Future asymptotics of vacuum Bianchi type

$V_{I_0}$ solutions

J. M. Heinzle and H. Ringström

REPORT No. 29, 2008/2009, fall
ISSN 1103-467X
ISRN IML-R- 29-08/09- SE+fall
Future asymptotics of vacuum Bianchi type VI$_0$ solutions

J. Mark Heinzle$^*$
Gravitational Physics, Faculty of Physics, University of Vienna, A-1090 Vienna, Austria
and
Mittag-Leffler Institute of the Royal Swedish Academy of Sciences
S-18260 Djursholm, Sweden

Hans Ringström$^†$
Department of Mathematics, Royal Institute of Technology, S-10044 Stockholm, Sweden

Abstract

In this paper, we present a thorough analysis of the future asymptotic dynamics of spatially homogeneous cosmological models of Bianchi type VI$_0$. Each of these models converges to a flat Kasner solution (Taub solution) for late times; we give detailed asymptotic expansions describing this convergence. In particular, we prove that the future asymptotics of Bianchi type VI$_0$ solutions cannot be approximated in any way by Bianchi type II solutions, which is in contrast to Bianchi type VIII and IX models (in the direction toward the singularity). The paper contains an extensive introduction where we put the results into a broader context. The core of these considerations consists in the fact that there exist regions in the phase space of Bianchi type VIII models where solutions can be approximated, to a high degree of accuracy, by type VI$_0$ solutions. The behavior of solutions in these regions is essential for the question of ‘locality’, i.e., whether particle horizons form or not. Since Bianchi type VIII models are conjectured to be important role models for generic cosmological singularities, our understanding of Bianchi type VI$_0$ dynamics might thus be crucial to help to shed some light on the important question of whether to expect generic singularities to be local or not.

$^*$Electronic address: mark.heinzle@univie.ac.at
$^†$Electronic address: hansr@kth.se
1 Introduction

In order to put the results of the present paper into a broader context, let us begin by quoting a passage from Misner [11] concerning Robertson-Walker spacetimes:

...if the 3°K background radiation were last scattered at a redshift $z = 7$, then the radiation coming to us from two directions in the sky separated by more than about $30^\circ$ was last scattered by regions of plasma whose prior histories had no causal relationship. [...] Robertson-Walker models therefore give no insight into why the observed microwave radiation from widely different angles in the sky has very precisely ($\lesssim 0.2\%$) the same temperature.

To resolve this 'horizon problem', Misner suggested to abandon Robertson-Walker models and to consider spacetimes of Bianchi type IX instead, which were conjectured by Belinskii, Khalatnikov, and Lifshitz [1, 10] and Misner and Chitre [4, 11] to be the paradigm in the understanding of spacetimes with 'generic singularities'. Misner argued in [11] that Bianchi type IX models might exhibit a behavior toward the singularity which is quite different from that of Robertson-Walker models. Based on the absence of particle horizons in the $x$-direction in the flat Kasner solution (Taub solution)

$$-dt^2 + t^2 dx^2 + dy^2 + dz^2$$

(1)
on $(0, \infty) \times T^3$, and the expectation that (generic) type IX solutions are recurrently well approximated by this solution (and its equivalent representations arising from permuting the axes), Misner conjectured that particle horizons should be absent in (generic) Bianchi type IX solutions and thus proposed a remedy for the horizon problem.

Our perspectives on the initial singularity of the Universe have changed quite drastically in the last forty years. In particular, the introduction of the concept of inflation, which is a formidable contestant to explain the causal structure of the early Universe, has brought about a return to Friedmann-Robertson-Walker models. Nevertheless, we consider the question of the causal structure close to the singularity of more general spacetimes to be still one of fundamental importance, the main reason being that it plays a crucial role in the study of singularities in the spatially inhomogeneous setting—we will explain this in more detail below.

**Particle horizons, PDE perspectives and heuristics.** In order to illustrate that the question of whether particle horizons form or not can be posed in a meaningful way in general classes of spacetimes, let us consider the maximal globally hyperbolic development, say $(M, g)$, of some initial data, where $M = (t_-, t_+) \times \Sigma$ and $\Sigma$ is the initial manifold. Let us, furthermore, assume that $\partial_t$ is future oriented timelike with respect to $g$, that $\{t\} \times \Sigma$ are spacelike Cauchy hypersurfaces for $t \in (t_-, t_+)$, and that all causal geodesics are incomplete to the past. Then $t = t_-$ can be thought of as a singularity. Let $\gamma$ be an inextendible causal curve in $(M, g)$, and let $\mathcal{A}_t$ be the subset of $\Sigma$ corresponding to the intersection $J^+(\gamma) \cap \{\{t\} \times \Sigma\}$. Note that it is necessary to control the initial data induced on this set in order to predict the behavior of the solution along the causal curve $\gamma$ all the way to the singularity. Various definitions of locality, phrased in terms of the properties of $\mathcal{A}_t$ as $t \to t_-$, are now conceivable. If $\mathcal{A}_t = \Sigma$ for all $t$ and all $\gamma$, then, clearly, it is reasonable to say that the singularity is not local and that particle horizons do not form. Nevertheless, to only use this as a criterion is somewhat too crude; in the case of (1) for instance, $\mathcal{A}_t$ is never all of $T^3$, regardless of the choice of curve. On the other hand, in that case, $\mathcal{A}_t$ has non-trivial topology, and it is not contained in a subset of $T^3$ homeomorphic to a 3-ball (cf. the notion of late time observers being oblivious to topology, introduced in [17] in the context of cosmological solutions with accelerated expansion). This is an indication that, even though particle horizons form in the sense that $\mathcal{A}_t \neq T^3$, the formation is partial. If one considers (1) as a metric on $(0, \infty) \times \mathbb{R}^3$, then $\mathcal{A}_t$ is unbounded, and $\mathbb{R}^3 \setminus \mathcal{A}_t$ has non-trivial topology, again signalling a partial absence of locality. The above discussion indicates that various definitions are possible, but we will not attempt to give a formal definition in the above degree of generality. Moreover, we will, in practice, assume that we have some preferred geometric structures (e.g., in the case of spatial homogeneity, a constant mean curvature foliation and a Gaussian time coordinate) which allow us, in an unambiguous way, to fix...
a Riemannian metric (or, possibly, a family of equivalent metrics) on $\Sigma$, with respect to which we can measure the size of the set $A_t$. If the set $A_t$ converges to a point for every inextendible causal curve, as $t \rightarrow t_-$, one could then define the singularity to be local. This is the definition we will use below.

The singularity theorems of Hawking and Penrose predict that there are, generically, singularities in the sense of causal geodesic incompleteness. However, these theorems do not provide any information concerning the formation of particle horizons, nor on the issue of curvature blow up; for example, is the Kretschmann scalar (i.e., the Riemann curvature tensor contracted with itself) unbounded in the incomplete directions of causal geodesics? A weaker question to ask would be: Does strong cosmic censorship hold? To answer such questions, it seems necessary to carry out a detailed analysis, which would entail, in all probability, considering Einstein’s equations from a PDE point of view. When taking the PDE perspective, the causal structure is very important. As an illustration, it is instructive to consider the proof of curvature blow up in the direction towards the singularity in spacetimes with $T^3$-Gowdy symmetry. In that case, the analysis is based on the fact that it is known a priori that particle horizons form, and it turns out to be crucial to consider regions of the spacetime that are as small as possible given the constraint that they still be large enough to make it possible to predict the behavior along a fixed causal curve going into the singularity (and that these regions shrink to zero in the approach to the singularity). For a more complete discussion, we refer the interested reader to [16], in particular the last section, and references cited therein. Needless to say, analyzing the causal structure will in general be part of the problem, i.e., it should not be expected to be possible to separate the problem of analyzing the causal structure from the problem of analyzing the asymptotics of solutions.

So far, the difficulty of the problem has prevented a rigorous investigation of the structure of singularities of generic spacetimes; however, there have been a great many heuristic studies, pioneered by the work of Belinskii, Khalatnikov, and Lifshitz [1, 2, 10] and Misner and Chitre [4, 11]. From these studies, a consistent picture emerges which describes generic singularities as spacelike, local, vacuum dominated, and oscillatory. In the Hamiltonian approach, the asymptotic behavior of the spacetime metric is described in terms of a ‘cosmological billiard’ motion in a (minisuper-)space bounded by infinitely high walls (‘billiard’), see Damour, Henneaux, and Nicolai [5, 6] and references therein. In the dual dynamical systems approach, the asymptotic dynamics are represented by a system of ordinary differential equations, one for each spatial point, on the ‘silent boundary’, see [7, 18] and references therein. It is important to note, however, that these heuristic approaches to generic singularities crucially rely on the assumption of asymptotic silence and locality: The assumption that particle horizons form guarantees localization in the spatial directions, which suggests that spatial derivatives in the PDEs are (generically) dominated by temporal derivatives, and the equations result in an asymptotic system of ODEs [18].

In this manner, these heuristic approaches to generic singularities consistently suggest that the asymptotic dynamics of generic spacetimes are intimately connected with the asymptotic dynamics of spatially homogeneous vacuum solutions (since these are described by ODEs that coincide with the asymptotic ODEs of generic cosmologies); of particular relevance are the oscillatory asymptotic dynamics of the Bianchi models of type VI$_{\pm 1/9}$, VIII, and IX. It is therefore natural (and essential) to consider these spatially homogeneous models and to investigate whether the singularities these models form are local or not. (Note that if these singularities were not local, our heuristic conceptions of generic spacetimes would be utterly inconsistent.)

**Bianchi class A.** We will here restrict our attention to the Bianchi class A vacuum spacetimes. These spacetimes can be defined as the maximal globally hyperbolic developments of left invariant vacuum initial data on unimodular Lie groups. The corresponding metrics can be written in the form (4), where the different Bianchi class A types are characterized by the values of $a_{\alpha}$, $\alpha = 1, 2, 3$, given by Table 1. Let us remind the reader that the Bianchi type IX solutions recollapse, i.e., they are past and future causally geodesically incomplete, and thus have both a future and a past singularity. However, all the other Bianchi class A solutions (except Minkowski space and quotients thereof, which we ignore from now on since they are non-singular) have the property that they are future causally geodesically complete and past causally geodesically incomplete. Turning to
the question of whether there is localization of the causal structure or not, it is natural to start
by considering the solutions with an additional local rotational symmetry (LRS). Only Bianchi
type I, II, VII₀, VIII, and IX admit an additional such symmetry (in the case of Bianchi type I,
an example of an LRS metric is given by (1)). The corresponding solutions do not have a local
singularity in the above sense (at least not w.r.t. the natural foliation given by the hypersurfaces
of spatial homogeneity). However, they are special in many ways: They are non-generic in their
respective classes, and they can be extended; in particular, they constitute examples of singularities
that are not curvature singularities; see, e.g., [12]. Excluding these examples, the remaining space-
times of Bianchi type I, II, VI₀, and VII₀ have local singularities. What remains to be considered
is thus solutions of Bianchi type VIII and IX that are not locally rotationally symmetric.

For the sake of definiteness, let us consider the past singularity, and let us assume that the met-
ric (4b) is defined on \((t_-, t_+) \times G\), where \(G\) is the unimodular Lie group under consideration
and \(t = t_-\) corresponds to the past singularity. Particle horizons form and the singularity is local (with
respect to the canonical constant mean curvature foliation) if and only if

\[
\sum_{i=1}^{3} \int_{t_0}^{t_+} \frac{1}{\sqrt{g_{ii}}} dt < \infty
\]

for some \(t_0 \in (t_-, t_+)\), where the \(g_{ii}\) are given in (4b). In the case of Bianchi type VIII and IX,
this condition can be reformulated to

\[
\int_{t_0}^{t_+} \left( \sqrt{|n_1n_2|} + \sqrt{|n_2n_3|} + \sqrt{|n_3n_1|} \right) dt < \infty,
\]

(2)

cf. (6) below. When analyzing the asymptotics, it is convenient to Hubble-normalize the variables,
cf. (7), i.e., one divides the traceless part of the second fundamental form and the variables \(n_\alpha,\)
\(\alpha = 1, 2, 3\), by the Hubble scalar (which is minus one third of the mean curvature). We will
denote the thus normalized traceless part of the second fundamental form by \((\Sigma_1, \Sigma_2, \Sigma_3)\) and
the normalized version of the \(n_\alpha\) by \(N_\alpha\). Furthermore, it is convenient to introduce a new time
coordinate by carrying out an analogous normalization, cf. (8). These variables were introduced in
[20], and describe the essential dynamics of all the Bianchi class A types; the \(N_\alpha\) should be
zero or non-zero and have signs according to Table 1. In this picture, the Kasner solutions, i.e.,
the Bianchi type I solutions, are given by demanding that all the \(N_\alpha\) be zero, and, due to the
Hamiltonian constraint, they constitute a circle of fixed points. On this circle, there are three
points that are referred to as the special points or Taub points; they are the ones corresponding
to the flat Kasner solution (1). Formulating the condition (2) in terms of the Hubble-normalized
variables and Hubble-normalized time, one obtains, in the case of Bianchi type VIII and IX, the
condition

\[
\int_{-\infty}^{\tau_0} \left| N_1 N_2 \right| + \left| N_2 N_3 \right| + \left| N_3 N_1 \right| d\tau < \infty.
\]

(3)

The paper [14] contains a proof of the statement that the integrand in (3), say \(I(\tau)\), converges
to zero as \(\tau \to -\infty\) for solutions of Bianchi type IX. Although the argument has been slightly
simplified in [8], the proof of the fact that \(I\) converges to zero is rather intricate, and proving (3) can
reasonably be expected to be much more difficult. In view of these difficulties, it seems reasonable
to turn to numerical methods. Starting with seemingly arbitrary initial data, numerically solving
the corresponding ODE indicates that \(I\) should converge to zero exponentially as \(\tau \to -\infty\),
and that the \(\alpha\)-limit set should coincide with the part of the boundary of the phase space where the
integrand equals zero (let us call this set the attractor and denote it by \(\mathcal{A}\)). Consequently, there
does not seem to be a problem. However, one can prove that there are no solutions with this
behavior; if the \(\alpha\)-limit set coincides with \(\mathcal{A}\), then \(I\) cannot converge to zero exponentially. This
indicates that the behavior of the solutions is quite subtle and that if one uses numerical techniques,
one has to be very careful. One of the reasons why the behavior is so subtle is that the solution
is expected to return an infinite number of times to regions of the phase space which are close to
the special points that correspond to solutions of the form (1), cf. the above arguments by Misner.
In fact, the expectation is that the solution will spend most of its time close to such points. Since
I(τ) need not decay in the vicinity of these points (let alone decay exponentially), it is not so clear that (3) holds.

**Bianchi type VIII.** Let us turn to Bianchi type VIII. Concerning this class of solutions, it is not even known whether I converges to zero or not. Furthermore, the behavior in the case of Bianchi type VIII is in some respects worse. For Bianchi type IX, it is known not only that I converges to zero, but also that the convergence is almost monotone in the sense that for any ε > 0, there is a δ > 0 such that if I(τ) ≤ δ at τ = τ₀, then I(τ) ≤ ε for all τ ≤ τ₀, cf. [14, Corollary 15.3, p. 471]. Proposition 6.2 of [15, p. 3802] proves that the analogous statement cannot hold in the case of Bianchi type VIII. The results of [13] imply that there is a sequence of times such that I converges to zero along it, but due to the absence of almost monotone convergence, this statement does not allow much in the way of conclusions. Proposition 6.2 of [15] is based on considerations of the future asymptotics of Bianchi type VI₀ solutions. Bianchi type VI₀ solutions all have an ω-limit set consisting of a fixed point in A; in fact, the ω-limit set always coincides with a special point, see also Lemma 3.1. Perturbing such a solution into the Bianchi type VIII class and going backward in time consequently results in a proof of the fact that there cannot be almost monotone convergence to the attractor in the case of Bianchi type VIII. This behavior should be contrasted with that of a Bianchi type VII₀ solution for which the ω-limit set is not contained in A. Given a Bianchi VII₀ solution, with N₁, N₂ > 0 say, the ω-limit set is a point on a line of fixed points in the Bianchi type VII₀ class (each element of which, incidentally, corresponds to a solution of the form (1)). In other words, the integrand I, considered for a Bianchi VII₀ solution, converges to a non-zero number to the future. Perturbing such a solution into the Bianchi type IX class and considering the behavior toward the past, the conclusion is that the integrand remains essentially constant. In the case of Bianchi type IX, the regions close to the lines of fixed points of Bianchi type VII₀ are the worst ones in that, there, it is hardest to prove decay of I going backward in time. Nevertheless, the above observation concerning Bianchi type VII₀ indicates that not that much growth can occur. In some sense, this is the basis for the proof of the fact that I converges to zero in the direction toward the singularity given in [14].

To recapitulate the above discussion, it is clear that if one wants to prove that (3) holds, i.e., that the singularity is local, the solutions in Bianchi class A that are most problematic are the ones of type VIII; at this stage there does not even seem to be any strong reason to conjecture that I converges to zero in that case. The problematic regions of the phase space (i.e., the regions where I can grow from being very small to being of a definite size) are close to the special points on the Kasner circle where the solution can be approximated by the future asymptotics of a Bianchi type VI₀ solution. A first step, admittedly a small one, in analyzing the problematic region consequently consists of carrying out a detailed analysis of the future asymptotic behavior in Bianchi type VI₀. This is the theme of the present paper.

**Bianchi type VI₀.** Computing the ω-limit set of Bianchi type VI₀ solutions is quite simple, see, e.g., Lemma 3.1, but for the purposes outlined above it is necessary to consider the behavior in greater detail. In [20], the function

\[ Z_{-1} = \frac{4\Sigma^2_{-1} + (N_2 + N_3)^2}{-N_2N_3} \]

was introduced; in the present paper we use an analogous function, ζ, see (20). (The variables in Z_{-1} are the variables of [20] which are somewhat different from the ones used in the present paper.) In [20], the source of this function is quoted as Bogoyavlensky, cf. [3, p. 63] (where the notation F_i is used). This function is monotone for a Bianchi type VI₀ solution, but on the ω-limit set, both the numerator and the denominator equal zero. Consequently, our knowledge concerning the ω-limit set does not allow us to draw any conclusions concerning the limit of Z_{-1}. Nevertheless, it is of great interest to know what this limit is. The reason for this is that Z_{-1} serves as a measure of the distance from Bianchi type II behavior: For a Bianchi type II solution, one of the variables N₂, N₃ is different from zero (and the other is zero), hence Z_{-1} = ∞. Therefore, Bianchi type VI₀ solutions (or type VIII solutions for that matter) are close to a type II solution (and can be approximated by such a solution) if and only if Z_{-1} is large. In the present paper, we prove that Z_{-1} converges to zero as τ → ∞ for every Bianchi type VI₀ solution, see Corollary 4.2. Consequently, even though
the ω-limit set is a special point on the Kasner circle, the behavior of a type VI₀ solution is very far that of type II solutions and cannot be approximated by (a sequence of) type II solutions. (This is in contrast to Bianchi type IX, where the asymptotics is in fact largely governed by sequences of type II solutions). The function $Z_{-1}$ serves as a quantitative measure of this discrepancy; in this paper, see Section 5, we illustrate in detail in which sense type VI₀ dynamics differ from type II dynamics. It seems reasonable to claim that $Z_{-1}$ is also an important quantity to keep track of when studying Bianchi type VIII solutions in the direction of the singularity, since the problematic behavior occurs when the solution is well approximated by a Bianchi type VI₀ solution for which $Z_{-1}$ goes from zero to some finite positive value. Therefore, by analyzing the asymptotics of Bianchi type VI₀ in great detail, we hope to develop some understanding for the problems that can arise in the case of Bianchi type VIII.

Outline. This paper is organized as follows: In Section 2, we introduce the equations for Bianchi class A models in the Hubble-normalized dynamical systems approach. In Section 3, we specialize to Bianchi type VI₀; we use adapted variables to represent the phase space $B_{VI₀}$ of type VI₀ models and we introduce the monotone function $\zeta (\sim Z_{-1})$, which is central to our analysis. Following these preliminary observations, we present a detailed and thorough analysis of the future asymptotics of Bianchi type VI₀ vacuum models in Section 4. The analysis is not quite straightforward; it is necessary to use methods tailored to the problem, since standard techniques from the theory of dynamical system fail due to the fact that the special point on the Kasner circle is a center fixed point. The main result of the paper is presented as Proposition 4.5, which gives the details of the future asymptotics. Finally, in Section 5, we illustrate the differences between type VI₀ dynamics and type II dynamics. We use units such that $c = 1 = 8\pi G$, where $c$ is the speed of light and $G$ the gravitational constant.

2 Basic equations

For a spatially homogeneous vacuum spacetime of Bianchi class A, there exists a symmetry-adapted (co-)frame $\{\hat{\omega}^1, \hat{\omega}^2, \hat{\omega}^3\}$ satisfying

$$d\hat{\omega}^1 = -\hat{n}_1 \hat{\omega}^2 \wedge \hat{\omega}^3, \quad d\hat{\omega}^2 = -\hat{n}_2 \hat{\omega}^3 \wedge \hat{\omega}^1, \quad d\hat{\omega}^3 = -\hat{n}_3 \hat{\omega}^1 \wedge \hat{\omega}^2$$

(4a)

such that the metric takes the form

$$\hat{g}^4 = -dt \otimes dt + g_{11}(t) \hat{\omega}^1 \otimes \hat{\omega}^1 + g_{22}(t) \hat{\omega}^2 \otimes \hat{\omega}^2 + g_{33}(t) \hat{\omega}^3 \otimes \hat{\omega}^3.$$  

(4b)

The different Bianchi types are characterized by different spatial structure constants $\hat{n}_1, \hat{n}_2, \hat{n}_3$; in Table 1 we list all Bianchi class A types. In the case of Bianchi type VI₀, we can, without loss of generality (by permuting the axes), settle for $\hat{n}_1 = +1, \hat{n}_2 = -1, \hat{n}_3 = 0$. For a representation of (4) in terms of a coordinate basis, see [19].

Let $k_{\alpha\beta}$ denote the second fundamental form, associated with (4), of the SH hypersurfaces $t = \text{const}$ and define

$$H = -\frac{1}{3} \text{tr} k \quad \text{and} \quad \sigma^\alpha_{\beta} = -k^\alpha_{\beta} + \frac{1}{3} \text{tr} k \delta^\alpha_{\beta} = \text{diag}(\sigma_1, \sigma_2, \sigma_3).$$

(5)

The quantity $H$ (‘Hubble scalar’) is related to the expansion of the normal congruence of the SH hypersurfaces, i.e., $d\sqrt{\text{det} g}/dt = 3H/\sqrt{\text{det} g}$. The tensor $\sigma_{\alpha\beta}$ is tracefree, i.e., $\sigma_1 + \sigma_2 + \sigma_3 = 0$; it can be interpreted as the shear. Furthermore, define

$$n_1(t) := \hat{n}_1 \frac{g_{11}}{\sqrt{\text{det} g}}, \quad n_2(t) := \hat{n}_2 \frac{g_{22}}{\sqrt{\text{det} g}}, \quad n_3(t) := \hat{n}_3 \frac{g_{33}}{\sqrt{\text{det} g}}.$$  

(6)

Given the ‘metric variables’, i.e., $(g_{\alpha\beta}, k_{\alpha\beta})$, one can construct the ‘orthonormal frame variables’, i.e., $(H, \sigma_1, \sigma_2, \sigma_3, n_1, n_2, n_3)$, and given the orthonormal frame variables, one can construct the metric variables (though this construction requires carrying out integrations of the shear variables for some of the Bianchi types). It is always understood that $\sigma_1 + \sigma_2 + \sigma_3 = 0$. 


In the Hubble-normalized dynamical systems approach we define dimensionless orthonormal frame variables according to
\[
\left(\Sigma_1, \Sigma_2, \Sigma_3, N_1, N_2, N_3\right) = \frac{1}{H} \left(\sigma_1, \sigma_2, \sigma_3, n_1, n_2, n_3\right).
\] (7)

In addition we introduce a new dimensionless time variable \(\tau\), where
\[
\frac{d}{d\tau} = H^{-1} \frac{d}{dt};
\] (8)
henceforth, a prime \(\prime\) denotes the derivative w.r.t. \(\tau\).

**Remark.** The variables (7) are well-defined except in the case of Bianchi type IX. The reason is that the Gauss constraint
\[
6H^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \frac{1}{2}(n_1^2 + n_2^2 + n_3^2) - n_1n_2 - n_2n_3 - n_3n_1
\]
guarantees that \(H\) remains positive if it is positive initially (with the exception of Bianchi type IX). In the case of Bianchi type IX, the variables cover half of the spacetime. In particular, Bianchi models (except those of type IX) that are expanding initially are expanding for all times.

The Einstein equations result in a system of differential equations for the Hubble-normalized variables; this system can be written, cf. [20],
\[
\Sigma_\alpha' = -2(1 - \Sigma^2)\Sigma_\alpha - 3S_\alpha, \quad \alpha = 1, 2, 3
\] (9a)
\[
N_\alpha' = 2(\Sigma^2 + \Sigma_\alpha) N_\alpha, \quad \alpha = 1, 2, 3 \quad (\text{no sum over } \alpha),
\] (9b)
where
\[
\Sigma^2 = \frac{1}{2}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2),
\] (10a)
\[
3S_\alpha = \frac{1}{2} \left[N_\alpha(2N_\alpha - N_\beta - N_\gamma) - (N_\beta - N_\gamma)^2\right], \quad (\alpha\beta\gamma) \in \{(123), (231), (312)\}.
\] (10b)

Apart from the trivial constraint \(\Sigma_1 + \Sigma_2 + \Sigma_3 = 0\), there exists the Gauss constraint
\[
\Sigma^2 + \frac{1}{6} \left[N_1^2 + N_2^2 + N_3^2 - 2N_1N_2 - 2N_2N_3 - 2N_3N_1\right] = 1.
\] (11)

Accordingly, the reduced state space is given as the space of all \((N_1, N_2, N_3, \Sigma_1, \Sigma_2, \Sigma_3)\) such that (11) holds. The existence interval for solutions to (9)–(11) is \((-\infty, \infty)\), with the exception of Bianchi type IX, in which case the existence interval is of the form \((-\infty, \tau_0)\) for some \(\tau_0 < \infty\), see, e.g., [14]. Since \(\Sigma_1 + \Sigma_2 + \Sigma_3 = 0\), the dimensionless state space of the Bianchi type VI\(_0\) vacuum models is 3-dimensional.

From these equations, the metric (4b) can be reconstructed by carrying out appropriate integrations. The pertinent equations are \(g'_{ii} = 2(1 + \Sigma_i)g_{ii}\) (with \(i = 1, 2, 3\)) and the equation \(H' = -(1 + 2\Sigma^2)H\) to recover cosmological time via (8).

<table>
<thead>
<tr>
<th>Bianchi type</th>
<th>(\hat{n}_\alpha)</th>
<th>(\hat{n}_\beta)</th>
<th>(\hat{n}_\gamma)</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>II</td>
<td>+</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>VI(_0)</td>
<td>+</td>
<td>−</td>
<td>0</td>
</tr>
<tr>
<td>VII(_0)</td>
<td>+</td>
<td>+</td>
<td>0</td>
</tr>
<tr>
<td>VIII</td>
<td>+</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>IX</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

Table 1: The Bianchi class A types are characterized by different signs of the structure constants \((\hat{n}_\alpha, \hat{n}_\beta, \hat{n}_\gamma)\), where \((\alpha\beta\gamma)\) is any permutation of \((123)\). In addition to the above representations, there are equivalent representations associated with an overall change of sign of the structure constants; e.g., another type VI\(_0\) representation is \((- + 0)\).
3 Adapting to Bianchi type VI₀

In Bianchi type VI₀, the permutation symmetry of the three spatial axes (exhibited by, e.g., type I and type IX models) is broken; the third axis is singled out. It is suggestive to globally solve \( \Sigma_1 + \Sigma_2 + \Sigma_3 = 0 \) by introducing variables according to

\[
\Sigma_+ = \frac{\Sigma_1 + \Sigma_2}{2} = \frac{\Sigma_3}{2}, \quad \Sigma_- = \frac{\Sigma_1 - \Sigma_2}{2\sqrt{3}},
\]

which yields \( \Sigma^2 = \Sigma^2_+ + \Sigma^2_- \). Likewise, we adapt to the constraint (11) by defining

\[
N_+ = \frac{N_1 + N_2}{2\sqrt{3}}, \quad N_- = \frac{N_1 - N_2}{2\sqrt{3}}.
\]

In these variables, the constraint (11) reads

\[
\Sigma^2_+ + \Sigma^2_- + N^2 = 1.
\]

The constraint (14) can be employed to globally solve for \( N_- \); since \( N_- \) is necessarily positive, we simply obtain

\[
N_- = \sqrt{1 - \Sigma^2_+ - \Sigma^2_-}.
\]

The range of the variable \( N_+ \) is restricted by the conditions \( N_1 > 0 \) and \( N_2 < 0 \); we obtain \(-N_- < N_+ < N_-\). Accordingly, in terms of the variables \((\Sigma_+, \Sigma_-, N_+), \) the Bianchi type VI₀ vacuum state space becomes an open ball, i.e.,

\[
B_{VI₀} = \left\{ (\Sigma_+, \Sigma_-, N_+) \mid \Sigma^2_+ + \Sigma^2_- + N^2 < 1 \right\}.
\]

The system of equations (9) takes the form

\[
\begin{align*}
\Sigma_+ &= -2(1 + \Sigma_+) (1 - \Sigma^2_+ - \Sigma^2_-) \\
\Sigma_- &= -2 \left[ \Sigma_-(1 - \Sigma^2_+ - \Sigma^2_-) - \sqrt{3} N_+ \sqrt{1 - \Sigma^2_+ - \Sigma^2_-} \right] \\
N_+ &= 2 \left[ N_+ \Sigma_+ + N_-(\Sigma^2_+ + \Sigma^2_-) - \sqrt{3} \Sigma_- \sqrt{1 - \Sigma^2_+ - \Sigma^2_-} \right]
\end{align*}
\]

on \( B_{VI₀} \). This system possesses a regular extension to the boundary \( \partial B_{VI₀} \), which is the 2-sphere \( \Sigma^2_+ + \Sigma^2_- + N^2 = 1 \); see Figure 1.

The induced system on \( \partial B_{VI₀} \) is associated with Bianchi types I and II. The equator of \( \partial B_{VI₀} \), i.e.,

\[
K^2 = \left\{ (\Sigma_+, \Sigma_-, N_+) \in \partial B_{VI₀} \mid N_+ = 0, \Sigma^2_+ + \Sigma^2_- = 1 \right\},
\]

is a circle of fixed points, the Kasner circle. Each fixed point on \( K^2 \) corresponds to a Kasner metric (Bianchi type I vacuum metric) [19]. On \( K^2 \) there exist points that are associated with locally rotationally symmetric solutions: \( Q_1, Q_2, Q_3 \) and \( T_1, T_2, T_3 \), see Figure 2. The ‘Taub point’ \( T_3 \) is of particular importance for our purposes; it is associated with the flat Taub solution (Bianchi type I representation of a part of Minkowski spacetime, cf. (1)). We refer to [19] for details.

The two hemispheres of \( \partial B_{VI₀} \) correspond to two different (but equivalent) representations of the Bianchi type II vacuum state space. The northern (+) and southern (−) hemispheres, \( B_\pm = \left\{ (\Sigma_+, \Sigma_-, N_+) \in \partial B_{VI₀} \mid \pm N_+ > 0 \right\}, \) correspond to \( N_1 > 0, N_2 = 0 \) (and \( N_3 = 0 \)), and \( N_1 = 0, N_2 < 0 \) (and \( N_3 = 0 \)) respectively. The orbits of the dynamical system (17) on \( B_+ \) and on \( B_- \) correspond to Bianchi type II vacuum solutions. These orbits form a family of straight lines when projected onto \((\Sigma_+, \Sigma_-)\)-space, see Figure 2.

**Lemma 3.1.** Let \( \gamma \) be an orbit in \( B_{VI₀} \). The \( \alpha \)-limit set of \( \gamma \) is one of the Kasner fixed points with \( \Sigma_+ > \frac{1}{3} \). (Conversely, each of these fixed points is the \( \alpha \)-limit for a one-parameter family of orbits.) The \( \omega \)-limit set of \( \gamma \) is the special point (Taub point) \( T_3 \) (where \( \Sigma_+ = -1, \Sigma_- = 0, N_+ = 0 \)).
The state space $B_{VI_0}$. The equator of $\partial B_{VI_0}$, $N_+ = 0$, corresponds to the Kasner circle. The northern hemisphere, $B_+$, which is given by $N_+ > 0$, is a representation of the type II state space; the same is true for the southern hemisphere, $B_-$, given by $N_+ < 0$. The orbit through the center is $\Gamma$.

Figure 2: Projection of the type II orbits of the hemispheres $B_+$ and $B_-$ onto $(\Sigma_+, \Sigma_-)$-space. The horizontal axis is the $\Sigma_+$ axis, the vertical axis is the $\Sigma_-$ axis.

**Proof.** Eq. (17a) implies that $\Sigma_+$ is a strictly monotonically decreasing function on $B_{VI_0} \setminus K^\circ$. Application of the monotonicity principle, see, e.g., [19], yields the possible $\alpha$- and $\omega$-limit sets of orbits in $B_{VI_0}$: $\alpha$-limit points must be contained on $K^\circ \setminus \{T_3\}$, $\omega$-limit points on $K^\circ \setminus \{Q_3\}$. The local analysis of the fixed points$^1$ on $K^\circ$ restricts the possibilities further and leads directly to the statement of lemma.

There exists one orbit of (17) that is central to our considerations:

$$\Gamma : \begin{cases} N_+ = 0, \\
\Sigma_- = 0. \end{cases}$$

(19)

Along this orbit, the system (17) reduces to the equation $\Sigma'_+ = -2(1 + \Sigma_+)^2(1 - \Sigma_+)$, which can be solved implicitly.

$^1$Strictly speaking, the local analysis of the fixed points on $K^\circ$ has to be performed using a set of variables in which the dynamical system is $C^1$ on the closure of the state space; the system (9) is well suited.
Consider the function
\[
\zeta = -1 - \frac{(\Sigma_1 - \Sigma_2)^2 + (N_1 - N_2)^2}{4N_1N_2} = \frac{N_1^2 + \Sigma_-^2}{1 - \Sigma_+^2 - \Sigma_-^2 - N_+^2},
\]
which is non-negative on \(\mathcal{B}_{VI_0}\). Up to a constant multiple, this function is the same as \(Z_{-1}\) which was defined in [20] (a paper which quotes [3] as its source). The same references defined a marginally different monotone function, called \(Z_1\), in the case of Bianchi type \(VII_0\). This function was employed in [14, 8] to analyze Bianchi type \(VII_0\) asymptotic dynamics (which was in turn the cornerstone for the analysis of Bianchi type IX Mixmaster dynamics). In the following we use the function \(\zeta\) as a building block in the analysis of Bianchi type \(VI_0\) asymptotics.

The condition \(\zeta = 0\) defines the orbit \(\Gamma\). The condition \(\zeta = z\) for \(\mathbb{R} \ni z > 0\) represents (the surface of) a prolate spheroid in \(\mathcal{B}_{VI_0}\),
\[
\Sigma_+^2 + (1 + z^{-1}) \Sigma_-^2 + (1 + z^{-1}) N_+^2 = 1.
\]
This spheroid is embedded in the unit ball \(\mathcal{B}_{VI_0}\) in a characteristic way: First, the principal axis of the prolate spheroid coincides with the orbit \(\Gamma\). Second, let \(R_s\) denote the radius of the disc that is orthogonal to the principal axis and arises as the intersection of the spheroid \(\zeta = z\) with the plane \(\Sigma_+ = \text{const}\); likewise, let \(R_d = (1 - \Sigma_+^2)^{1/2}\) denote the radius of the disc that arises as the intersection of the unit ball \(\mathcal{B}_{VI_0}\) with the plane \(\Sigma_+ = \text{const}\). Then \(R_s/R_d = (1 + 1/z)^{-1/2}\), i.e., the surface of the spheroid \(\zeta = z\) is at a constant relative distance \(R_s/R_d\) from the central orbit (axis) \(\Gamma\).

While the function \(\zeta\) is zero along the orbit \(\Gamma\), \(\zeta\) is positive and strictly monotonically decreasing along every other orbit in \(\mathcal{B}_{VI_0}\). To see this we use (17) to compute
\[
\zeta' = -4 \frac{\Sigma_+ (1 + \Sigma_+)}{1 - \Sigma_+^2 - \Sigma_-^2 - N_+^2} \leq 0 \tag{22a}
\]
\[
\zeta''|_{\Sigma_- = 0} = 0 \tag{22b}
\]
\[
\zeta'''|_{\Sigma_- = 0} = -96 \frac{(1 + \Sigma_+)(1 - \Sigma_+^2)N_+^2}{1 - \Sigma_+^2 - N_+^2} < 0. \tag{22c}
\]

Remark. When calculating derivatives of \(\zeta\), it is convenient to use that
\[
(1 - \Sigma_+^2 - \Sigma_-^2 - N_+^2)' = 4(\Sigma_+^2 + \Sigma_-^2 + \Sigma_+)(1 - \Sigma_+^2 - \Sigma_-^2 - N_+^2),
\]
which follows straightforwardly from \(1 - \Sigma_+^2 - \Sigma_-^2 - N_+^2 = \frac{4}{9} N_1 N_2\), cf. (13) and (14), and from (9b).

The monotonicity of \(\zeta\) can alternatively be expressed as follows: If \(\zeta|_{\gamma(\tau_0)} = \zeta_0\) for an orbit \(\gamma(\tau)\) at \(\tau = \tau_0\), then the orbit is contained within the spheroid \(\zeta = \zeta_0\) for all \(\tau > \tau_0\). The surface \(\zeta = \zeta_0\) thus defines a ‘channel’ in which the orbit must be contained for all \(\tau > \tau_0\); this ‘channel’ directs the orbit to \(T_3\), cf. Lemma 3.1.

4 Future asymptotics of Bianchi type \(VI_0\) vacuum models

The analysis of Section 3 suggests to introduce adapted variables. We define the variable
\[
\hat{\Sigma}_+ = 1 + \Sigma_+ \tag{23}
\]
which is positive on \(\mathcal{B}_{VI_0}\); in addition, we introduce adapted ‘spheroidal coordinates’ \(\zeta\) and \(\theta\) instead of \(\Sigma_-\) and \(N_+\) via
\[
\Sigma_- = \sqrt{\frac{1 - \Sigma_+^2}{1 + \zeta^{-1}}} \cos \theta, \tag{24a}
\]
\[
N_+ = -\sqrt{\frac{1 - \Sigma_+^2}{1 + \zeta^{-1}}} \sin \theta; \tag{24b}
\]
note that 1 − Σ₂ = (2 − Σ₂)Σ₂. The variable transformation (Σ⁺, Σ⁻, N₂) → (Σ⁺, ζ, θ) is one-to-one on \( B_{VI₀} \setminus \Gamma \) when we disregard translations by integer multiples of 2π in θ. Surfaces ζ = const represent the prolate spheroids; Σ⁺ = const its cross sections (orthogonal to the principal axis Γ); the variable θ is an angular variable in these sections.

Another beneficial change concerns the time variable: We introduce a new time variable σ through

\[
\frac{d}{d\sigma} = \frac{1}{\Sigma^+} \frac{d}{d\tau}.
\]

By Lemma 3.1, for every orbit in \( B_{VI₀} \), \( \Sigma^+ \) goes to zero as \( \tau \to \infty \), where

\[
\tilde{\Sigma}^+_+ = -2(1 - \Sigma^+_2 - \Sigma^-_2)\Sigma^+_+,
\]

see (17a). Consequently, \( 1 - \Sigma^+_2 - \Sigma^-_2 \notin L^1(\tau_0, \infty) \) for every \( \tau_0 \in \mathbb{R} \). Since

\[
0 < 1 - \Sigma^+_2 - \Sigma^-_2 \leq 1 - \Sigma^+_2 = (2 + \tilde{\Sigma}^+_+)\Sigma^+_+ \leq 2\tilde{\Sigma}^+_+,
\]

we infer that \( \tilde{\Sigma}^+_+ \notin L^1(\tau_0, \infty) \) for every \( \tau_0 \in \mathbb{R} \). Since \( \tilde{\Sigma}^+_+ \) is positive and

\[
\sigma(\tau) = \sigma(\tau_0) + \int_{\tau_0}^{\tau} \tilde{\Sigma}^+_+(s) \, ds,
\]

we conclude that \( \tau \to \infty \) corresponds to \( \sigma \to \infty \).

Using \((\tilde{\Sigma}^+_+, \zeta, \theta)\) as variables and \( \sigma \) as ‘time’, the dynamical system (17) becomes

\[
\begin{align*}
\frac{d\tilde{\Sigma}^+_+}{d\sigma} &= -2\tilde{\Sigma}^+_+(2 - \tilde{\Sigma}^+_+) \left[ 1 + \zeta \sin^2 \theta \right] (1 + \zeta)^{-1}, \quad (26a) \\
\frac{d\theta}{d\sigma} &= 2\sqrt{3} \tilde{\Sigma}^+_+^{1/2} (2 - \tilde{\Sigma}^+_+)^{1/2} (1 + \zeta \sin^2 \theta)^{1/2} (1 + \zeta)^{-1/2} + \sin 2\theta, \quad (26b) \\
\frac{d\zeta}{d\sigma} &= -4\zeta \cos^2 \theta. \quad (26c)
\end{align*}
\]

This system (together with the exceptional orbit Γ) completely describes the Bianchi type VI₀ dynamics. In the following, we present a detailed analysis of the system (26) and the properties of its solutions.

**Lemma 4.1.** Consider a solution \((\tilde{\Sigma}^+_+(\sigma), \theta(\sigma), \zeta(\sigma))\) of the dynamical system (26) in \( B_{VI₀} \setminus \Gamma \). Let \( \sigma_0 \in \mathbb{R} \) and let

\[
g(\sigma) := \int_{\sigma_0}^{\sigma} \cos^2(\theta(s)) \, ds.
\]

Then there exists a \( \beta > 0 \) such that

\[
\frac{\sigma}{2} - \beta \leq g(\sigma) \leq \frac{\sigma}{2} + \beta
\]

for all \( \sigma \geq \sigma_0 \).

**Remark.** Since \( \log \zeta(\sigma)^{-1/4} = \log \zeta(\sigma_0)^{-1/4} + g(\sigma) \) by (26c), the inequality (27) gives a first estimate on the asymptotic behavior of \( \zeta(\sigma) \) as \( \sigma \to \infty \), see Corollary 4.2.

**Proof.** Due to the fact that \( \tilde{\Sigma}^+_+(\sigma) \to 0 \) as \( \sigma \to \infty \) and the fact that \( \zeta(\sigma) \) converges as \( \sigma \to \infty \), we can assume \( \sigma_0 \) to be large enough that there is a \( c_0 > 1 \) such that

\[
\frac{1}{c_0} \tilde{\Sigma}^+_+^{-1/2} = \frac{d\theta}{d\sigma} \leq c_0 \tilde{\Sigma}^+_+^{-1/2}
\]

for all \( \sigma \geq \sigma_0 \), cf. (26b). Using a simple trigonometric identity and changing variables leads to

\[
g(\sigma) = \frac{1}{2} \int_{\sigma_0}^{\sigma} \left[ 1 + \cos(2\theta(s)) \right] \, ds = \frac{1}{2}(\sigma - \sigma_0) + \frac{1}{2} \int_{\theta(\sigma_0)}^{\theta(\sigma)} \cos(2\theta) \left( \frac{d\theta}{d\sigma} \right)^{-1} \, d\theta.
\]


Inserting (28) we obtain
\[
\frac{1}{2c_0} \int_{\vartheta(\sigma_0)}^{\vartheta(\sigma)} \cos(2\theta) \tilde{\Sigma}_+^{1/2} \, d\theta \leq g(\sigma) - \frac{1}{2} (\sigma - \sigma_0) \leq \frac{c_0}{2} \int_{\vartheta(\sigma_0)}^{\vartheta(\sigma)} \cos(2\theta) \tilde{\Sigma}_+^{1/2} \, d\theta .
\] (30)

Here we consider \( \tilde{\Sigma}_+ \) to be a function of \( \vartheta \), where we have
\[
\frac{d\tilde{\Sigma}_+}{d\sigma} = \frac{d\tilde{\Sigma}_+}{d\vartheta} \left( \frac{d\vartheta}{d\sigma} \right)^{-1} < 0 ,
\]
assuming \( \sigma \) to be large enough, cf. (28). Let us estimate the integral in (30).

If \( 2\theta_n = -\pi/2 + 2n\pi \), with \( n \in \mathbb{N} \) sufficiently large, then
\[
\int_{\theta_n}^{	heta_{n+1}} \tilde{\Sigma}_+^{1/2} \cos(2\theta) \, d\theta \geq 0 .
\] (31)

If it were not for the factor \( \tilde{\Sigma}_+^{1/2} \), the integral would be zero. The reason for the inequality (31) is that the infimum of this factor over the subset of the interval of integration where the integrand is positive equals the supremum of this factor over the subset of the interval of integration where the integrand is negative. For similar reasons, if \( 2\bar{\theta}_n = \pi/2 + 2n\pi \), \( n \in \mathbb{N} \), then
\[
\int_{\bar{\theta}_n}^{ar{\theta}_{n+1}} \tilde{\Sigma}_+^{1/2} \cos(2\theta) \, d\theta \leq 0 .
\]

Therefore, since the integral
\[
\int_{\vartheta(\sigma_0)}^{\vartheta(\sigma)} \tilde{\Sigma}_+^{1/2} \cos(2\theta) \, d\theta
\]
can be written as a sum of non-negative integrals or as a sum of non-positive integrals, up to an error corresponding to, at worst, one interval at the beginning of the interval of integration and one interval at the end of the interval of integration, we conclude that
\[
\int_{\vartheta(\sigma_0)}^{\vartheta(\sigma)} \tilde{\Sigma}_+^{1/2} \cos(2\theta) \, d\theta = O\left( \tilde{\Sigma}_+^{1/2}(\sigma_0) \right) .
\]

Since this estimate is uniform in \( \sigma \), we are led to the conclusion that
\[
\int_{\vartheta(\sigma_0)}^{\vartheta(\sigma)} \tilde{\Sigma}_+^{1/2} \cos(2\theta) \, d\theta
\]
converges as \( \sigma \to \infty \), and that there is a constant, say \( A \), such that
\[
\int_{\vartheta(\sigma_0)}^{\vartheta(\sigma)} \tilde{\Sigma}_+^{1/2} \cos(2\theta) \, d\theta = A + O\left( \tilde{\Sigma}_+^{1/2}(\sigma) \right) .
\] (32)

In particular, the conclusions of the lemma follow. \( \square \)

**Corollary 4.2.** Along every orbit in \( B_{\text{VI}_0} \) we have \( \zeta \to 0 \) as \( \sigma \to \infty \) (or \( \tau \to \infty \)).

**Proof.** For the orbit \( \Gamma \) the statement is trivial. For every other orbit we use Lemma 4.1. Since \( \log \zeta(\sigma)^{-1/4} = \log \zeta(\sigma_0)^{-1/4} + g(\sigma) \), Lemma 4.1 implies that there exists a \( c_1 > 1 \) such that
\[
c_1^{-1} e^{-2\sigma} \leq \zeta(\sigma) \leq c_1 e^{-2\sigma}
\] (33)
for all \( \sigma \geq \sigma_0 \). \( \square \)
Lemma 4.1, in combination with the fact that \( \tilde{\Sigma}_+ \to 0 \) \( (\sigma \to \infty) \), suggests that the asymptotic behavior of solutions of (26) is approximately determined by the system

\[
\begin{align}
\frac{d\tilde{\Sigma}_+}{d\sigma} &= -4\tilde{\Sigma}_+, \\
\frac{d\vartheta}{d\sigma} &= 2\sqrt{6}\tilde{\Sigma}_+^{-1/2}, \\
\frac{d\zeta}{d\sigma} &= -2\zeta.
\end{align}
\]

(34a, 34b, 34c)

This statement is made precise in the following lemma.

**Lemma 4.3.** Consider a solution \((\tilde{\Sigma}_+(\sigma), \vartheta(\sigma), \zeta(\sigma))\) of the dynamical system (26) in \( B_{VI_0} \setminus \Gamma \). Then there are positive constants \( a, b = \sqrt{6/a}, \) and \( c \) such that

\[
\begin{align}
\tilde{\Sigma}_+(\sigma) &= a e^{-4\sigma} \left[ 1 + O(e^{-2\sigma}) \right], \\
\vartheta(\sigma) &= b e^{+2\sigma} \left[ 1 + O(\sigma e^{-2\sigma}) \right], \\
\zeta(\sigma) &= c e^{-2\sigma} \left[ 1 + O(e^{-2\sigma}) \right] \quad (\sigma \to \infty).
\end{align}
\]

(35a, 35b, 35c)

**Proof.** Due to Lemma 4.1, see also (33), we have \( \zeta = O(e^{-2\sigma}) \). Eq. (26a) thus reads

\[
\frac{d\tilde{\Sigma}_+}{d\sigma} = -2\tilde{\Sigma}_+(2 - \tilde{\Sigma}_+)[1 + O(e^{-2\sigma})],
\]

whence \( \tilde{\Sigma}_+ \) is (at least) of order \( O(e^{-2\sigma}) \), since \( \tilde{\Sigma}_+ \to 0 \) \( (\sigma \to \infty) \). Reinserting this information into (26a), we thus have

\[
\frac{d\tilde{\Sigma}_+}{d\sigma} = -4\tilde{\Sigma}_+[1 + O(e^{-2\sigma})],
\]

so that

\[
\tilde{\Sigma}_+(\sigma) = \exp[-4\sigma + A + O(e^{-2\sigma})]
\]

for some constant \( A \); Eq. (35a) follows. Combining this observation with (26b), we conclude that

\[
\frac{d\vartheta}{d\sigma} = 2\sqrt{6}\tilde{\Sigma}_+^{-1/2}[1 + O(e^{-2\sigma})],
\]

and (35b) ensues by a simple integration. Finally, consider

\[
g(\sigma) = \frac{1}{2}(\sigma - \sigma_0) = \frac{1}{2} \int_{\vartheta(\sigma_0)}^{\vartheta(\sigma)} \cos(2\theta) \left( \frac{d\vartheta}{d\sigma} \right)^{-1} d\theta = \frac{1}{4\sqrt{6}} \int_{\vartheta(\sigma_0)}^{\vartheta(\sigma)} \cos(2\theta) \tilde{\Sigma}_+^{1/2}[1 + O(e^{-2\sigma(\theta)})] d\theta
\]

\[
= \frac{1}{4\sqrt{6}} \int_{\vartheta(\sigma_0)}^{\vartheta(\sigma)} \cos(2\theta) \tilde{\Sigma}_+^{1/2} d\theta + \frac{1}{2} \int_{\sigma_0}^{\sigma} \cos(2\vartheta(\sigma)) O(e^{-2\sigma}) [1 + O(e^{-2\sigma})] d\sigma, \quad (36)
\]

where we have changed back the integration variable in the second integral. Due to (32), we have

\[
\int_{\vartheta(\sigma_0)}^{\vartheta(\sigma)} \frac{1}{4\sqrt{6}} \tilde{\Sigma}_+^{1/2} \cos(2\theta) d\theta = B + O(\tilde{\Sigma}_+^{1/2}(\sigma)) = B + O(e^{-2\sigma}),
\]

for some constant \( B \), where we have used the fact that \( \tilde{\Sigma}_+(\sigma) = O(e^{-4\sigma}) \). The second integral in (36) is of the form \( C + O(e^{-2\sigma}) \). To conclude,

\[
g(\sigma) = \frac{1}{2}\sigma + D + O(e^{-2\sigma}),
\]

and (35c) follows. \( \Box \)

**Lemma 4.4.** Consider a solution in \( B_{VI_0} \setminus \Gamma \). Then there is a \( \zeta_0 > 0 \) such that

\[
\begin{align}
\tilde{\Sigma}_+(\tau) &= \frac{1}{4}\tau^{-1}[1 + O(\tau^{-1/2})], \\
\vartheta(\tau) &= 2\sqrt{6}\tau^{1/2} + O(\log \tau), \\
\zeta(\tau) &= \zeta_0 \tau^{-1/2}[1 + O(\tau^{-1/2})],
\end{align}
\]

(37a, 37b, 37c)

as \( \tau \to \infty \).
Proof. Using (25) and (35a), we find that
\[ \sigma = \frac{1}{4} \log \tau + \frac{1}{4} \log(4a) + O(\tau^{-1/2}) \quad (\tau \to \infty). \]

Thus the statements of the lemma are essentially immediate consequences of observations already made.

It follows from Lemma 4.4 that there exists a positive constant \( r_0 \) such that
\[ \frac{\sqrt{1 - \Sigma_0^2(\tau)}}{\sqrt{1 + \zeta^{-1}(\tau)}} = r_0 \tau^{-3/4} [1 + O(\tau^{-1/2})] \]
as \( \tau \to \infty \). Therefore, we finally obtain

**Proposition 4.5.** Consider a solution in \( B_{V10} \setminus \Gamma \). Then there is an \( r_0 > 0 \) such that the asymptotic behaviors given by

\[
\begin{align*}
\Sigma_+ &= -1 + \frac{1}{4} \tau^{-1} \left[ 1 + O(\tau^{-1/2}) \right], \\
\Sigma_- &= r_0 \tau^{-3/4} \cos \vartheta(\tau) \left[ 1 + O(\tau^{-1/2}) \right], \\
N_+ &= -r_0 \tau^{-3/4} \sin \vartheta(\tau) \left[ 1 + O(\tau^{-1/2}) \right], \\
N_- &= \frac{1}{\sqrt{2}} \tau^{-1/2} \left[ 1 + O(\tau^{-1/2}) \right].
\end{align*}
\]

**Remark.** Every solution in \( B_{V10} \) has an asymptotic representation of the form (38); for the orbit \( \Gamma \), we set \( r_0 = 0 \). (In the case of \( \Gamma \), a simplified analysis in fact leads to somewhat better error estimates.) Note that there is merely one free parameter in (38) — at this level of approximation, it is a one-parameter family of orbits in \( B_{V10} \) that has the same leading asymptotics, cf. Figure 3.

**Remark.** From the results of Proposition 4.5, the asymptotic behavior of the variables \( (\Sigma_1, \Sigma_2, \Sigma_3) \) and \( (N_1, N_2) \) follows directly. It is thus straightforward to obtain the asymptotic expressions for the metric via the transformations of Section 2.

## 5 Comparison with type II dynamics

In this section, we discuss one important consequence of the result (38): Type VI future asymptotic dynamics are totally unrelated to type II dynamics. Let us elaborate.

Consider a type II orbit on the hemisphere \( B_\pm \) of \( \partial B_{V10} \), see Figure 1. When projected onto the \( (\Sigma_+, \Sigma_-) \)-plane, such an orbit is represented by a straight line \( \Sigma_+ \leq n \Sigma_- = -1 + \sqrt{3} n \) with \( n \in [0, \sqrt{3}] \); \( n = 0 \) corresponds to the point \( T_3 \), \( n = 1/\sqrt{3} \) to the orbit \( T_1 \mapsto Q_1 \) [\( T_2 \mapsto Q_2 \)], and \( n = \sqrt{3} \) to the point \( T_2 \) [\( T_1 \)] in the case of the northern [southern] hemisphere; see Figure 2 and, e.g., [19] for further details. Consequently, a type II orbit on \( B_\pm \) is the intersection of the plane \( \Sigma_+ \leq n \Sigma_- = -1 + \sqrt{3} n \) with \( B_\pm \).

A good representation of the structure of the flow on \( \partial B_{V10} \) is obtained by projecting the type II orbits onto the \( (\Sigma_-, N_+) \)-plane. Let \( n \) be small, i.e., we restrict our attention to orbits in the vicinity of the Taub point \( T_3 \). In that case, \( \Sigma_- / N_+ \) is monotonically increasing and exhausts the interval \( (-\infty, \infty) \). Hence we can parametrize an orbit on \( B_\pm \) close to \( T_3 \) by \( \theta \) in such a way that \( \Sigma_- / N_+ = -\cot \theta \), where \( \theta \) has the range \( (-\pi, 0) \) and \( (0, \pi) \) for the northern and southern hemisphere, respectively; accordingly, the orbit is represented by the parametric curve \( (\Sigma_+, \Sigma_-, N_+) = (\sqrt{1 - r(\theta)^2}, r(\theta) \cos \theta, -r(\theta) \sin \theta) \). Using \( \Sigma_+ \leq n \Sigma_- = -1 + \sqrt{3} n \) and the relation \( \Sigma_- = r(\theta) \cos \theta \), one obtains a second order equation for \( r(\theta) \). This equation has the (positive) solution
\[ r(\theta) = (2\sqrt{3})^{1/2} n^{1/2} + n \cos \theta + O(n^{3/2}) \]
Figure 3: This figure shows the part $\Sigma_+ < -1/3$ of the state space $B_{VI_0}$. The outer shell is $\partial B_{VI_0}$, which can be represented as $(\Sigma_+^2 + N_+^2)^{1/2} = (1 - \Sigma_+^2)^{1/2}$; in the vicinity of $T_3$ this corresponds to $(\Sigma_+^2 + N_+^2)^{1/2} \simeq \sqrt{2}(1 + \Sigma_+)^{1/2}$. The two 'cones' are surfaces $(\Sigma_+^2 + N_+^2)^{1/2} = a(1 + \Sigma_+)^{3/4}$ for two different values of the constant $a$. By (38), a type $VI_0$ orbit that is contained on such a cone initially remains (approximately) on that cone for all later times; in fact, asymptotically, a type $VI_0$ orbit spirals in towards $T_3$ along such a 'cone'.

for small $n$. Introducing $c_0 = (2\sqrt{3}n)^{-1/2}$, this can be expressed

$$r(\theta) = \left( c_0 \pm \frac{1}{2\sqrt{3}} \cos \theta \right)^{-1} + O(c_0^{-3}).$$

(39)

In practice, we will ignore the lower order terms in what follows.

Concatenating type II orbits (in the natural manner so that the $\omega$-limit point of orbit number $k - 1$ coincides with the $\alpha$-limit point of orbit number $k$), we obtain a sequence of type II orbits. Using (39) we see that the projection of this sequence onto the $(\Sigma_-, N_+)$-plane is represented by the spiral $(\Sigma_-, N_+) = r(\theta)(\cos \theta, -\sin \theta)$ with $\theta > \theta_0$ for some $\theta_0 \in \mathbb{R}$ and

$$r(\theta) = \frac{2\sqrt{3}}{c_1 + \text{cs}(\theta)},$$

(40)

where cs($\theta$) is defined as $\int_0^\theta |\sin(s)|ds$ and $c_1$ is a (large) constant. Since cs($\theta$) = $(2/\pi) \theta [1+O(\theta^{-1})]$, an asymptotic representation of this spiral is

$$r(\theta) = \frac{\sqrt{3}\pi}{\theta} [1 + O(\theta^{-1})].$$

(41)

This 'type II spiral' is depicted in Figure 4(a).

The Kasner parameter, $u$, is a parameter that (uniquely) characterizes the fixed points in a vicinity of the Taub point $T_3$ (which is itself represented by $u = \infty$); see, e.g., [19] for details. The connection between the Kasner parameter $u$ and the radius $r$ is very simple: $u = \sqrt{3} r^{-1} [1+O(r^{-1})]$
5 COMPARISON WITH TYPE II DYNAMICS

Figure 4: Subfigure (a) depicts the projection of the flow on $\partial B_{VI_0}$ (in a vicinity of the Taub point $T_3$) onto $(\Sigma_-, N_+)$-space. Each hemicircle is the projection of one type II orbit; the line $N_+ = 0$ is the projection of the Kasner circle. Concatenating type II orbits in the natural way results in a spiral, see (41). Subfigure (b) depicts the projection of an actual orbit in $B_{VI_0}$ onto $(\Sigma_-, N_+)$-space. The resulting spiral, which is described by (43), is completely different from the ‘type II spiral’ in subfigure (a) — the asymptotic evolution of type VI$_0$ models is unrelated to type II dynamics.

(see, e.g., [7]). Consequently, setting $\theta = k\pi$ with $k \in \mathbb{N}$ (so that $k$ numbers the type II orbits of the sequence of orbits), Eq. (41) leads to

$$u_k = u|_{\theta = k\pi} = \frac{\sqrt{3}}{r(k\pi)} = k + u_0.$$ (42)

This is the well-known result for a sequence of type II orbits (Kasner sequence), which corresponds to the $k$-fold iteration of the Kasner map $u \mapsto u + 1$ with initial value $u_0$.

The type VI$_0$ dynamics are different. Consider an arbitrary orbit in $B_{VI_0}$. The asymptotic dynamics of this orbit are given by Proposition 4.5. From (37b) and (38), with $\rho_0 = 2\pi^{3/4} r_0$, we thus obtain $(\Sigma_-, N_+) = r(\theta)(\cos \vartheta, -\sin \vartheta)$ with

$$r(\theta) = \rho_0 \theta^{-3/2} \left[1 + O(\theta^{-1/2})\right].$$ (43)

This represents a spiral in the $(\Sigma_-, N_+)$-plane which is completely different from the type II spiral (41); see Figure 4(b). Setting $\vartheta = k\pi$, $k \in \mathbb{N}$, leads to $r(k\pi) \propto k^{-3/2} \left[1 + O(k^{-1}\log k)\right]$; when we further define $u_k$, in analogy with the above, by $u_k = \sqrt{3}/r(k\pi)$, then

$$u_k = u_0 k^{3/2} \left[1 + O(k^{-1}\log k)\right].$$ (44)

Hence the asymptotic evolution of the (analog of the) Kasner parameter is radically different from the type II (Kasner map) evolution.

Remark. A solution in $B_{VI_0}$ whose initial data is chosen extremely close to $\partial B_{VI_0}$ shadows a sequence of type II orbits for some time. Accordingly, such a solution behaves like (41), cf. Figure 4(a), for some time. However, eventually, as proved by Proposition 4.5, the solution deviates from this type of evolution and behaves asymptotically like (43), cf. Figure 4(b).

Remark. A different way of illustrating the differences between type II and type VI$_0$ evolution is to track the key quantity $|N_1/N_2|$. For every type VI$_0$ orbit we obtain from (38) that

$$\left|\frac{N_1}{N_2}\right| = 1 - n_0 \tau^{-1/4} \sin \vartheta(\tau) + O(\tau^{-1/2})$$ (45)

The $O(\cdot)$-term yields a constant for general reasons.
as \( \tau \to \infty \) for some constant \( n_0 \). Hence, \(|N_1/N_2|\) oscillates around 1, but the amplitude of the oscillations converges to zero as \( \tau \to \infty \). In contrast, if a type VI\(_0\) were asymptotic to a sequence of type II orbits, \(|N_1/N_2|\) would oscillate with growing amplitudes: \( \lim \inf |N_1/N_2| = 0 \) and \( \lim \sup |N_1/N_2| = \infty \).

By the results of [14], see also [8, 9], it is known, in a sense that has to remain vague here, that Bianchi type IX solutions are approximated by sequences of type II solution as the singularity is approached. For Bianchi type VIII it is unclear whether this result holds analogously, cf. also the remarks in the introduction. It will be the role type VI\(_0\) plays in the context of type VIII dynamics that will decide whether singularities associated with Bianchi type VIII are local and whether they are oscillatory in the same sense (type II oscillations) as singularities of type IX. In any case, answering these questions will have broad ramifications for our understanding of the nature of generic singularities.

Acknowledgments

We gratefully acknowledge the hospitality of the Mittag-Leffler Institute, which was afforded us during the programme entitled “Geometry, Analysis, and General Relativity”. H. R. is supported the Göran Gustafsson Foundation and is a Royal Swedish Academy of Sciences Research Fellow supported by a grant from the Knut and Alice Wallenberg Foundation.

References


