actually, we shall show next that $\hat{H}(\alpha)$ in $L^2(\mathbb{R}; d\omega_\alpha)$ is a model for all such operators.

Moreover, since
\[
((H - z)g, U_\alpha(\cdot - z)^{-1})_{\mathcal{H}} = \int_\mathbb{R} d\omega_\alpha(\lambda)(\lambda - z)(U_{\alpha}^{-1}g)(\lambda)(\lambda - z)^{-1} = 0, \quad g \in \text{dom}(H), \quad z \in \mathbb{C}\setminus\mathbb{R},
\]
one infers that $U_\alpha(\cdot - z)^{-1} \in \text{dom}(H^*)$. Since $\text{dom}(H)$ is dense in $\mathcal{H}$,
\[
\ker(\hat{H}(\alpha)^* - z) = \{c(\cdot - z)^{-1} \mid c \in \mathbb{C}\}, \quad z \in \mathbb{C}\setminus\mathbb{R},
\]
where
\[
H^* = U_\alpha \hat{H}(\alpha)^* U_{\alpha}^{-1}.
\]

Note. If $u_+$ is not cyclic for $H_\alpha$ then introduce appropriate reducing subspaces.

Next we show the model character of $(\hat{H}_\alpha, \hat{H}(\alpha), \hat{H}_\alpha)$ following the approach outlined by Donoghue ’65.
Theorem 11

Let $H$ be a densely defined, closed, symmetric operator with deficiency indices $(1, 1)$ and normalized deficiency vectors $u_\pm \in \ker(H^* \mp i)$, $\|u_\pm\|_\mathcal{H} = 1$ in some separable complex Hilbert space $\mathcal{H}$. Let $H_\alpha$ be a self-adjoint extension of $H$ with simple spectrum (i.e., $u_+$ is a cyclic vector for $H_\alpha$). Then the pair $(H, H_\alpha)$ in $\mathcal{H}$ is unitarily equivalent to the pair $(\hat{H}(\alpha), \hat{H}_\alpha)$ in $\hat{\mathcal{H}}$ effected by the unitary operator $U_\alpha$. Conversely, given a measure $d\tilde{\omega}$ satisfying

$$\int_\mathbb{R} \frac{d\tilde{\omega}(\lambda)}{1+\lambda^2} = 1, \quad \int_\mathbb{R} d\tilde{\omega}(\lambda) = \infty,$$

define the self-adjoint operator $\tilde{H}$ of multiplication by $\lambda$ in $\tilde{\mathcal{H}} = L^2(\mathbb{R}; d\tilde{\omega})$,

$$(\tilde{H}g)(\lambda) = \lambda g(\lambda), \quad g \in \text{dom}(\tilde{H}) = L^2(\mathbb{R}; (1 + \lambda^2)d\tilde{\omega}),$$

and the linear operator $H$ in $\tilde{\mathcal{H}}$,

$$\text{dom}(H) = \{ g \in \text{dom}(\tilde{H}) \mid \int_\mathbb{R} d\tilde{\omega}(\lambda) g(\lambda) = 0 \}, \quad H = \tilde{H}\big|_{\text{dom}(H)}.$$

Then $H$ is a densely defined, closed, symmetric operator in $\tilde{\mathcal{H}}$ with deficiency indices $(1, 1)$ and deficiency spaces

$$\ker(H^* \mp i) = \{ c(\lambda \mp i)^{-1} \mid c \in \mathbb{C} \}.$$
Returning to the self-adjoint extension $H_\alpha$ of the symmetric operator $H$ with deficiency indices $(1,1)$, introduce the Nevanlinna–Herglotz function, the Donoghue $m$-function,

$$m^{D_\alpha}_\alpha(z) = \int_\mathbb{R} d\omega^{D_\alpha}_\alpha(\lambda)((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1})$$

$$= z + (1 + z^2)(u_+, (H_\alpha - z)^{-1}u_+)$$

(the last equality being a simple consequence of $\int_\mathbb{R} d\omega_\alpha(\lambda) (1 + \lambda^2)^{-1} = 1$) one verifies

$$m^{D_\beta}_\beta(z) = \frac{-\sin(\beta - \alpha) + \cos(\beta - \alpha) m^{D_\alpha}_\alpha(z)}{\cos(\beta - \alpha) + \sin(\beta - \alpha) m^{D_\alpha}_\alpha(z)}, \quad \alpha, \beta \in [0, \pi).$$

A comparison of the linear fractional transformations suggests invoking

$$a(\alpha, \beta) = \begin{pmatrix} \cos(\beta - \alpha) & \sin(\beta - \alpha) \\ -\sin(\beta - \alpha) & \cos(\beta - \alpha) \end{pmatrix} \in A_2, \quad \alpha, \beta \in [0, \pi).$$

Moreover, since $m_\gamma(i) = i$ for all $\gamma \in [0, \pi)$, Theorem 8 applies (with $a(\alpha, \beta)_{1,1} = a(\alpha, \beta)_{2,2} = \cos(\beta - \alpha), \quad a(\alpha, \beta)_{1,2} = -a(\alpha, \beta)_{2,1} = \sin(\beta - \alpha)$).
Connection between **Weyl–Titchmarsh** and **Donoghue m-functions**: Since $m_{\alpha}^{Do}(i) = i$,

$$m_{\alpha}^{Do}(z) = \left[ m_{\alpha}^{WT}(z) - \text{Re}(m_{\alpha}^{WT}(i)) \right] / \text{Im}(m_{\alpha}^{WT}(i))$$

$$= d_{\alpha} + \int_{\mathbb{R}} d\omega_{\alpha}^{Do}(\lambda) \left[ (\lambda - z)^{-1} - \lambda(\lambda^2 + 1)^{-1} \right].$$

Comparing with

$$m_{\alpha}^{WT}(z) = c_{\alpha} + \int_{\mathbb{R}} d\rho_{\alpha}^{WT}(\lambda) \left[ (\lambda - z)^{-1} - \lambda(\lambda^2 + 1)^{-1} \right],$$

one infers the equivalence of measures,

$$d\omega_{\alpha}^{Do}(\cdot) \sim d\rho_{\alpha}^{WT}(\cdot) \sim dE_{H_{\alpha}}(\cdot), \quad \alpha \in [0, \pi).$$

Moreover,

$$\text{supp} \left( d\omega_{\alpha}^{Do}(\cdot) \right) = \text{supp} \left( d\rho_{\alpha}^{WT}(\cdot) \right) = \text{supp} \left( dE_{H_{\alpha}}(\cdot) \right) = \sigma(H_{\alpha}), \quad \alpha \in [0, \pi).$$
Suppose that $H$ is nonnegative, $H \geq 0$, we now characterize the Friedrichs and Krein extensions, $H_F$ and $H_K$, of $H$ in terms of the (Nevanlinna–Herglotz) Donoghue $m$-functions $m^D\alpha(z)$.

Then all nonnegative, self-adjoint extensions $\tilde{H}$ of $H$ satisfy

$$0 \leq (H_F - \mu)^{-1} \leq (\tilde{H} - \mu)^{-1} \leq (H_K - \mu)^{-1}, \quad \mu \in (-\infty, 0),$$

i.e., $\tilde{H}$ are “sandwiched” between $H_K$ and $H_F$.

Hence $H$ has a unique nonnegative self-adjoint extension if and only if $H_K = H_F$. 
Theorem 12

(i) Let $H_\alpha = H_{\alpha}^* \geq 0$. Then $H_\alpha = H_F$ for some $\alpha \in [0, \pi)$ if and only if
$$\lim_{\lambda \downarrow -\infty} m_\alpha^{D o}(\lambda) = -\infty.$$  

(ii) Let $H_\beta = H_\beta^* \geq 0$. Then $H_\beta = H_K$ for some $\beta \in [0, \pi)$ if and only if
$$\lim_{\lambda \uparrow 0} m_\beta^{D o}(\lambda) = \infty.$$  

(iii) Let $H_\gamma = H_\gamma^* \geq 0$. Then $H_\gamma = H_F = H_K$ for some $\gamma \in [0, \pi)$ if and only if
$$\lim_{\lambda \downarrow -\infty} m_\gamma^{D o}(\lambda) = -\infty \quad \text{and} \quad \lim_{\lambda \uparrow 0} m_\gamma^{D o}(\lambda) = \infty.$$  

(iv) Suppose $\alpha_F \in [0, \pi)$ corresponds to $H_{\alpha_F} = H_F$, $\beta_K \in [0, \pi)$ to $H_{\beta_K} = H_K$, and $\gamma \in [0, \pi)$. Then

$$\lim_{\lambda \downarrow -\infty} m_\gamma^{D o}(\lambda) = \cot(\gamma - \alpha_F) = -\int_\mathbb{R} d\omega_\gamma^{D o}(\lambda) \lambda (1 + \lambda^2)^{-1}, \quad \gamma \neq \alpha_F,$$

$$\lim_{\lambda \uparrow 0} m_\gamma^{D o}(\lambda) = \cot(\gamma - \alpha_K) = \int_\mathbb{R} d\omega_\gamma^{D o}(\lambda) [\lambda^{-1} - \lambda (1 + \lambda^2)^{-1}], \quad \gamma \neq \alpha_K,$$

$$\int_\mathbb{R} d\omega_\gamma^{D o}(\lambda) \lambda^{-1} = \cot(\gamma - \alpha_K) - \cot(\gamma - \alpha_F), \quad \gamma \neq \alpha_F, \; \gamma \neq \alpha_K.$$
The **model approach** applies to certain $2 \times 2$ Hamiltonian systems on a half-line, $[x_0, \infty)$, regular at $x_0 \in \mathbb{R}$, such as:

- **Sturm-Liouville operators** (up to three coefficients)
- **Dirac-type** operators
- **Jacobi** operators
- **CMV operators** (they are a bit different, unitary, etc.), but analogous constructions exist.
Some Literature Hints:

A very large number of mathematicians contributed to the subjects mentioned in this scalar-valued part. While it is impossible to list them here, the following references are useful in connection with the topics discussed:

General references on scalar Nevanlinna–Herglotz functions (boundary values of analytic functions, etc.):


Some Literature Hints (contd.):


Some Literature Hints (contd.):

Aronszajn–Donoghue and Simon–Wolff theory (supports of measures, etc.):


Some Literature Hints (contd.):


**Self-adjoint extensions, including Friedrichs and Krein extensions, and rank-one perturbations (just a few hints which contain detailed reference lists to the enormous amount of literature on this subject):**


Some Literature Hints (contd.):


The following paper contains an exhaustive list of references and an extended version of the originally published article was uploaded to the dropbox:

The (self-adjoint) matrix-valued measures needed below will be of the type
\[ \Sigma(M) = \int_M d\Omega(\lambda)(1 + \lambda^2)^{-1}, \quad \Sigma = (\Sigma_{j,k})_{1 \leq j, k \leq n}, \quad \Omega = (\Omega_{j,k})_{1 \leq j, k \leq n}, \]
where \( \Sigma_{j,k}, 1 \leq j, k \leq n \) are complex (and hence finite) Borel measures on \( \mathbb{R} \) and \( \Omega_{j,k}, 1 \leq j, k \leq n \) are complex-valued set functions defined on the bounded Borel subsets of \( \mathbb{R} \) with the properties,

(i) \( \Omega(X) = (\Omega_{j,k}(X))_{1 \leq j, k \leq n} \subset M_n(\mathbb{C}) \) is nonnegative, \( \Omega(X) \geq 0 \), for all bounded Borel sets \( X \subset \mathbb{R} \), and \( \Omega(\phi) = 0 \).

(ii) \( \Omega_{j,k}(\bigcup_{\ell \in \mathbb{N}} X_\ell) = \sum_{\ell \in \mathbb{N}} \Omega_{j,k}(X_\ell), \ 1 \leq j, k \leq n \) for each sequence of disjoint Borel sets \( \{X_\ell\}_{\ell \in \mathbb{N}} \subset \mathbb{R} \) with \( \bigcup_{\ell \in \mathbb{N}} X_\ell \) bounded.

Each diagonal element \( \Sigma_{j,j}, 1 \leq j \leq n \) defines a positive (finite) Borel measure on \( \mathbb{R} \). In addition, we denote by
\[ \sigma^{tr} = \text{tr}_{\mathbb{C}^n}(\Sigma) = \Sigma_{1,1} + \cdots + \Sigma_{n,n} \]
the (scalar) trace measure of \( \Sigma \) and note that
\[ \sigma^{tr}(X) = 0 \quad \text{if and only if} \quad \Sigma(X) = 0 \]
for all Borel sets \( X \subset \mathbb{R} \) since
\[ \Sigma_{j,k} \ll \Sigma_{j,j} + \Sigma_{k,k} \ll \sigma^{tr}, \quad 1 \leq j, k \leq n. \]
We will use the standard Lebesgue decomposition of matrix-valued measures with respect to Lebesgue measure on $\mathbb{R}$, in particular, we will use the fact that $\Omega = \Omega_{ac}$ is purely absolutely continuous with respect to Lebesgue measure $d\lambda$ if and only if $d\Omega(\lambda) = P(\lambda)d\lambda$ for some nonnegative locally integrable matrix $P$ on $\mathbb{R}$.

For $M \in \mathbb{C}^{n \times n}$, one defines $\text{Im}(M) = (2i)^{-1}[M - M^*]$.

**Definition 13**

$M : \mathbb{C}_+ \to \mathbb{C}^{n \times n}$ is called a matrix-valued Nevanlinna–Herglotz function (in short, a N-H matrix) if $M$ is analytic on $\mathbb{C}_+$ and $\text{Im}(M(z)) \geq 0$ for all $z \in \mathbb{C}_+$.

We exclusively continue $M$ to $\mathbb{C}_-$ by reflection, i.e.,

$$M(z) = M(\bar{z})^*, \quad z \in \mathbb{C}_-. $$
Theorem 14

Let $M(z) \in M_n(\mathbb{C})$ be a matrix-valued Nevanlinna–Herglotz function. Then

(i) Each diagonal element $M_{j,j}(z)$, $1 \leq j \leq n$ of $M(z)$ is a (scalar) Herglotz function.

(ii) $M(z)$ has finite normal limits $M(\lambda \pm i0) = \lim_{\varepsilon \downarrow 0} M(\lambda \pm i\varepsilon)$ for a.e. $\lambda \in \mathbb{R}$.

(iii) If each diagonal element $M_{j,j}(z)$, $1 \leq j \leq n$ of $M(z)$ has a zero normal limit on a fixed subset of $\mathbb{R}$ having positive Lebesgue measure, then $M(z) = C_0$, where $C_0 = C_0^*$ is a constant self-adjoint $n \times n$ matrix with vanishing diagonal elements.

(iv) There exists a matrix-valued measure $\Omega$ on the bd. Borel subsets of $\mathbb{R}$ s.t. 
$$\int_{\mathbb{R}} (c, d\Omega(\lambda)c)_{\mathbb{C}^n} (1 + \lambda^2)^{-1} < \infty$$
for all $c \in \mathbb{C}^n$ having the representation

$$M(z) = C + Dz + \int_{\mathbb{R}} d\Omega(\lambda)((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}), \quad z \in \mathbb{C}_+,$$

$$C = \text{Re}(M(i)), \quad D = \lim_{\eta \uparrow \infty} \left( \frac{1}{i\eta} M(i\eta) \right) \geq 0.$$
Theorem 14 (contd.)

(v) The Stieltjes inversion formula for \( \Omega \) reads

\[
\frac{1}{2} \Omega(\{\lambda_1\}) + \frac{1}{2} \Omega(\{\lambda_2\}) + \Omega((\lambda_1, \lambda_2)) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \int_{\lambda_1}^{\lambda_2} d\lambda \Im(M(\lambda + i\varepsilon)).
\]

(vi) The absolutely continuous part \( \Omega_{ac} \) of \( \Omega \) is given by

\[
d\Omega_{ac}(\lambda) = \pi^{-1} \Im(M(\lambda + i0))d\lambda.
\]

(vii) Any poles of \( M \) are simple and located on the real axis, the residues at poles being nonpositive matrices (of rank \( r \in \{1, \ldots, n\} \)).
Additional results and the exponential Nevanlinna–Herglotz representation:

**Theorem 15**

Let \( M(z) \in M_n(\mathbb{C}) \) be a matrix-valued Nevanlinna–Herglotz function. Then

(i) For all \( \lambda \in \mathbb{R} \),

\[
\lim_{\varepsilon \downarrow 0} \varepsilon \text{Re}(M(\lambda + i\varepsilon)) = 0,
\]

\[
\Omega(\{\lambda\}) = \lim_{\varepsilon \downarrow 0} \varepsilon \text{Im}(M(\lambda + i\varepsilon)) = -i \lim_{\varepsilon \downarrow 0} \varepsilon M(\lambda + i\varepsilon).
\]

(ii) Let \( L > 0 \) and suppose \( 0 \leq \text{Im}(M(z)) \leq LI_n \) for all \( z \in \mathbb{C}_+ \). Then \( D = 0 \), \( \Omega \) is purely absolutely continuous, \( \Omega = \Omega_{ac} \), and

\[
0 \leq \frac{d\Omega(\lambda)}{d\lambda} = \pi^{-1} \lim_{\varepsilon \downarrow 0} \text{Im}(M(\lambda + i\varepsilon)) \leq \pi^{-1} LI_n \text{ for a.e. } \lambda \in \mathbb{R}.
\]

(iii) Assume \( M(z) \) is invertible for all \( z \in \mathbb{C}_+ \). Then there exist \( \Xi_{j,k} \in L^\infty(\mathbb{R}) \), \( 1 \leq j, k \leq n \), \( 0 \leq \Xi \leq I_n \) a.e., such that

\[
\ln(M(z)) = K + \int_{\mathbb{R}} d\lambda \Xi(\lambda)((\lambda - z)^{-1} - \lambda(1 + \lambda^2)^{-1}), \quad z \in \mathbb{C}_+,
\]

\[
K = \text{Re}(\ln(M(i))),
\]

where

\[
\Xi(\lambda) = \pi^{-1} \lim_{\varepsilon \downarrow 0} \text{Im}(\ln(M(\lambda + i\varepsilon))) \text{ for a.e. } \lambda \in \mathbb{R}.
\]
Linear fractional transformations: Define

\[ J_{2n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \quad \mathcal{A}_{2n} = \{ A \in \mathbb{C}^{n \times n} \mid A^* J_{2n} A = J_{2n} \}. \]

This leads to

\[ M_A(z) = (A_{2,1} + A_{2,2} M(z))(A_{1,1} + A_{1,2} M(z))^{-1}, \quad A \in \mathcal{A}_{2n}, \quad z \in \mathbb{C}_+, \]

assuming \( \ker(A_{1,1} + A_{1,2} M(z)) = \{0\} \) for all \( z \in \mathbb{C}_+ \).
In order to capture spectral multiplicities in the matrix-valued case in connection with applications to differential and difference operators we introduce the sets 

\[ (1 \leq r \leq n) \]

\[ S_{\Omega_{ac},r} = \{ \lambda \in \mathbb{R} \mid \lim_{\epsilon \downarrow 0} M(\lambda + i\epsilon) \text{ exists finitely, } \text{rank}(\text{Im}(M(\lambda + i0))) = r \}, \]

\[ S_{\Omega_{ac}} = \bigcup_{r=1}^{n} S_{\Omega_{ac},r}, \]

\[ S_{\Omega_{pp},r} = \{ \lambda \in \mathbb{R} \mid \text{rank}(\lim_{\epsilon \downarrow 0} \epsilon M(\lambda + i\epsilon)) = r \}, \quad 1 \leq r \leq n, \]

\[ S_{\Omega_{pp}} = \bigcup_{r=1}^{n} S_{\Omega_{pp},r}, \]

\[ S_{\Omega_{s}} = \{ \lambda \in \mathbb{R} \mid \lim_{\epsilon \downarrow 0} \text{Im}(\text{tr}_{\mathbb{C}^n}(M(\lambda + i\epsilon))) = +\infty \}, \]

\[ S_{\Omega_{sc}} = \{ \lambda \in S_{\Omega_{s}} \mid \lim_{\epsilon \downarrow 0} \epsilon \text{tr}_{\mathbb{C}^n}(M(\lambda + i\epsilon)) = 0 \}, \]

\[ S_{\Omega} = S_{\Omega_{ac}} \cup S_{\Omega_{s}}. \]

Thus, \( S_{\Omega_{ac},r}, S_{\Omega_{pp},r'}, S_{\Omega_{sc}} \) are all disjoint for any \( 1 \leq r, r' \leq n. \)
As in the scalar case we define the equivalence classes $E(\Omega_{ac})$ and $E_r(\Omega_{ac})$ of $S_{\Omega_{ac}}$ and $S_{\Omega_{ac},r}$, $1 \leq r \leq n$ with respect to the equivalence relation $S \sim T$ if and only if $\mu(S \triangle T) = 0 = \nu(S \triangle T)$ (with $\nu$ representing Lebesgue measure on $\mathbb{R}$ and $\mu = \Omega_{ac}$).

The following result is analogous to Theorem 7 in the scalar case and can be reduced to it by studying the trace measure $\omega^{tr}$ of $\Omega$.

**Theorem 16**

Let $M$ be a matrix-valued Nevanlinna–Herglotz function. Then

(i) $S_{\Omega_{ac}}$ is a minimal support of $\Omega_{ac}$.

(ii) $S_{\Omega_{sc}}$ is a minimal support of $\Omega_{sc}$.

(iii) $S_{\Omega_{pp}}$ is the smallest support of $\Omega_{pp}$.

(iv) $S_{\Omega}$ is a minimal support of $\Omega$.

(v) If in addition $M(z)$ is invertible for all $z \in \mathbb{C}_+$, then

$$\tilde{S}_{ac} = \{ \lambda \in S_{\Omega_{ac}} \mid \ln(M(\lambda + i0)) \text{ exists finitely and } 0 < \text{tr}(\Xi(\lambda)) < n \}$$

is a minimal support of $\Omega_{ac}$. 
Theorem 17

Let \( M(z) \) and \( M_A(z) = (A_{2,1} + A_{2,2}M(z))(A_{1,1} + A_{1,2}M(z))^{-1} \), \( A = (A_{p,q})_{1 \leq p, q \leq 2} \in A_{2n} \) be Nevanlinna–Herglotz matrices and assume that \( \ker(A_{1,1} + A_{1,2}M(z)) = \{0\} \) for all \( z \in \mathbb{C}_+ \). Let \( \Omega \) and \( \Omega_A \) be the measures associated with \( M(z) \) and \( M_A(z) \). Then

(i) For all \( A \in A_{2n} \),
\[
\mathcal{E}_r(\Omega_{A,ac}) = \mathcal{E}_r(\Omega_{ac}), \quad 1 \leq r \leq n,
\]
\[
\mathcal{E}(\Omega_{A,ac}) = \mathcal{E}(\Omega_{ac}),
\]
i.e., \( \mathcal{E}_r(\Omega_{A,ac}), 1 \leq r \leq n \) and \( \mathcal{E}(\Omega_{A,ac}) \) are independent of \( A \in A_{2n} \) (and hence denoted by \( \mathcal{E}_{ac,r}, 1 \leq r \leq n \) and \( \mathcal{E}_{ac} \) below) and \( \Omega_{A,ac} \sim \Omega_{ac} \) for all \( A \in A_{2n} \).

(ii) Suppose \( \Omega_B \) is a discrete point measure, \( \Omega_B = \Omega_{B,d} \), for some \( B \in A_{2n} \). Then \( \Omega_A = \Omega_{A,d} \) is a discrete point measure for all \( A \in A_{2n} \).

(iii) Define
\[
S = \{ \lambda \in \mathbb{R} \mid \text{there is no } A \in A_{2n} \text{ for which } \text{Im}(M_A(\lambda + i0)) \text{ exists and equals } 0 \}.
\]
Then \( S \in \mathcal{E}_{ac} \).
Applications of Matrix-Valued N-H Functions:

With this in place all applications discussed in the scalar context extend to the matrix-valued case:

- The model approach to finite-rank perturbations.
- The model approach to self-adjoint extensions of symmetric operators with deficiency indices \((n, n), n \in \mathbb{N}\).
- Fractional transformations of \(n \times n\) Nevanlinna-Herglotz functions, the Donoghue \(M\)-functions, associated to self-adjoint extensions.
- Characterizations of Friedrichs and Krein extensions in terms of these \(n \times n\) Donoghue \(M\)-functions.
- Certain \(2n \times 2n, n \in \mathbb{N}\), Hamiltonian systems on a half-line \([x_0, \infty)\), regular at \(x_0 \in \mathbb{R}\), including Schrödinger operators with \(n \times n\) matrix-valued potentials, \(2n \times 2n\) Dirac-type operators, etc.
• Certain $2n \times 2n$, $n \in \mathbb{N}$, Hamiltonian systems on $\mathbb{R}$:

BUT, how to put the two half-lines, $(-\infty, x_0]$ and $[x_0, \infty)$ together to get the entire real line $\mathbb{R}$?

Associate $m_{+,\alpha}^{D_0}(\cdot, x_0)$, $H_{+,\alpha}(x_0)$ with $[x_0, \infty)$ and $m_{-,\alpha}^{D_0}(\cdot, x_0)$, $H_{-,\alpha}(x_0)$ with $(-\infty, x_0]$ (we use the convention $\pm m_{\pm,\alpha}^{D_0}(\cdot)$ are N-H fcts.), then something like

$$
\begin{pmatrix}
-m_{-,\alpha}^{D_0}(\cdot, x_0) & 0 \\
0 & m_{+,\alpha}^{D_0}(\cdot, x_0)
\end{pmatrix}
$$

would correspond to

$$
H_{-,\alpha}(x_0) \oplus H_{+,\alpha}(x_0) \text{ in } L^2((-\infty, x_0]) \oplus L^2([x_0, \infty)) \sim L^2(\mathbb{R}).
$$

BUT, that’s NOT what we want! We don’t want direct sums with b.c.’s but an operator $H$ in $L^2(\mathbb{R})$ that feels no artificial boundary condition (parametrized by $\alpha$ ... ) at an artificial point $x_0 \in \mathbb{R}$!

The solution for this is somewhat intricate!
It’s a bit simpler for Weyl–Titchmarsh $m$-functions, so we give that first:

$$M_{\alpha}^{WT}(z, x_0) = \frac{1}{m_{-,\alpha}^{WT}(z, x_0) - m_{+,\alpha}^{WT}(z, x_0)}$$

$$\times \left( \frac{1}{[m_{-,\alpha}^{WT}(z, x_0) + m_{+,\alpha}^{WT}(z, x_0)]/2} \begin{bmatrix} [m_{-\alpha}^{WT}(z, x_0) + m_{+,\alpha}^{WT}(z, x_0)]/2 & m_{-,\alpha}^{WT}(z, x_0) \\ m_{-,\alpha}^{WT}(z, x_0) m_{+,\alpha}^{WT}(z, x_0) \end{bmatrix} \right)$$

$$= C_{\alpha} + \int_{\mathbb{R}} d\Omega_{\alpha}^{WT}(\lambda, x_0) [(\lambda - z)^{-1} - \lambda(\lambda^2 + 1)^{-1}], \quad \alpha \in [0, \pi), \ x_0 \in \mathbb{R}.$$ 

This is indeed a $2 \times 2$ Nevanlinna–Herglotz function and

$$d\Omega_{\alpha}^{WT}(\cdot, x_0) \sim dE_H(\cdot), \quad \text{supp} \left( d\Omega_{\alpha}^{WT}(\cdot, x_0) \right) = \text{supp} \left( dE_H(\cdot) \right) = \sigma(H),$$

$$\alpha \in [0, \pi), \ x_0 \in \mathbb{R}.$$ 

One can show (explicit formulas exist!) that,

$$M_{\alpha}^{Do}(z, x_0) = T_{\alpha}(x_0)^* M_{\alpha}^{WT}(z, x_0) T_{\alpha}(x_0) + D_{\alpha}, \quad \alpha \in [0, \pi), \ x_0 \in \mathbb{R},$$

where $D_{\alpha}(x_0)^* = D_{\alpha}(x_0)$ and $T_{\alpha}(x_0)$ are $z$-independent, with $T_{\alpha}(x_0)$ invertible.
Some Literature Hints:

Various references for the matrix-valued case:


The following paper contains an exhaustive list of references and an extended version of the originally published article was uploaded to the dropbox:

Let $\mathcal{H}$ be a separable, complex Hilbert space with inner product denoted by $(\cdot, \cdot)_{\mathcal{H}}$, identity operator abbreviated by $I_{\mathcal{H}}$. Denote $\mathbb{C}_\pm = \{ z \in \mathbb{C} | \pm \text{Im}(z) > 0 \}$ and $\text{Im}(M) = (M - M^*)/(2i)$, $\text{Re}(M) = (M + M^*)/2$, $M \in B(\mathcal{H})$ (the Banach space of linear, bounded operators on $\mathcal{H}$).

**Definition 18**

The map $M : \mathbb{C}_+ \to B(\mathcal{H})$ is called a bounded operator-valued Nevanlinna–Herglotz function on $\mathcal{H}$ (in short, a bounded Nevanlinna–Herglotz operator on $\mathcal{H}$) if $M$ is analytic on $\mathbb{C}_+$ and $\text{Im}(M(z)) \geq 0$ for all $z \in \mathbb{C}_+$.

We exclusively continue $M$ to $\mathbb{C}_-$ by reflection, i.e.,

$$M(z) = M(\bar{z})^*, \quad z \in \mathbb{C}_-. $$
Next we recall the definition of a bounded operator-valued measure:

**Definition 19**

Let $\mathcal{H}$ be a separable, complex Hilbert space. A map $\Sigma : \mathcal{B}(\mathbb{R}) \to \mathcal{B}(\mathcal{H})$, with $\mathcal{B}(\mathbb{R})$ the Borel $\sigma$-algebra on $\mathbb{R}$, is called a **bounded, nonnegative, operator-valued measure** if the following conditions $(i)$ and $(ii)$ hold:

$(i)$ $\Sigma(\emptyset) = 0$ and $0 \leq \Sigma(B) \in \mathcal{B}(\mathcal{H})$ for all $B \in \mathcal{B}(\mathbb{R})$.

$(ii)$ $\Sigma(\cdot)$ is strongly countably additive (i.e., with respect to the strong operator topology in $\mathcal{H}$), i.e.,

$$\Sigma(B) = \text{s-lim}_{N \to \infty} \sum_{j=1}^{N} \Sigma(B_j)$$

whenever $B = \bigcup_{j \in \mathbb{N}} B_j$, with $B_k \cap B_\ell = \emptyset$ for $k \neq \ell$, $B_k \in \mathcal{B}(\mathbb{R})$, $k, \ell \in \mathbb{N}$.

$\Sigma(\cdot)$ is called an **(operator-valued) spectral measure** (or an **orthogonal operator-valued measure**) if additionally the following condition $(iii)$ holds:

$(iii)$ $\Sigma(\cdot)$ is projection-valued (i.e., $\Sigma(B)^2 = \Sigma(B)$, $B \in \mathcal{B}(\mathbb{R})$) and $\Sigma(\mathbb{R}) = I_\mathcal{H}$.

Due to monotonicity considerations, taking the limit in the strong operator topology in $(*)$ is equivalent to taking the limit with respect to the weak operator topology in $\mathcal{H}$.
Definition 19 (contd.)

Let \( f \in \mathcal{H} \) and \( B \in \mathcal{B}(\mathbb{R}) \). Then the vector-valued measure \( \Sigma(\cdot)f \) has **finite variation on** \( B \), denoted by \( V(\Sigma f; B) \), if

\[
V(\Sigma f; B) = \sup \left\{ \sum_{j=1}^{N} \| \Sigma(B_j)f \|_{\mathcal{H}} \right\} < \infty,
\]

where the supremum is taken over all finite sequences \( \{B_j\}_{1 \leq j \leq N} \) of pairwise disjoint subsets on \( \mathbb{R} \) with \( B_j \subseteq B \), \( 1 \leq j \leq N \). In particular, \( \Sigma(\cdot)f \) has **finite total variation** if \( V(\Sigma f; \mathbb{R}) < \infty \).

Theorem 20

Let \( M \) be a bounded operator-valued Nevanlinna–Herglotz function in \( \mathcal{H} \). Then the following assertions hold:

(i) For each \( f \in \mathcal{H} \), \( (f, M(\cdot)f)_{\mathcal{H}} \) is a (scalar) Nevanlinna–Herglotz function.

(ii) Suppose that \( \{e_j\}_{j \in \mathbb{N}} \) is a complete orthonormal system in \( \mathcal{H} \) and that for some subset of \( \mathbb{R} \) having positive Lebesgue measure, and for all \( j \in \mathbb{N} \), \( (e_j, M(\cdot)e_j)_{\mathcal{H}} \) has zero normal limits. Then \( M \equiv 0 \).
Theorem 20 (contd.)

(iii) There exists a bounded, nonnegative $B(\mathcal{H})$-valued measure $\Omega$ on $\mathbb{R}$ such that the Nevanlinna representation

$$M(z) = C + Dz + \int_{\mathbb{R}} d\Omega(\lambda) \left[ \frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right], \quad z \in \mathbb{C}_+,$$

$$\tilde{\Omega}((-\infty, \lambda]) = \text{s-lim}_{\epsilon \downarrow 0} \int_{-\infty}^{\lambda + \epsilon} d\Omega(t) (t^2 + 1)^{-1}, \quad \lambda \in \mathbb{R},$$

$$\tilde{\Omega}(\mathbb{R}) = \text{Im}(M(i)) - D = \int_{\mathbb{R}} d\Omega(\lambda) (\lambda^2 + 1)^{-1} \in B(\mathcal{H}),$$

$$C = \text{Re}(M(i)), \quad D = \text{s-lim}_{\eta \uparrow \infty} \frac{1}{i\eta} M(i\eta) \geq 0,$$

holds in the strong sense in $\mathcal{H}$. Here $\tilde{\Omega}(B) = \int_B (1 + \lambda^2)^{-1} d\Omega(\lambda), \ B \in \mathcal{B}(\mathbb{R})$.

(iv) Let $\lambda_1, \lambda_2 \in \mathbb{R}, \ \lambda_1 < \lambda_2$. Then the Stieltjes inversion formula for $\Omega$ reads

$$\Omega((\lambda_1, \lambda_2])f = \pi^{-1} \text{s-lim}_{\delta \downarrow 0} \text{s-lim}_{\epsilon \downarrow 0} \int_{\lambda_1 + \delta}^{\lambda_2 + \delta} d\lambda \text{Im}(M(\lambda + i\epsilon))f, \quad f \in \mathcal{H}.$$
Theorem 20 (contd.)

(v) Any isolated poles of $M$ are simple and located on the real axis, the residues at poles being nonpositive bounded operators in $B(\mathcal{H})$.

(vi) For all $\lambda \in \mathbb{R}$,

$$s\text{-lim}_{\varepsilon \downarrow 0} \varepsilon \text{Re}(M(\lambda + i\varepsilon)) = 0, \quad (*)$$

$$\Omega(\{\lambda\}) = s\text{-lim}_{\varepsilon \downarrow 0} \varepsilon \text{Im}(M(\lambda + i\varepsilon)) = -i s\text{-lim}_{\varepsilon \downarrow 0} \varepsilon M(\lambda + i\varepsilon). \quad (**)$$

(vii) If in addition $M(z) \in B_\infty(\mathcal{H})$, $z \in \mathbb{C}_+$, then the measure $\Omega$ is countably additive with respect to the $B(\mathcal{H})$-norm, and the Nevanlinna representation and the Stieltjes inversion formula as well as $(*)$, $(**)$ hold with the limits taken with respect to the $\| \cdot \|_{B(\mathcal{H})}$-norm.
(viii) Let $f \in \mathcal{H}$ and assume in addition that $\Omega(\cdot)f$ is of finite total variation. Then for a.e. $\lambda \in \mathbb{R}$, the normal limits $M(\lambda + i0)f$ exist in the strong sense and

$$s\text{-lim}_{\varepsilon \downarrow 0} M(\lambda + i\varepsilon)f = M(\lambda + i0)f = H(\Omega(\cdot)f)(\lambda) + i\pi \Omega'(\lambda)f,$$

where $H(\Omega(\cdot)f)$ denotes the $\mathcal{H}$-valued Hilbert transform

$$H(\Omega(\cdot)f)(\lambda) = \text{p.v.} \int_{-\infty}^{\infty} d\Omega(t)f \frac{1}{t - \lambda} = s\text{-lim}_{\delta \downarrow 0} \int_{|t - \lambda| \geq \delta} d\Omega(t)f \frac{1}{t - \lambda}.$$

As usual, the normal limits in Theorem 20 can be replaced by nontangential ones.
The exponential Nevanlinna–Herglotz representation in the operator-valued context is discussed next:

**Theorem 21**

Suppose $M$ is a bounded operator-valued Nevanlinna–Herglotz function in $\mathcal{H}$ and $M(z_0)^{-1} \in B(\mathcal{H})$ for some (and hence for all) $z_0 \in \mathbb{C}_+$.  

(i) There exists a family of bounded self-adjoint weakly (Lebesgue) measurable operators $\{\Xi(\lambda)\}_{\lambda \in \mathbb{R}} \subset B(\mathcal{H})$,

\[
0 \leq \Xi(\lambda) \leq I_{\mathcal{H}} \quad \text{for a.e. } \lambda \in \mathbb{R},
\]

such that

\[
\log(M(z)) = C + \int_{\mathbb{R}} d\lambda \, \Xi(\lambda) \left[ \frac{1}{\lambda - z} - \frac{\lambda}{\lambda^2 + 1} \right], \quad z \in \mathbb{C}_+,
\]

\[
C = C^* \in B(\mathcal{H}),
\]

the integral being taken in the weak sense.
Theorem 21 (contd.)

(ii) Moreover, if \( \text{Im}(\log(M(z_0))) \in \mathcal{B}_1(\mathcal{H}) \) for some (and hence for all) \( z_0 \in \mathbb{C}_+ \), then

\[
0 \leq \Xi(\lambda) \in \mathcal{B}_1(\mathcal{H}) \text{ for a.e. } \lambda \in \mathbb{R},
\]

\[
0 \leq \text{tr}_\mathcal{H}(\Xi(\cdot)) \in L^1_{\text{loc}}(\mathbb{R}; d\lambda), \quad \int_{\mathbb{R}} d\lambda \left(1 + \lambda^2\right)^{-1}\text{tr}_\mathcal{H}(\Xi(\lambda)) < \infty,
\]

and

\[
\text{tr}_\mathcal{H}(\text{Im}(\log(M(z)))) = \text{Im}(z) \int_{\mathbb{R}} d\lambda \text{tr}_\mathcal{H}(\Xi(\lambda))|\lambda - z|^{-2}, \quad z \in \mathbb{C}_+.
\]
Applications of Operator-Valued N-H Functions:

- Applies to Schrödinger, Dirac-type, and Jacobi operators with operator-valued coefficients.
- Applies to Dirichlet-to-Neumann (more generally, Robin-to-Robin) maps for elliptic PDEs.
- Krein-type resolvent formulas connecting self-adjoint extensions of symmetric operators (applies abstractly and is applicable to PDEs).
Some Literature Hints:

General references on operator-valued Nevanlinna–Herglotz functions (boundary values of operator-valued analytic functions, etc.):


Some Literature Hints (contd.):


Some Literature Hints (contd.):


Some Literature Hints (contd.):


Self-adjoint extensions, including Friedrichs and Krein extensions (just a few hints which contain detailed reference lists to the enormous amount of literature on this subject):

Some Literature Hints (contd.):


Some Literature Hints (contd.):


**Krein-type resolvent formulas (just a few hints which contain detailed reference lists to the enormous amount of literature on this subject):**


Some Literature Hints (contd.):

Dirichlet-to-Neumann (Robin-to-Robin) maps (just a few hints which contain detailed reference lists to the enormous amount of literature on this subject):


Some Literature Hints (contd.):


Some Literature Hints (contd.):


Some Literature Hints (contd.):


Some Literature Hints (contd.):

The following papers contain exhaustive lists of references and were uploaded to the dropbox:


Further Comments:

The following comments address a few issues that were raised in connection with this talk:

(i) The effect of compositions of Nevanlinna–Herglotz functions on the measure has been studied in:


(Thanks to Maxence Cassier for reminding me of these references.)

See also,

Further Comments (contd.):

(ii) For Bochner's Theorem in the infinite-dimensional context and Naimark's Dilation Theorem, see,


(iii) Concerning the notion of gaps in connection with symmetric operators other than $(-\infty, 0)$, see, e.g.:


Further Comments (contd.):

(iv) **Stieltjes functions** are studied in detail, e.g., in:


(Thanks to Jonathan Eckhardt for the reference hint.)

### Definition

$f$ is a Stieltjes function if the following conditions (i)–(iii) hold:

(i) $f$ is analytic on $\mathbb{C}\setminus[0, \infty)$.

(ii) $\text{Im}(f(z)) > 0$, $z \in \mathbb{C}_+$.

(iii) $f(x) \geq 0$, $x \in (-\infty, 0)$. 
Further Comments (contd.):

**Theorem**

The following conditions (i)–(iv) are equivalent:

(i) $f$ is a Stieltjes function.

(ii) $f$ is a N-H function and so is $zf(z)$.

(iii) $f$ is a N-H function and so is $zf(z^2)$.

(iv) $f : \mathbb{C} \setminus [0, \infty) \to \mathbb{C}$ has the representation,

$$f(z) = d + \int_{[0, \infty)} d\omega(\lambda)(\lambda - z)^{-1}, \quad z \in \mathbb{C} \setminus [0, \infty),$$

$$d = \lim_{\lambda \downarrow -\infty} f(\lambda) \in [0, \infty), \quad \int_{[0, \infty)} \frac{d\omega(\lambda)}{|\lambda| + 1} < \infty.$$