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Abstract. In this paper it is shown that given a finite dimensional linear system with polynomial output that there exists a linear system with the same outputs. This linearization is a subsystem of the classical Carleman linearization. It is further shown that there is a relationship between the observability of the linearization and the observability of the polynomial system.

1. Introduction

In this note we consider the problem of the output behavior of linear systems with polynomial output. The system that is considered is

\[
\frac{d}{dt} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}
\]

and

\[
p(x_1,\cdots,x_n) = \sum_{\sum k_i \leq k} a_{k_1,k_2,\cdots,k_n} x_1^{k_1} \cdots x_n^{k_n}
\]

The arguments are restricted to the case of a single scalar output. Our main theorem is

**Theorem 1.1.** Let \( \dot{x} = Ax \) and \( y = p(x) \) where \( x \in \mathbb{R}^n \) and \( p(x) \) is a polynomial of \( n \) variables and is the sum of homogeneous polynomials \( p_j \) of degree \( m_j \leq k \). Then there exists an observable linear system of degree \( M \) where \( M \leq N = \sum m_j \left( \frac{n + m_j - 1}{m_j} \right) \) with linear observation, \( \dot{z} = Hz, y = cz, \) and a polynomial map \( F \) from \( \mathbb{R}^n \) to \( \mathbb{R}^N \) such that

1. \( \dot{x} = Ax \) whenever \( z \in \text{Image}(F) \)

and further more

2. If \( \dot{z} = Hz, y = cz \) is observable and \( F \) is one to one then \( \dot{x} = Ax, y = p(x) \) is observable.

We also determine the dimension of the minimal linear system that has the same output as the polynomial system. From this theorem we conclude that polynomial output does not yield any behavior not realizable from a linear system with linear output.
2. Realization

We first write \( p(x_1, \cdots, x_n) \) as a sum of homogeneous polynomials of degree at most \( k \), i.e. that for \( m_j \leq k \)

\[
p_{m_j}(x_1, \cdots, x_n) = \sum_{k_i=m_j} a_{k_1k_2 \cdots k_n} x_1^{k_1} \cdots x_n^{k_n}.
\]

and

\[
p(x_1, \cdots, x_n) = \sum_{m_j} p_{m}(x_1, \cdots, x_n)
\]

We, at this point, do not assume any special structure on the matrix \( A \).

Let

\[
S_{m_j}^n = \{ x_1^{k_1} \cdots x_n^{k_n} : \sum_{i=1}^n k_i = m_j \}
\]

and let

\[
W_{m_j}^n = \text{span} \ S_{m_j}^n.
\]

Define

\[
W_k^n = \oplus \sum_{m_j} W_{m_j}^n.
\]

\textbf{Lemma 2.1.} \( W_k^n \) is closed under differentiation along the trajectories of \( \dot{x} = Ax \).

\textbf{Proof:} It suffices to show that the derivative of each element of \( S_{m_j} \) is in \( W_{m_j} \). Let \( r(x) \in S_{m_j} \), then we have

\[
\frac{d}{dt} r(x) = \sum_{i=1}^n \frac{\partial}{\partial x_i} r(x) \frac{d}{dt} x_i(t)
\]

\[
= \sum_{i=1}^n \frac{\partial}{\partial x_i} r(x) e_i Ax
\]

where \( e_i \) is the \( i \)th unit vector. Now \( \frac{\partial}{\partial x_i} r(x) \) is homogeneous of degree \( m_j - 1 \) and \( e_i Ax \) is homogeneous of degree 1 and the lemma is proved.

Now assume that each \( S_{m_j}^n \) is ordered using the lexicographic ordering and let the \( \cup_{m_j} S_{m_j}^n \) be ordered with lexicographic ordering extended by letting the first element of the ordering be the \( m_j \)s. Let \( z \) be the vector of the totally ordered \( \cup S_{m_j}^n \). We have shown in the lemma that there exists a matrix \( H \) so that

\[
\dot{z}(t) = H z(t)
\]

It is well known, [3], that the cardinality of \( S_{m_j}^n \) is

\[
\binom{m_j + n - 1}{m_j}
\]

We see that the dimension of the space \( W_k^n \) is

\[
N = \sum_{m_j} \binom{m_j + n - 1}{m_j}
\]

and grows at a non-polynomial rate.

Now since the observation is the sum of homogeneous polynomials we can write

\[
p(x_1, x_2, \cdots , x_n) = cz
\]
where the coefficients are the coefficients of the monomials in \( p \). So we have constructed a linear system

\[
\dot{z} = Hz \quad y = cz
\]

and for initial conditions for \( z \) that correspond to the initial conditions for the original system

\[
\dot{x} = Ax \quad y = p(x)
\]

we have that

\[
 cz(t) = p(x(t)).
\]

We state this as the following theorem.

**Theorem 2.2.** Let the system \( \dot{x} = Ax, \ y = p(x) \) be given. There exists a finite dimensional linear system, \( \dot{z} = Hz, \ y = cz \) that linearizes the above system in sense that the output trajectories of \( \dot{x} = Ax, \ y = p(x) \) are a subset of the output trajectories of \( \dot{z} = Hz, \ y = cz \).

In order to make the above statement about initial data precise we define a function \( F \) with domain \( \mathbb{R}^n \) and range \( \mathbb{R}^N \) by

\[
F(a) = z(a)
\]

where this notation is intended to indicate that each monomial is evaluated at the point \( a \). Note that \( F \) is the sum of homogeneous polynomial maps.

**Lemma 2.3.** The function \( F \) is one to one if and only if there exists an \( m_j \) such that \( m_j = 2r + 1 \).

**Proof:** First assume that each of the \( m_j \)'s is even. Then we have \( F(-x) = F(x) \) and so \( F \) is not one to one. Now suppose that there exists an \( m_j \) that is odd and assume that \( F(a) = F(b) \). Then since all monomials of degree \( m_j \) are present we have \( a_i^{m_j} = b_i^{m_j} \) for each \( i \). Since \( m_j \) is odd and \( a \) and \( b \) are real we have that \( a_i = b_i \) and the lemma is proved.

Thus we have the two systems

\[
\begin{align*}
\dot{x} &= Ax, \quad x(0) = a \quad \dot{z} = Hz \quad z(0) = F(a) \\
y &= p(x) \quad y = c(x)
\end{align*}
\]

with identical outputs.

**Remark 2.4.** Recall how the initial data is recovered from a linear system, \( \dot{x} = Ax \) with linear output \( y = cx \). The standard approach is to differentiate \( y \) \( n-1 \) times at 0 and then solve the resulting system of linear equations. With polynomial output the above procedure is quite similar. We are in effect differentiating \( p(x) \) \( N \) times and eliminating the monomials in order to obtain a differential equation in the output \( y \). The Carleman linearization, \cite{dayawansa1995}, provides a systematic way to do the elimination.

### 3. Observability

In this section we consider the relation between the observability of the system with polynomial output with the observability of the linearization. Dayawansa and Martin examined this problem from another point of view in \cite{dayawansa1995}. There they were concerned with the existence of a polynomial output that observes the system. In this paper we are concerned with the observability of a given polynomial output. Starkov and Helmke in \cite{starkov2002} considered the problem of discrete observability of such system and used similar decomposition results as we use in this paper.

The following proposition is straightforward.

**Proposition 3.1.** If \( \dot{z} = Hz, \ y = cz \) is observable and \( F \) is one to one then \( \dot{x} = Ax, \ y = p(x) \) is observable.
Proof: The fact that the large system is observable says that each monomial is recovered and if the initial value is in the image of $F$ then $F^{-1}$ essentially solves the monomials for the values of the $x_i$, i.e. $p(x)$ and its derivatives along trajectories can used to solve the nonlinear equations.

3.1. A is diagonal. Let

$$\dot{x} = Ax, \quad y = p(x)$$

with $x \in \mathbb{R}^n$ be such that $A$ has distinct eigenvalues, real or complex. Assume that the system is observable and that $A$ has been diagonalized. Also assume that $p(x)$ is homogeneous of degree $k$. Let

$$\dot{z} = Hz, \quad y = cz$$

$z \in \mathbb{R}^N$ be the Carleman linearization of the small system.

Lemma 3.2. The Carleman linearization is observable if and only if 1) every monomial occurs in $p(x)$ and 2) the mapping

$$(i_1, \ldots, i_n) \mapsto \sum_{j=1}^{n} \lambda_j i_j$$

is one to one.

Proof: Let $m_j(x) = x_1^{i_1}x_2^{i_2} \cdots x_n^{i_n}$ be any monomial of degree $k$. Assume that the monomials have been totally ordered. We construct the derivative of $m(x)$ along trajectories of $\dot{x} = Ax$. Since $A$ is diagonal we have that

$$Dm_j(x) = i_1\lambda_1m_j(x) + \cdots + i_n\lambda_nm_j(x) = \left(\sum_{k=1}^{n} i_k\lambda_k\right)m_j(x) = a_jm_j(x).$$

Thus $H$ is diagonal. So we can construct the observability Grammian for the the system.

$$F = \begin{pmatrix} c_1 & c_2 & \cdots & c_{N-1} & c_N \\ c_1a_1 & c_2a_2 & \cdots & c_{N-1}a_{N-1} & c_Na_N \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_1a_1^N & c_2a_2^N & \cdots & c_{N-1}a_{N-1}^N & c_Na_N^N \end{pmatrix}$$

We see that $F$ can be factored as

$$F = \begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ a_1 & a_2 & \cdots & a_{N-1} & a_N \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1^N & a_2^N & \cdots & a_{N-1}^N & a_N^N \end{pmatrix} \begin{pmatrix} c_1 & 0 & \cdots & 0 & 0 \\ 0 & c_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & c_N \end{pmatrix}$$

The matrix $F$ is nonsingular if and only if both of the factors are nonsingular. The matrix on the left is a Vandermonde matrix so it is nonsingular if and only if the $a_i$ are distinct and the matrix on the right is nonsingular if and only if the $c_i$ are all nonzero. We state this as a proposition.

Proposition 3.3. Let the matrix $A$ be $n \times n$ diagonal with distinct eigenvalues $\lambda_i$ and let $y = p(x)$ where $p$ is homogeneous of degree $k$. Then the Carleman linearization of the system with the state vector being the vector of all monomials of degree $k$ has state matrix $H$ diagonal with the elements $n_1\lambda_1 + \cdots + n_n\lambda_n$ where $n_1 + n_2 + \cdots + n_n = k$. The system $\dot{z} = Hz, \quad y = cz$ is observable if and only if each $c_i \neq 0$ and the mapping

$$(n_1, n_2, \ldots, n_n) \rightarrow \lambda_1n_1 + \cdots + \lambda_nn_n$$

is one to one over the set of all partitions of $k$ into nonnegative parts.

The following corollary is of some interest.

Corollary 3.4. The Carleman linearization of $\dot{x} = Ax, \quad y = p(x)$ where $p$ is a homogeneous polynomial of degree $k$, is observable for every $k$ if and only if the eigenvalues of $A$ are linearly independent over the rational numbers and the polynomial is the nonzero sum of every monomial of degree $k$. 

This condition has arisen in the paper of Dayawansa and Martin, [1], as a condition for the existence of a polynomial output that makes the system observable, in Strakov and Helmke as a condition for discrete observability and in Sternberg, [6] for the linearization of nonlinear system around an equilibrium.

3.2. A has a Single Jordan Block. We assume that

\[
A = \begin{pmatrix}
  a & 1 & 0 & 0 & \cdots & 0 \\
  0 & a & 1 & 0 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & \cdots & a & 1 \\
  0 & 0 & a & \cdots & \cdots & \cdots 
\end{pmatrix}
\]

and that \(x = (x_1, \ldots, x_n)^T\). We define the vector of monomials of degree \(k\) to be \(z = (x_1^k, x_1^{k-1}x_2, \ldots, x_n^k)\) where the ordering of the monomials is the lexicographic ordering on the exponents. We now construct the Carleman linearization of the system \(\dot{x} = Ax\). Let \(m(x) = x_1^{k_1}x_2^{k_2} \cdots x_n^{k_n}\) where \(k_1 + \cdots + k_n = k\). Now

\[
D(m(x)) = k_1am(x) + k_1a_1x_1^{k_1-1}x_2^{k_2+1} \cdots x_n^{k_n} \\
+ k_2am(x) + k_2a_1x_1^{k_1}x_2^{k_2-1} \cdots x_n^{k_n} \\
+ \cdots + k_na_m(x)
\]

We can now write the matrix \(H = ka + N\) where the matrix \(N\) is strictly upper triangular. The question is now to determine the Jordan structure. We can assume that \(a = 0\). The total Jordan structure is not easy to calculate and the question we want to answer is "When are there at least two Jordan blocks?". This is easy to answer. Let \(y(t) = x_1^T\). solving the equation \(\dot{x} = Ax\) we have that \(x_1(t) = \sum_{i=1}^n b_i t^{i-1} / (i-1)!\) and hence is a polynomial of degree at most \(n-1\). Therefore \(\frac{d^{n+k-1}}{dx_{1}^{n+k-1}} x_1^k(t) = 0\) thus we conclude that there is no Jordan block of dimension greater than \(nk - k + 1\). Now the dimension of \(H\) is \(\left(\begin{array}{c} n + k - 1 \\ k \end{array}\right)\). We must determine when

\(nk - k + 1 \leq \left(\begin{array}{c} n + k - 1 \\ k \end{array}\right)\).

When \(n = 1\) we have that \(1 \leq 1\) for arbitrary \(k\). When \(n = 2\) we have \(k + 1 = k + 1\). However when \(n = 3\) we have \(2k + 1 < \left(\frac{3 + 1 - 1}{2}\right)\) unless \(k = 0\) or \(k = 1\). For larger \(n\) the right hand side increasing factorially and the left hand side linearly. A simple induction argument suffices to formally prove the statement. Thus only for \(n = 1\) and \(n = 2\) do we have a single Jordan block when \(k > 1\). We state these facts as a proposition.

Proposition 3.5. If \(A\) has a single Jordan block with eigenvalue \(a\) then for monomials of degree \(k\), \(H = akI + N\) where \(N\) is strictly upper triangular. \(H\) has a single Jordan block if and only if \(n = 1\) or \(n = 2\) and \(k\) is arbitrary or \(k = 0\) or \(k = 1\) and \(n\) is arbitrary.

Now we consider the observability of \(\dot{z} = Hz\), \(y = cz\). We first note that if \(n > 2\) then the system cannot be observable for any \(c\) since there are at least two Jordan blocks with the same eigenvalue. So we suppose that \(n = 2\). In this case

\[
z = (x_1^k, x_1^{k-1}x_2, x_1^{k-2}x_2^2, \ldots, x_2^k)^T
\]

and hence

\[
H = \begin{pmatrix}
  ak & 1 & 0 & 0 & \cdots & 0 & 0 \\
  0 & ak & 2 & 0 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  0 & 0 & 0 & ak & k \\
  0 & 0 & 0 & ak & \cdots & \cdots & \cdots
\end{pmatrix}
\]
Now this system is observable if and only if the system with $a = 0$ is observable. Thus we can construct the observability matrix as

\[
\begin{pmatrix}
c_1 & c_2 & c_3 & \cdots & c_{k+1} \\
0 & c_1 & 2c_2 & \cdots & kc_k \\
0 & 0 & 2c_1 & \cdots & (k-1)kc_{k-1} \\
0 & \cdots & k!c_1
\end{pmatrix}.
\]

Thus the system is observable if and only if $c_1 \neq 0$. We state this as a proposition.

**Proposition 3.6.** If $n = 2$ and $k > 0$ and the matrix $A$ is a single Jordan block then the system $\dot{z} = H$, $y = cx$ is observable if and only if $c_1 \neq 0$.

### 3.3. The general case

Let the matrix $A$ be in Jordan canonical form and let the state vector $x$ be partitioned accordingly, i.e.,

\[
\frac{d}{dt} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} A_1 & & \\ & \ddots & \\ & & A_n \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}
\]

Let

\[x_i = (x_{i1}, \ldots, x_{im_i}).\]

We assume that the maximal degree of $p(x)$ is $k$. Let $M$ be the space spanned by the set of all monomials of degree less than or equal to $k$ in the variables of $x$. Define an action of the ring of polynomials in one variable, $R[x]$, on $M$ by defining for every monomial $m(x)$

\[Dm(x) = \sum_{i=1}^{m} \frac{\partial}{\partial x_{ij}} \frac{d}{dt} x_{ij}.\]

This is of course just the Carleman linearization. Now extend this action to any polynomial in $D$. Thus with this action $M$ becomes an $R[x]$ module.

Define

\[M_i = \text{span}\{m(x_i) : m(x_i) \text{ a monomial of degree less than or equal to } k \text{ in the variables } x_i\}\]

**Lemma 3.7.** The space $M_i$ is a submodule of $M$.

Define

\[M_i^j = \text{span}\{m(x_i) \in M_i : \text{degree}(m(x_i)) = j\}.\]

Again we have that $M_i^j$ is a submodule of $M$ and in fact $M_j = \bigoplus M_i^j$ where $\bigoplus$ indicates direct sum. Note that $M \neq \bigoplus M_i$ but $\bigoplus M_i$ is a direct summand of $M$. We see immediately that if $A$ has a Jordan block of dimension greater than 2 then the Carleman linearization cannot be observable. For the rest of this section we assume that the Jordan form of $A$ consists of blocks of dimension 1 and 2 and we further assume that 0 occurs as an eigenvalue in at most one Jordan block.

Now let

\[m(x) = (x_1^{k_1} \cdots x_{r_1}^{k_1})(x_{11}^{k_{11}} x_{12}^{k_{12}}) \cdots (x_{s_1}^{k_{s_1}} x_{s_2}^{k_{s_2}})\]
where $Dx_i = \lambda_i x_i$, $Dx_{i1} = \lambda_{i1} x_{i1} + x_{i2}$ and $Dx_{i2} = \lambda_i x_{i2}$. Now we have

$$Dm(x) = \left( \sum_{i=1}^{r} k_i \lambda_i m(x) \right) + \left( \sum_{i=1}^{s} \lambda_{i1} k_{i1} m(x) \right) + \left( \sum_{i=1}^{s} \lambda_{i2} k_{i2} m(x) \right) + \sum_{i=1}^{s} k_{i1} x_{i1}^{k_{i1}} x_{i2}^{k_{i2}} x_{s_1}^{k_{s_1}} x_{s_2}^{k_{s_2}}$$

Thus we have the following theorem.

**Theorem 3.9.** If the eigenvalues of the matrix $A$ are linearly independent over the rational numbers then the system $\dot{z} = Hz$, $y = cz$ is observable for all $c$ except for a set of $c$ algebraic co-dimension 1.
Attacking the question of observability in this manner is rather delicate. Linearizing spaces are built up and then cut down. This is close to the manner of realization. A realization is constructed and then is reduced by factoring out the unobservable subspace. Much of the time it is difficult to find the unobservable state explicitly. There is another way to attack the problem.

**Theorem 3.10.** Let $M$ be the minimal $R[x]$-module that contains the tuple of monomials found in $p(x)$. This module linearizes the system $\dot{x} = Ax, \ y = p(x)$ and is observable if and only if $\dot{x} = Ax, \ y = p(x)$ is observable.

The theorem is basically a tautology. The problem with applying this theorem is that it is difficult to construct $M$.

**References**


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