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# $C^{(n)}$-Cardinals

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# $C^{(n)}$-CARDINALS

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Abstract. For each natural number $n$, let $C^{(n)}$ be the closed and unbounded proper class of ordinals $\alpha$ such that $V_\alpha$ is a $\Sigma_n$-elementary substructure of $V$. A $C^{(n)}$-cardinal is a cardinal $\kappa$ that is the critical point of an elementary embedding $j : V \to M$, $M$ transitive, with $j(\kappa) \in C^{(n)}$. We show that the $C^{(n)}$-cardinals form a hierarchy that refines the usual large cardinal hierarchy, and we give characterizations of various natural reflection principles in terms of such cardinals. In particular, building on results of [1], we give sharper formulations of Vopěnka's Principle in terms of $C^{(n)}$-extendible cardinals.

For each natural number $n$, let $C^{(n)}$ denote the *club* (i.e., closed and unbounded) proper class of ordinals $\alpha$ that are $\Sigma_n$*-correct* in the universe $V$ of all sets, meaning that $V_\alpha$ is a $\Sigma_n$-elementary substructure of $V$, and written $V_\alpha \preceq_n V$. Thus, $C^{(0)}$ is the class of all ordinals. But if $V_\alpha \preceq_1 V$, then $\alpha$ is already an uncountable strong limit cardinal, for if $\beta < \alpha$, then the sentence

$$\exists\gamma\exists f(\gamma \text{ an ordinal} \wedge f : \gamma \to V_\beta \text{ is onto})$$

is $\Sigma_1$ in the parameter $V_\beta$, and therefore it must hold in $V_\alpha$. Further, if $\alpha \in C^{(1)}$, then $V_\alpha = H(\alpha)$. Thus $C^{(1)}$ is precisely the class of all uncountable cardinals $\alpha$ such that $V_\alpha = H(\alpha)$. It follows that $C^{(1)}$ is $\Pi_1$ definable, for $\alpha \in C^{(1)}$ iff $\alpha$ is a cardinal and

$$\forall M(M \text{ a transitive model of } ZFC^* \wedge \alpha \in M \to M \models V_\alpha = H(\alpha)).$$

(Here $ZFC^*$ denotes a sufficiently large finite fragment of $ZFC$.) The point is that if $\alpha \in C^{(1)}$ and $M$ is a transitive model of $ZFC^*$ that contains $\alpha$, then if in $M$ we could find some transitive $x \in V_\alpha \setminus H(\alpha)$, we would have that $|x| \geq \alpha$. But since in $V$ the cardinality of $x$ is less than $\alpha$, because $V_\alpha = H(\alpha)$, this would mean that, in $V$, $\alpha$ is not a cardinal, which is absurd.

More generally, since the truth predicate $\models_n$ for $\Sigma_n$ sentences (for $n \geq 1$) is $\Sigma_n$ definable (see [3], section 0.2), and since the relation $x = V_y$ is $\Pi_1$, the class $C^{(n)}$ is $\Pi_n$ definable: $\alpha \in C^{(n+1)}$ iff

$$\alpha \in C^{(n)} \wedge \forall\varphi(x) \in \Sigma_{n+1}\forall a \in V_\alpha(\models_{n+1} \varphi(a) \to V_\alpha \models \varphi(a)).$$

Let us remark that $C^{(n)}$, for $n \geq 1$, cannot be $\Sigma_n$ definable. Otherwise, if $\alpha$ is the least ordinal in $C^{(n)}$, then the sentence "there is some ordinal in $C^{(n)}$" would be $\Sigma_n$ and so it would hold in $V_\alpha$, yielding an ordinal in $C^{(n)}$ smaller than $\alpha$, which is impossible.

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The classes $C^{(n)}$, $n \geq 1$, form a *basis* for definable club proper classes of ordinals, in the sense that every $\Sigma_n$ club proper class of ordinals contains $C^{(n)}$. More generally, every club proper class $C$ of ordinals that is $\mathbf{\Sigma_n}$ (i.e., $\Sigma_n$-definable with parameters) contains all $\alpha \in C^{(n)}$ that are greater than the rank of the parameters involved in any $\Sigma_n$ definition of $C$.

Finally, note that since the least ordinal in $C^{(n)}$ does not belong to $C^{(n+1)}$, $C^{(n+1)} \subset C^{(n)}$, all $n$.

When considering non-trivial elementary embeddings $j : V \to M$, with $M$ transitive, one would like to have some control over where the image $j(\kappa)$ of the critical point $\kappa$ goes. A especially interesting case is when one wants $V_{j(\kappa)}$ to reflect some specific property of $V$ or, more generally, when one wants $j(\kappa)$ to belong to a particular definable club proper class of ordinals. Now, since the $C^{(n)}$, $n \in \omega$, form a basis for such classes, the problem can be reformulated as follows: when can we have $j(\kappa) \in C^{(n)}$, for a given $n \in \omega$? This prompts the following definition.

Let us say that a cardinal $\kappa$ is *$C^{(n)}$-measurable* if there is an elementary embedding $j : V \to M$, some transitive class $M$, with critical point $crit(j) = \kappa$ and with $j(\kappa) \in C^{(n)}$.

Observe that if $j : V \to M \cong Ult(V,\mathcal{U})$, $M$ transitive, is the ultrapower elementary embedding obtained from a non-principal $\kappa$-complete ultrafilter $\mathcal{U}$ on $\kappa$, then $2^\kappa < j(\kappa) < (2^\kappa)^+$ (see [3]). Hence, since $V_{j(\kappa)} \preceq_1 V$ implies that $j(\kappa)$ is a (strong limit) cardinal, $j$ cannot witness the $C^{(1)}$-measurability of $\kappa$. Nonetheless, by using iterated ultrapowers (see [2], 19.15), one has that for every cardinal $\alpha > 2^\kappa$, the $\alpha$-th iterated ultrapower embedding $j_\alpha : V \to M_\alpha \cong Ult(V,\mathcal{U}_\alpha)$, where $\mathcal{U}_\alpha$ is the $\alpha$-th iterate of $\mathcal{U}$, has critical point $\kappa$ and $j_\alpha(\kappa) = \alpha$. So, if $\kappa$ is measurable, then for each $n$ we can always find an elementary embedding $j : V \to M$, $M$ transitive, with $j(\kappa) \in C^{(n)}$. We have thus shown the following.

**Proposition 0.1.** *Every measurable cardinal is $C^{(n)}$-measurable, for all $n$.*

A similar situation occurs in the case of strong cardinals.

Let us say that a cardinal $\kappa$ is *$C^{(n)}$-strong* if for every $\lambda > \kappa$, $\kappa$ is *$\lambda$-$C^{(n)}$-strong*, that is, there exists an elementary embedding $j : V \to M$, $M$ transitive, with critical point $\kappa$, and such that $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $j(\kappa) \in C^{(n)}$. Equivalently (see [3] 26.7), $\kappa$ is $\lambda$-$C^{(n)}$-strong iff there exists a $(\kappa, \beta)$-extender $E$, for some $\beta > |V_\lambda|$, with $V_\lambda \subseteq M_E$ and $\lambda < j_E(\kappa) \in C^{(n)}$.

Suppose now that $j : V \to M$ witnesses the $\lambda$-strongness of $\kappa$, with $j(\kappa)$ not necessarily in $C^{(n)}$. Let $E$ be the $(\kappa, j(\kappa))$-extender obtained from $j$, and let $j_E : V \to M_E$ be the corresponding $\lambda$-strong embedding (see [3]). Then in $M_E$, $E' := j_E(E)$ is a $(j_E(\kappa), j_E(j(\kappa)))$-extender, which gives rise to an elementary embedding $j_{E'} : M_E \to M_{E'}$ with critical point $j_E(\kappa)$. Still in $M_E$, let $\mathcal{U}$ be the $j_E(\kappa)$-complete ultrafilter on $j_E(\kappa)$ derived from $j_{E'}$, i.e.,

$$\mathcal{U} = \{X \subseteq j_E(\kappa) : j_E(\kappa) \in j_{E'}(X)\}$$

and let $j_{\mathcal{U}} : M_E \to M$ be the corresponding elementary embedding. Then we can iterate $j_{\mathcal{U}}$ $\alpha$-times, for some $\alpha \in C^{(n)}$ greater than $2^{j_E(\kappa)}$, so that if $j_\alpha : M_E \to M_\alpha$ is the resulting elementary embedding, then $j_\alpha(j_E(\kappa)) = \alpha$.

Letting $k := j_\alpha \circ j_E$, we have that $k : V \to M_\alpha$ is a $\lambda$-strong elementary embedding with critical point $\kappa$ and with $k(\kappa) \in C^{(n)}$. Thus we have shown the following.

**Proposition 0.2.** *Every $\lambda$-strong cardinal is $\lambda$-$C^{(n)}$-strong, for all $n$. Hence, every strong cardinal is $C^{(n)}$-strong, for every $n$.*

Thus for measurable or strong cardinals $\kappa$, the requirement that $j(\kappa) \in C^{(n)}$ for the corresponding elementary embeddings $j : V \to M$ does not yield stronger large cardinal notions. But, as we shall see next, the situation changes completely in the case of superstrong embeddings, that is, when $j$ is such that $V_{j(\kappa)} \subseteq M$.

## 1. $C^{(n)}$-SUPERSTRONG CARDINALS

In the case of superstrong cardinals $\kappa$, the requirement that $j(\kappa) \in C^{(n)}$, for $n > 1$, produces a hierarchy of ever stronger large cardinal principles.

**Definition 1.1.** *A cardinal $\kappa$ is* $C^{(n)}$-superstrong *if there exists an elementary embedding $j : V \to M$, $M$ transitive, with critical point $\kappa$, $V_{j(\kappa)} \subseteq M$, and $j(\kappa) \in C^{(n)}$.*

Note that in the definition above, from the fact that $j(\kappa) \in C^{(n)}$ and $V_{j(\kappa)} \subseteq M$ it already follows that $\kappa \in C^{(n)}$. Thus, every $C^{(n)}$-superstrong cardinal belongs to $C^{(n)}$.

**Proposition 1.2.** *If $j : V \to M$ is an elementary embedding such that $V_{j(\kappa)} \subseteq M$, where $\kappa = crit(j)$, then $j(\kappa) \in C^{(1)}$. Hence, every superstrong cardinal is $C^{(1)}$-superstrong.*

*Proof.* Since $\kappa \in C^{(1)}$, $M$ thinks that $j(\kappa) \in C^{(1)}$, i.e., $M$ thinks $j(\kappa)$ is a strong limit cardinal and $V_{j(\kappa)} = H(j(\kappa))$. But then, since $(V_{j(\kappa)})^M = V_{j(\kappa)}$, $j(\kappa)$ is, in $V$, a strong limit cardinal with $V_{j(\kappa)} = H(j(\kappa))$, and therefore $j(\kappa) \in C^{(1)}$. □

Observe that for $n \geq 1$, the sentence "$\kappa$ is $C^{(n)}$-superstrong" is $\Sigma_{n+1}$, for $\kappa$ is $C^{(n)}$-superstrong iff

$$\exists\beta\exists\mu\exists E(\beta < \mu \wedge \beta, \mu \in C^{(n)} \wedge E \text{ is a } (\kappa,\beta)\text{-extender} \wedge E \in V_\mu \wedge$$
$$V_\mu \models \text{“}V_{j_E(\kappa)} \subseteq M_E\text{”}).$$

**Proposition 1.3.** *For every $n \geq 1$, if $\kappa$ is $C^{(n+1)}$-superstrong, then there is a $\kappa$-complete normal ultrafilter $\mathcal{U}$ over $\kappa$ such that*

$$\{\alpha < \kappa : \alpha \text{ is } C^{(n)}\text{-superstrong}\} \in \mathcal{U}.$$

*Hence, the first $C^{(n)}$-superstrong cardinal $\kappa$, if it exists, is not $C^{(n+1)}$-superstrong.*

*Proof.* Suppose $\kappa$ is $C^{(n+1)}$-superstrong, witnessed by a normal $(\kappa,\beta)$-extender $E$ with associated elementary embedding $j_E = j : V \to M$ such that $\beta = j(\kappa)$ and $V_{j(\kappa)} \subseteq M$. Since $j(\kappa) \in C^{(n+1)}$,

$$V_{j(\kappa)} \models \text{“}\kappa \text{ is } C^{(n)}\text{-superstrong”}.$$

And since $\kappa \in C^{(n)}$, $M \models$ "$j(\kappa) \in C^{(n)}$". Hence, since $V_{j(\kappa)} = (V_{j(\kappa)})^M$, and since "$\kappa$ is $C^{(n)}$-superstrong" is a $\Sigma_{n+1}$ statement, we have:

$$M \models \text{"}\kappa \text{ is } C^{(n)}\text{-superstrong"}.$$

Therefore, by normality of $E$, the set $\{\{\alpha\} : \alpha < \kappa \wedge \alpha$ is $C^{(n)}$-superstrong$\}$ belongs to $E_{\{\kappa\}}$. Hence, $\{\alpha < \kappa : \alpha$ is $C^{(n)}$-superstrong$\}$ belongs to the standard $\kappa$-complete normal ultrafilter $\mathcal{U} := \{X \subseteq \kappa : \kappa \in j(X)\}$. □

The following Proposition gives an upper bound on the relative position of $C^{(n)}$-superstrong cardinals in the large cardinal hierarchy.

**Proposition 1.4.** *If $\kappa$ is $2^\kappa$-supercompact and belongs to $C^{(n)}$, then there is a $\kappa$-complete normal ultrafilter $\mathcal{U}$ over $\kappa$ such that the set of $C^{(n)}$-superstrong cardinals smaller than $\kappa$ belongs to $\mathcal{U}$.*

*Proof.* Let $j : V \to M$ be an elementary embedding coming from a normal ultrafilter $\mathcal{V}$ on $\mathcal{P}_\kappa(2^\kappa)$. Let $\bar{j} := j \restriction V_{\kappa+1}$. So, $\bar{j} : V_{\kappa+1} \to M_{j(\kappa)+1}$ is elementary and $\bar{j} \in M$. Hence $M \models$ "$\bar{j} : V_{\kappa+1} \to V_{j(\kappa)+1}$ is elementary". Since $\kappa \in C^{(n)}$, also $M \models$ "$j(\kappa) \in C^{(n)}$". Thus, $M \models$ "$\kappa$ is $\kappa+1$-$C^{(n)}$-extendible". Hence, $\{x \in \mathcal{P}_\kappa(2^\kappa) : ot(x \cap \kappa)$ is $ot(x\cap\kappa)+1$-$C^{(n)}$-extendible$\} \in \mathcal{V}$. Letting $\mathcal{U}$ be the projection of $\mathcal{V}$ on $\kappa$, we have

$$\{\alpha < \kappa : \alpha \text{ is } \alpha + 1\text{-}C^{(n)}\text{-extendible}\} \in \mathcal{U}.$$

Now as in [3], Proposition 26.11 (a), one can show that if $\alpha$ is $\alpha+1$-$C^{(n)}$-extendible, then $\alpha$ is $C^{(n)}$-superstrong. □

## 2. $C^{(n)}$-EXTENDIBLE CARDINALS

Recall that a cardinal $\kappa$ is $\lambda$–*extendible* if there is an elementary embedding $j : V_\lambda \to V_\mu$, some $\mu$, with critical point $\kappa$ and such that $j(\kappa) > \lambda$. And $\kappa$ is *extendible* if it is $\lambda$–extendible for all $\lambda > \kappa$.

The following lemma implies that every extendible cardinal is supercompact (see also [3], 23.6).

**Lemma 2.1** (M. Magidor [4])**.** *Suppose $j : V_\lambda \to V_\mu$ is elementary, $\lambda$ is a limit ordinal, and $\kappa$ is the critical point of $j$. Then $\kappa$ is $< \lambda$-supercompact.*

**Definition 2.2.** *For a cardinal $\kappa$ and $\lambda > \kappa$, we say that $\kappa$ is* $\lambda$-$C^{(n)}$-extendible *if there is an elementary embedding $j : V_\lambda \to V_\mu$, some $\mu$, with critical point $\kappa$, and such that $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$.*

*We say that $\kappa$ is* $C^{(n)}$–extendible *if it is $\lambda$-$C^{(n)}$-extendible for all $\lambda > \kappa$.*

It follows from the next Proposition that a cardinal is extendible if and only if it is $C^{(1)}$-extendible.

**Proposition 2.3.** *Every extendible cardinal is $C^{(1)}$-extendible.*

*Proof.* Suppose $\kappa$ is extendible and $\lambda$ is greater than $\kappa$. Pick $\lambda' \geq \lambda$ such that $\lambda' \in C^{(1)}$. Let $j : V_{\lambda'} \to V_\mu$ be an elementary embedding with $crit(j) = \kappa$ and $j(\kappa) > \lambda'$. Since $\lambda'$ is a cardinal and $V_{\lambda'} = H(\lambda')$, by elementarity we must also have that $\mu$ is a cardinal and $V_\mu = H(\mu)$. Hence $\mu \in C^{(1)}$. And since, again by elementarity, $V_\mu \models j(\kappa) \in C^{(1)}$, we have that $j(\kappa) \in C^{(1)}$. □

In the definition above, notice that if $\kappa, \lambda, \mu \in C^{(n)}$, then the requirement $j(\kappa) \in C^{(n)}$ follows automatically. Also, if $\kappa$ is $C^{(n)}$-extendible, then it follows easily from the fact that $j(\kappa) \in C^{(n)}$ and $j$ is elementary that $\kappa \in C^{(n)}$. But more is true.

**Proposition 2.4.** *If $\kappa$ is $C^{(n)}$-extendible, then $\kappa \in C^{(n+2)}$.*

*Proof.* By induction on $n$. For $n = 0$, since $\kappa \in C^{(1)}$, we only need to show that if $\exists x \varphi(x)$ is a $\Sigma_2$ sentence, where $\varphi$ is $\Pi_1$ and has parameters in $V_\kappa$, that holds in $V$, then it holds in $V_\kappa$. So suppose $a$ is such that $\varphi(a)$ holds in $V$. Let $\lambda > \kappa$ be such that $a \in V_\lambda$, and let $j : V_\lambda \to V_\mu$ be elementary, with critical point $\kappa$ and with $j(\kappa) > \lambda$. Then $V_{j(\kappa)} \models \varphi(a)$, and so by elementarity $V_\kappa \models \exists x \varphi(x)$.

Now suppose $\kappa$ is $C^{(n)}$-extendible and $\exists x \varphi(x)$ is a $\Sigma_{n+2}$ sentence, where $\varphi$ is $\Pi_{n+1}$ and has parameters in $V_\kappa$. If $\exists x \varphi(x)$ holds in $V_\kappa$, then since by induction hypothesis $\kappa \in C^{(n+1)}$, we have that $\exists x \varphi(x)$ holds in $V$. Now suppose $a$ is such that $\varphi(a)$ holds in $V$. Let $\lambda > \kappa$ be such that $a \in V_\lambda$, and let $j : V_\lambda \to V_\mu$ be elementary, with critical point $\kappa$ and with $j(\kappa) > \lambda$. Then since $j(\kappa) \in C^{(n)}$, we have $V_{j(\kappa)} \models \varphi(a)$, and so by elementarity $V_\kappa \models \exists x \varphi(x)$. □

Let us observe that for any $\alpha < \lambda$, the relation "$\alpha$ is $\lambda$-$C^{(n)}$-extendible" is $\Sigma_{n+1}$ (for $n \geq 1$), for it holds if and only if

$$\exists \mu \exists j (j : V_\lambda \to V_\mu \wedge j \text{ elementary } \wedge crit(j) = \alpha \wedge j(\alpha) > \lambda \wedge j(\alpha) \in C^{(n)}).$$

Hence, "$x$ is a $C^{(n)}$-extendible cardinal" is a $\Pi_{n+2}$ property of $x$.

**Proposition 2.5.** *For every $n \geq 1$, if $\kappa$ is $C^{(n)}$-extendible and $\kappa+1$-$C^{(n+1)}$-extendible, then the set of $C^{(n)}$-extendible cardinals is unbounded below $\kappa$. Hence, the first $C^{(n)}$-extendible cardinal $\kappa$, if it exists, is not $\kappa+1$-$C^{(n+1)}$-extendible. In particular, the first extendible cardinal $\kappa$ is not $\kappa+1$-$C^{(2)}$-extendible.*

*Proof.* Suppose $\kappa$ is $C^{(n)}$-extendible and $\kappa+1$-$C^{(n+1)}$-extendible, witnessed by $j : V_{\kappa+1} \to V_{j(\kappa)+1}$. Since $j(\kappa) \in C^{(n+1)}$,

$$V_{j(\kappa)} \models \text{“}\kappa \text{ is } C^{(n)}\text{-extendible”}.$$

Hence, for every $\alpha < \kappa$,

$$V_{j(\kappa)} \models \text{“}\exists \beta > \alpha (\beta \text{ is } C^{(n)}\text{-extendible})\text{”},$$

since this is witnessed by $\kappa$. By elementarity of $j$, for every fixed $\alpha < \kappa$, there is $\beta > \alpha$ such that,

$$V_\kappa \models \text{“}\beta > \alpha \wedge \beta \text{ is } C^{(n)}\text{-extendible”}.$$

And since, by Proposition 2.4, $\kappa \in C^{(n+2)}$, $\beta$ is $C^{(n)}$-extendible in $V$. □

**Proposition 2.6.** *For every $n$, if there exists a $C^{(n+2)}$-extendible cardinal, then there exist a proper class of $C^{(n)}$–extendible cardinals.*

*Proof.* By the last Proposition, if $\kappa$ is $C^{(n+2)}$-extendible, then the set of $C^{(n)}$-extendible cardinals is unbounded below $\kappa$. Now the Proposition follows easily from the fact that if $\kappa$ is $C^{(n+2)}$-extendible, then $\kappa \in C^{(n+4)}$ (Proposition 2.4), and the fact that being $C^{(n)}$-extendible is a $\Pi_{n+2}$-property. □

Note that the existence of a $C^{(n+1)}$-extendible cardinal $\kappa$ (for $n \geq 1$) does not imply the existence of a $C^{(n)}$-extendible cardinal greater than $\kappa$. For if $\lambda$ is such a cardinal, then $V_\lambda \models$ "$\kappa$ is $C^{(n+1)}$-extendible".

The following Proposition shows that $C^{(n)}$-extendible cardinals are much larger than $C^{(n)}$-superstrong cardinals.

**Proposition 2.7.** *If $\kappa$ is $\kappa+1$-$C^{(n)}$-extendible and belongs to $C^{(n)}$, then $\kappa$ is $C^{(n)}$-superstrong, and there is a $\kappa$-complete normal ultrafilter $\mathcal{U}$ over $\kappa$ such that the set of $C^{(n)}$-superstrong cardinals smaller than $\kappa$ belongs to $\mathcal{U}$.*

*Proof.* As in [3], Proposition 26.11 (a). □

## 3. $C^{(n)}$-extendible cardinals and Vopěnka's Principle

This section builds on results from [1], giving sharper characterizations of Vopěnka's Principle in terms of $C^{(n)}$-extendible cardinals.

Recall that *Vopěnka's Principle (VP)* states that for every proper class $\mathcal{C}$ of structures of the same type, there exist $A \neq B$ in $\mathcal{C}$ such that $A$ is elementarily embeddable into $B$.

VP can be formulated in the first-order language of set theory as an axiom schema, i.e., as an infinite set of axioms, one for each formula with two free variables. Formally, for each such formula $\varphi(x,y)$ one has the axiom:

$$\forall x[(\forall y \forall z(\varphi(x,y) \wedge \varphi(x,z) \rightarrow y \text{ and } z \text{ are structures of the same type}) \wedge$$
$$\forall \alpha \in OR\, \exists y(rank(y) > \alpha \wedge \varphi(x,y)) \rightarrow$$
$$\exists y \exists z(\varphi(x,y) \wedge \varphi(x,z) \wedge y \neq z \wedge \exists e(e : y \rightarrow z \text{ is elementary}))].$$

Henceforth, $VP$ will be understood as this axiom schema.

The theory $ZFC + VP$ implies, for instance, that the class of extendible cardinals is stationary, i.e., every definable club proper class contains an extendible cardinal ([4]). And its consistency is known to follow from the consistency of $ZFC$ plus the existence of an almost-huge cardinal (see [3], or [2]). We will give the exact consistency strength below.

Let us consider the following variants of VP, the first apparently much stronger than the second.

**Definition 3.1.** *If $\Gamma$ is one of $\mathbf{\Sigma_n}$, $\mathbf{\Pi_n}$, $\mathbf{\Delta_n}$, for $n \in \omega$, and $\kappa$ is an infinite cardinal, then we write $VP(\kappa, \Gamma)$ for the following assertion:*

For every $\Gamma$ proper class $\mathcal{C}$ of structures of the same type $\tau$ such that both $\tau$ and the parameters of some $\Gamma$-definition of $\mathcal{C}$, if any, belong to $H(\kappa)$, $\mathcal{C}$ *reflects below* $\kappa$, i.e., for every $B \in \mathcal{C}$, there exists $A \in \mathcal{C} \cap H(\kappa)$ that is elementarily embeddable into $B$.

*If $\Gamma$ is one of $\mathbf{\Sigma_n}$, $\mathbf{\Pi_n}$, $\mathbf{\Delta_n}$, or $\Sigma_n$, $\Pi_n$, $\Delta_n$, for $n \in \omega$, we write $VP(\Gamma)$ for the following statement:*

For every $\Gamma$ proper class $\mathcal{C}$ of structures of the language of set theory with one (equivalently, finitely-many) additional 1-ary relation symbol, there exist distinct $A$ and $B$ in $\mathcal{C}$ with an elementary embedding of $A$ into $B$.

Clearly, for every $\Gamma$, $VP(\kappa, \Gamma)$ for some $\kappa$ implies $VP(\Gamma)$.

VP for $\mathbf{\Sigma_1}$ classes follows from ZFC. In fact, the following holds.

**Theorem 3.2.** *$VP(\kappa, \mathbf{\Sigma_1})$ holds for every uncountable cardinal $\kappa$.*

*Proof.* Fix an uncountable cardinal $\kappa$ and a class $\mathcal{C}$ of structures of the same type $\tau \in H(\kappa)$, definable by a $\Sigma_1$ formula with parameters in $H(\kappa)$.

Given $B \in \mathcal{C}$, let $\lambda$ be a regular cardinal greater than $\kappa$, with $B \in H(\lambda)$, and let $N$ be an elementary substructure of $H(\lambda)$, of cardinality less than $\kappa$, which contains $B$ and the transitive closure of $\{\tau\}$ together with the parameters involved in some $\Sigma_1$ definition of $\mathcal{C}$.

Let $A$ and $M$ be the transitive collapses of $B$ and $N$, respectively, and let $j : M \to N$ be the collapsing isomorphism. Then $A \in H(\kappa)$, and $j \restriction A : A \to B$ is an elementary embedding. Observe that $j(\tau) = \tau$. So, since $\Sigma_1$ formulas are upwards absolute for transitive models, and since $M \models A \in \mathcal{C}$, we have that $A \in \mathcal{C}$. □

In contrast to the last Theorem, Vopěnka's Principle for $\Pi_1$ proper classes implies the existence of very large cardinals.

**Theorem 3.3.**

(1) *If $VP(\Pi_1)$ holds, then there exists a supercompact cardinal.*
(2) *If $VP(\mathbf{\Pi_1})$ holds, then there is a proper class of supercompact cardinals.*

*Proof.* (1). Let $\mathcal{C}$ be the class of structures of the form $\langle V_{\lambda+2}, \in, \alpha, \lambda\rangle$, where $\lambda$ is the least limit ordinal greater than $\alpha$ such that no $\kappa \leq \alpha$ is $< \lambda$-supercompact.

We claim that $\mathcal{C}$ is $\Pi_1$ definable without parameters. For $X \in \mathcal{C}$ if and only if $X = \langle X_0, X_1, X_2, X_3\rangle$, where

(1) $X_2$ is an ordinal.
(2) $X_3$ is a limit ordinal greater than $X_2$.
(3) $X_0 = V_{X_3+2}$
(4) $X_1 = \in \restriction X_0$
(5) And the following holds in $\langle X_0, X_1\rangle$:
    (a) $\forall \kappa \leq X_2(\kappa$ is not $< X_3$-supercompact)
    (b) $\forall \mu(\mu$ limit$\wedge X_2 < \mu < X_3 \to \exists \kappa \leq X_2(\kappa$ is $< \mu$-supercompact)).

If there is no supercompact cardinal, then $\mathcal{C}$ is a proper class. So by $VP(\Pi_1)$, there exist $\langle V_{\lambda+2}, \in, \alpha, \lambda\rangle \neq \langle V_{\mu+2}, \in, \beta, \mu\rangle$ and an elementary embedding

$$j : \langle V_{\lambda+2}, \in, \alpha, \lambda\rangle \to \langle V_{\mu+2}, \in, \beta, \mu\rangle.$$

Since $j$ must send $\alpha$ to $\beta$ and $\lambda$ to $\mu$, $j$ is not the identity. Hence by Kunen's theorem (see [3]) we must have $\lambda < \mu$, and therefore also $\alpha < \beta$. So, $j$ has critical point some $\kappa \leq \alpha$. It now follows by Lemma 2.1 that $\kappa$ is $< \lambda$-supercompact. But this is impossible because $\langle V_{\lambda+2}, \in, \alpha, \lambda\rangle \in \mathcal{C}$.

(2). Fixing an ordinal $\xi$, to show that there is a supercompact cardinal greater than $\xi$, we argue as above. The only difficulty now is to ensure that $\kappa > \xi$. But this can be achieved by letting $\mathcal{C}$ be the class of structures of the form $\langle V_{\lambda+2}, \in, \alpha, \lambda, \{\gamma\}_{\gamma\leq\xi}\rangle$, where $\alpha > \xi$ and $\lambda$ is the least limit ordinal greater than $\alpha$ such that no $\kappa \leq \alpha$ is $< \lambda$-supercompact. The class $\mathcal{C}$ is now $\Pi_1$ definable with $\xi$ as an additional parameter. □

Let us say that a limit ordinal $\lambda$ *captures* a proper class $\mathcal{C}$ if the ranks of elements of $\mathcal{C}$ are unbounded below $\lambda$.

Note that if $\mathcal{C}$ is $\mathbf{\Pi_n}$, then every $\lambda$ in $C^{(n+1)}$ greater than the rank of the parameters involved in a $\Pi_n$ definition of $\mathcal{C}$ captures $\mathcal{C}$. Notice also that every cardinal in $C^{(2)}$ belongs to $Lim(C^{(1)})$ – i.e., is a limit point of $C^{(1)}$ – and captures all $\Pi_1$ proper classes. However, the least ordinal $\lambda$ in $Lim(C^{(1)})$ that captures all $\Pi_1$ proper classes is strictly less than the least ordinal in $C^{(2)}$. The point is that, fixing an enumeration $\langle \varphi_n(x) : n < \omega \rangle$ of all $\Pi_1$ formulas that define proper classes, the sentence

$$\exists\lambda\exists x(\lambda \in Lim(C^{(1)}) \wedge x = V_\lambda \wedge \forall n(V_\lambda \models \forall\alpha\exists\beta > \alpha\exists a(rk(a) > \beta \wedge \varphi_n(a)))$$

is $\Sigma_2$ in the parameter $\langle \varphi_n(x) : n < \omega \rangle$.

**Proposition 3.4.** *Suppose $\mathcal{C}$ is a $\Pi_1$ proper class of structures of the same type, and $\kappa$ is $<\lambda$-supercompact, for some $\lambda \in Lim(C^{(1)})$ greater than $\kappa$ that captures $\mathcal{C}$. Then VP holds for $\mathcal{C}$.*

*Proof.* Since $\lambda$ captures $\mathcal{C}$, in $V_\lambda$ there exist elements of $\mathcal{C}$ of arbitrarily high rank. So, since $\lambda \in Lim(C^{(1)})$, we can find $\delta < \lambda$ such that $V_\delta = H(\delta)$, and $B \in \mathcal{C} \cap V_\delta$ of rank greater than $\kappa$. Let $j : V \to M$ be an elementary embedding with critical point $\kappa$, with $j(\kappa) > \delta$, and $M$ closed under $\delta$-sequences. Since $B \in M$ and $\mathcal{C}$ is $\Pi_1$-definable, $M \models$ "$B \in \mathcal{C}$". And since $M$ is closed under $\delta$-sequences, the elementary embedding $j \restriction B : B \to j(B)$ belongs to $M$. Thus,

$$M \models \text{“}\exists A \in \mathcal{C}\, \exists e(rank(A) < j(\kappa) \wedge e : A \to j(B) \text{ is elementary}),$$

since this is witnessed by $B$ and $j \restriction B$.

By elementarity, the same must hold in $V$, namely,

$$\exists A \in \mathcal{C}\, \exists e(rank(A) < \kappa \wedge e : A \to B \text{ is elementary}),$$

which is what we wanted. □

**Theorem 3.5** ([1])**.** *Suppose that $\mathcal{C}$ is a $\mathbf{\Sigma_2}$ class of structures of the same type, and suppose that there exists a supercompact cardinal $\kappa$ larger than the rank of the parameters that appear in some $\Sigma_2$ definition of $\mathcal{C}$. Then for every $B \in \mathcal{C}$ there exists $A \in \mathcal{C} \cap V_\kappa$ that is elementarily embeddable into $B$.*

*Proof.* Fix a $\Sigma_2$ formula $\varphi(x, y)$ and a set $b$ such that $\mathcal{C} = \{a : \varphi(a, b)\}$, and suppose that $\kappa$ is a supercompact cardinal with $b \in V_\kappa$. Fix $B \in \mathcal{C}$, and let $\lambda \in C^{(2)}$ be greater than rank$(B)$. Let $j\colon V \to M$ be an elementary embedding with $M$ transitive and critical point $\kappa$, such that $j(\kappa) > \lambda$ and $M$ is closed under $\lambda$-sequences. Thus, $B$ and $j \restriction B : B \to j(B)$ are in $M$, and also $V_\lambda \in M$. Hence $V_\lambda \preceq_1 M$.

Since $V_\lambda \preceq_2 V$, $V_\lambda \models \varphi(B, b)$. And since $\Sigma_2$ formulas are upwards absolute between $V_\lambda$ and $M$, $M \models \varphi(B, b)$.

Thus, in $M$ it is true that there exists $X \in M_{j(\kappa)}$ such that $\varphi(X, b)$, namely $B$, and there exists an elementary embedding $e\colon X \to j(B)$, namely $j \restriction B$. Therefore, by elementarity, the same holds in $V$; that is, there exists $X \in V_\kappa$ such that $\varphi(X, b)$, and there exists an elementary embedding $e\colon X \to B$. □

The following corollaries give characterizations of Vopěnka's principle for $\Pi_1$ and $\Sigma_2$ classes (and therefore also for $\Delta_2$ classes) in terms of supercompactness. The equivalence of (2) and (3) was already proved in [1].

**Corollary 3.6.** *The following are equivalent:*

(1) $VP(\Pi_1)$.
(2) $VP(\kappa, \mathbf{\Sigma}_2)$, *for some* $\kappa$.
(3) *There exists a supercompact cardinal.*

**Corollary 3.7.** *The following are equivalent:*

(1) $VP(\mathbf{\Pi}_1)$.
(2) $VP(\kappa, \mathbf{\Sigma}_2)$, *for a proper class of cardinals* $\kappa$.
(3) *There exists a proper class of supercompact cardinals.*

Recall from Definition 3.1 that a cardinal $\kappa$ *reflects* a class of structures $\mathcal{C}$ if for every $B \in \mathcal{C}$ there exists $A \in \mathcal{C} \cap V_\kappa$ which is elementary embeddable into $B$. The following theorem is due to Magidor [4], which follows from his characterization of supercompact cardinals, namely: $\kappa$ is supercompact if and only if for a proper class of $\lambda$ greater than $\kappa$ there exists $\alpha < \kappa$ and an elementary embedding $j : V_\alpha \to V_\lambda$ such that $\kappa$ is the image under $j$ of the critical point (see [4, 3]).

**Theorem 3.8** (Magidor [4]). *If $\kappa$ is the least cardinal that reflects the* $\Pi_1$ *proper class* $\mathcal{C}$ *of structures of the form* $\langle V_\lambda, \in\rangle$, *then $\kappa$ is supercompact.*

*Proof.* For each $\lambda$ greater than $\kappa$ there is $\alpha < \kappa$ and an elementary embedding

$$j_\lambda : \langle V_\alpha, \in\rangle \to \langle V_\lambda, \in\rangle.$$

Let $\alpha$ be be the least ordinal for which there is such an embedding for a proper class of $\lambda$s. We may assume that the $j_\lambda$ are not the identity (otherwise, $V_\alpha$ would be an elementary substructure of $V$) and that the critical point of all these embeddings is the same, say $\beta$. Moreover, we may assume that the image of $\beta$ is always the same, for otherwise for a proper class of $\lambda$s, $j_\lambda \restriction \beta$ would witness that $V_\beta$ is an elementary substructure of $V_{j_\lambda(\beta)}$, with the $j_\lambda(\beta)$s forming a proper class, which in turn would imply that $V_\beta$ is an elementary substructure of $V$.

So let $\delta$ be least such that for a proper class of $\lambda$s the $\alpha$ is the same, $j_\lambda$ is not the identity, and the image of the critical point is $\delta$. By Magidor's characterization of supercompactness, $\delta$ is supercompact. Thus $\delta \geq \kappa$, because $\delta$ reflects $\mathcal{C}$, by Theorem 3.5, and $\kappa$ is the least cardinal that does this. So suppose, aiming for a contradiction, that $\delta > \kappa$. Then by Theorem 3.5, $\delta$ reflects the proper class of structures of the form $\langle V_\lambda, \in, \gamma\rangle$, where $\gamma$ is some ordinal less than $\lambda$, which is $\Pi_1$. Arguing similarly as above we have that for a proper class of $\lambda$s there are fixed $\gamma < \alpha < \kappa$ with an embedding from $\langle V_\alpha, \in, \gamma\rangle$ into $\langle V_\lambda, \in, \kappa\rangle$ whose image of the critical point is less or equal than $\kappa$, contradicting the minimality of $\delta$. □

From the last two Theorems we obtain a characterization of the first supercompact cardinal in terms of a natural form of reflection.

**Corollary 3.9.** *A cardinal $\kappa$ reflects all* $\Pi_1$ *proper classes of structures of the same type if and only if it is greater or equal than the first supercompact cardinal.*

David Asperó pointed out the following parameterized form of the Corollary above. One direction follows from Theorem 3.5 and by observing that the property of reflecting $\mathbf{\Pi}_1$ classes is closed under limits. The other direction can be proved similarly as in the Theorem above, but working, for any fixed $\xi < \kappa$, with the class of structures of the form $\langle V_\lambda, \in, \{\eta\}_{\eta<\xi}\rangle$, where $\beta$ is some ordinal less than $\lambda$, which is $\Pi_1$ definable in the parameter $\alpha$.

**Corollary 3.10.** *A cardinal $\kappa$ reflects all $\mathbf{\Pi}_1$ proper classes of structures of the same type if and only if either $\kappa$ is a supercompact cardinal or a limit of supercompact cardinals.*

We will prove similar results for classes of higher complexity, for which we shall need $C^{(n)}$-extendible cardinals.

**Theorem 3.11.** *For every $n \geq 1$, if $\kappa$ is a $C^{(n)}$–extendible cardinal, then $VP(\kappa, \mathbf{\Sigma}_{n+2})$ holds.*

*Proof.* Fix a $\Sigma_{n+2}$ formula $\exists x\varphi(x, y, z)$, where $\varphi$ is $\Pi_{n+1}$, such that for some set $b \in V_\kappa$,

$$\mathcal{C} := \{B : \exists x\varphi(x, B, b)\}$$

is a proper class of structures of the same type.

Fix $B \in \mathcal{C}$ and let $\lambda \in C^{(n+2)}$ be greater than $\kappa$ and the ranks of $b$ and $B$. Thus,

$$V_\lambda \models \exists x\varphi(x, B, b).$$

Let $j : V_\lambda \to V_\mu$ be an elementary embedding with critical point $\kappa$, with $j(\kappa) > \lambda$, and $j(\kappa) \in C^{(n)}$. Note that $B$ and $j \restriction B : B \to j(B)$ are in $V_\mu$.

As $\kappa, \lambda \in C^{(n+2)}$ (see Proposition 2.4), it follows that $V_\kappa \preceq_{n+2} V_\lambda$. And this implies that $V_{j(\kappa)} \preceq_{n+2} V_\mu$. Indeed, we have that

$$V_\lambda \models \text{“}\forall x \in V_\kappa \forall\theta \in \Sigma_{n+2}(V_\kappa \models \theta(x) \leftrightarrow \models_{n+2} \theta(x))\text{”}.$$

Hence, by elementarity,

$$V_\mu \models \text{“}\forall x \in V_{j(\kappa)} \forall\theta \in \Sigma_{n+2}(V_{j(\kappa)} \models \theta(x) \leftrightarrow \models_{n+2} \theta(x))\text{”},$$

which implies $V_{j(\kappa)} \preceq_{n+2} V_\mu$.

Since $j(\kappa) \in C^{(n)}$, we also have $V_\lambda \preceq_{n+1} V_{j(\kappa)}$, and therefore $V_\lambda \preceq_{n+1} V_\mu$. It follows that $V_\mu \models \exists x\varphi(x, B, b)$.

Thus, in $V_\mu$ it is true that there exists $X \in V_{j(\kappa)}$ such that $X \in \mathcal{C}$, namely $B$, and there exists an elementary embedding $e : X \to j(B)$, namely $j \restriction B$. Therefore, by elementarity of $j$, the same is true in $V_\lambda$, that is, there exists $X \in V_\kappa$ such that $X \in \mathcal{C}$, and there exists an elementary embedding $e : X \to B$. Let $A \in V_\kappa$ be such an $X$, and let $e : A \to B$ be an elementary embedding. Since $\lambda \in C^{(n+2)}$, $A \in \mathcal{C}$, and we are done. □

The next Theorem will yield a strong converse to Theorem 3.11.

The notion of $C^{(n)}$-extendibility used in [1] has the following (apparently) stronger form – let us call it $C^{(n)+}$-extendibility: For $\lambda \in C^{(n)}$, a cardinal $\kappa$ is $\lambda$-$C^{(n)+}$-*extendible* if it is $\lambda$-$C^{(n)}$-extendible, witnessed by some $j : V_\lambda \to V_\mu$ which, in addition to satisfying $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$, it also satisfies that $\mu \in C^{(n)}$.

$\kappa$ is $C^{(n)+}$-*extendible* if it is $\lambda$-$C^{(n)+}$-extendible for every $\lambda > \kappa$ with $\lambda \in C^{(n)}$.

Every extendible cardinal is $C^{(1)+}$-extendible (see [1], or the proof of Proposition 2.3 above).

**Theorem 3.12.** *Suppose $n \geq 1$. If $VP(\Pi_{n+1})$ holds, then there exists a $C^{(n)+}$-extendible cardinal.*

*Proof.* Suppose there are no $C^{(n)+}$-extendible cardinals. Then the class function $F$ on the ordinals given by:

$F(\alpha)$ = the least $\lambda \in C^{(n+1)}$ greater than $\alpha$ such that $\alpha$ is not $\lambda$-$C^{(n)+}$-extendible,

is defined for all ordinals $\alpha$.

Let $C = \{\eta > 0 : \forall \alpha < \eta \; F(\alpha) < \eta\}$. So $C$ is a closed unbounded proper class of ordinals contained in $C^{(n+1)}$.

We claim that $C$ is $\Pi_{n+1}$ definable, without parameters. First note that $F$ is $\Pi_{n+1}$ definable, for $\lambda = F(\alpha)$ iff

(1) $\lambda \in C^{(n+1)}$
(2) $\alpha < \lambda$
(3) $\forall \beta > \lambda (\beta \in C^{(n)} \rightarrow V_\beta \models (\alpha \text{ is not } \lambda\text{-}C^{(n)+}\text{-extendible}))$, and
(4) $V_\lambda \models \forall \lambda' > \alpha (\lambda' \in C^{(n+1)} \rightarrow (\alpha \text{ is } \lambda'\text{-}C^{(n)+}\text{-extendible}))$.

The point is that, for any $\alpha < \lambda'$, the relation "$\alpha$ is $\lambda'$-$C^{(n)+}$-extendible" is $\Sigma_{n+1}$, for it holds iff

$$\exists \mu \exists j (j : V_{\lambda'} \rightarrow V_\mu \wedge j \text{ elementary} \wedge crit(j) = \alpha \wedge j(\alpha) > \lambda' \wedge j(\alpha), \mu \in C^{(n)}).$$

So it holds in $V$ if and only if it holds in $V_\lambda$, for any $\lambda \in C^{(n+1)}$ greater than $\lambda'$. And if it holds in $V_\beta$, with $\beta \in C^{(n)}$, then it holds in $V$. Moreover, since $\lambda \in C^{(n+1)}$, for every $\lambda' < \lambda$ we have $\lambda' \in C^{(n+1)}$ if and only if $V_\lambda \models \lambda' \in C^{(n+1)}$.

It follows that $C$ is also $\Pi_{n+1}$ definable.

For each ordinal $\alpha$, let $\lambda_\alpha$ be the least limit point of $C$ greater than $\alpha$. We have that $x = \lambda_\alpha$ iff $x$ is an ordinal greater than $\alpha$ that belongs to $C$ and such that

(1) $V_x \models \forall \beta \exists \gamma (\gamma > \beta \wedge \gamma \in C)$
(2) $V_x \models \forall \beta (\beta > \alpha \rightarrow \exists \gamma < \beta \forall \eta (\gamma < \eta < \beta \rightarrow \eta \notin C))$,

which shows that the function $\alpha \longmapsto \lambda_\alpha$ is $\Pi_{n+1}$ definable.

Consider now the proper class $\mathcal{C}$ of structures $\mathcal{A}_\alpha$ of the form

$$\langle V_{\lambda_\alpha}, \in, \alpha, \lambda_\alpha, C \cap \alpha + 1 \rangle,$$

where $\alpha \in C$.

We claim that $\mathcal{C}$ is $\Pi_{n+1}$ definable. We have: $X \in \mathcal{C}$ if and only if $X = \langle X_0, X_1, X_2, X_3, X_4 \rangle$, where

(1) $X_2 \in C$
(2) $X_3 = \lambda_{X_2}$
(3) $X_0 = V_{X_3}$
(4) $X_1 = \in \restriction X_0$
(5) $X_4 = C \cap X_2 + 1$

We have already seen that (1) and (2) are $\Pi_{n+1}$ expressible. And the same is true for (3) and (4), as one can easily see. As for (5), note that $X_4 = C \cap \alpha + 1$ holds in $V$ iff it holds in $V_{X_3}$.

So by $VP(\Pi_{n+1})$ there exist $\alpha \neq \beta$ and an elementary embedding

$$j : \mathcal{A}_\alpha \rightarrow \mathcal{A}_\beta.$$

Since $j$ must send $\alpha$ to $\beta$, $j$ is not the identity. So $j$ has critical point some $\kappa \leq \alpha$.

We claim that $\kappa \in C$. Otherwise, $\gamma := sup(C \cap \kappa) < \kappa$. Let $\delta$ be the least ordinal in $C$ greater than $\gamma$ such that $\delta < \lambda_\alpha$. So $\kappa < \delta \leq \alpha$. Since $\delta$ is definable from $\gamma$ in $\mathcal{A}_\alpha$, and since $j(\gamma) = \gamma$, we must also have $j(\delta) = \delta$. But then $j \restriction V_{\delta+2} : V_{\delta+2} \to V_{\delta+2}$ is a nontrivial elementary embedding, contradicting Kunen's Theorem (see [3]).

Hence by elementarity, $j(\kappa) \in C$.

Since $\alpha \in C$, we have $\kappa < F(\kappa) < \alpha$. Thus,

$$j \restriction V_{F(\kappa)} : V_{F(\kappa)} \to V_{j(F(\kappa))}$$

is elementary, with critical point $\kappa$.

And since $j(\kappa) \in C$, $F(\kappa) < j(\kappa)$. Moreover, by elementarity of $j$, $V_{\lambda_\beta}$ satisfies that $j(F(\kappa))$ belongs to $C^{(n)}$, so since $\lambda_\beta \in C^{(n+1)}$ this is true in $V$. This shows that $j \restriction V_{F(\kappa)}$ witnesses that $\kappa$ is $F(\kappa)$-$C^{(n)+}$-extendible. But this is impossible, by definition of $F$. □

The proof of the last theorem can easily be adapted to prove the parameterized version: if $VP(\mathbf{\Pi}_{n+1})$ holds, then there is a proper class of $C^{(n)}$–extendible cardinals. (See the proof of Proposition 3.12 (2).)

Note that from the last two theorems it follows that there exists a $C^{(n)}$-extendible cardinal if and only if there exists a $C^{(n)+}$-extendible cardinal. The following corollaries summarize the results above (one may replace $C^{(n)}$-extendible by $C^{(n)+}$-extendible).

**Corollary 3.13.** *The following are equivalent:*

(1) $VP(\Pi_2)$.
(2) $VP(\kappa, \mathbf{\Sigma}_3)$*, for some* $\kappa$.
(3) *There exists an extendible cardinal.*

**Corollary 3.14.** *The following are equivalent for* $n \geq 1$*:*

(1) $VP(\Pi_{n+1})$.
(2) $VP(\kappa, \mathbf{\Sigma}_{n+2})$*, for some* $\kappa$.
(3) *There exists a* $C^{(n)}$*-extendible cardinal.*

**Corollary 3.15.** *The following are equivalent:*

(1) $VP(\Pi_n)$*, for every* $n$.
(2) $VP(\kappa, \mathbf{\Sigma}_n)$*, for a proper class of cardinals* $\kappa$*, and for every* $n$.
(3) $VP$
(4) *For every* $n$*, there exists a* $C^{(n)}$*-extendible cardinal.*

We shall next obtain characterizations of $C^{(n)}$-extendible cardinals in terms of natural reflection principles.

**Theorem 3.16.** *If* $\kappa$ *is the least cardinal that reflects all* $\Pi_{n+1}$*-classes of structures of the same type, then* $\kappa$ *is* $C^{(n)}$*-extendible.*

*Proof.* Suppose otherwise. Then by 3.11 there is no $C^{(n)}$-extendible cardinal less or equal than $\kappa$.

Consider the class $\mathcal{C}$ of structures of the form $\langle V_\xi, \in, \lambda, \alpha, C^{(n)} \cap \xi \rangle$, where $\alpha < \lambda < \xi$, and

(1) $\lambda \in C^{(n)}$,
(2) $\xi \in Lim(C^{(n)})$,
(3) the cofinality of $\xi$ is uncountable,
(4) $\forall \beta < \xi \, \forall \mu (\exists j (j : V_\lambda \to V_\mu \wedge crit(j) = \alpha \wedge j(\alpha) = \beta) \to$
$\exists j' \exists \mu' (j' : V_\lambda \to V'_\mu \wedge \mu' < \xi \wedge crit(j') = \alpha \wedge j'(\alpha) = \beta))$, and
(5) $\lambda$ witnesses that no ordinal less or equal than $\alpha$ is $\lambda$-$C^{(n)}$-extendible.

Clearly, $\mathcal{C}$ is a $\Pi_{n+1}$ definable proper class. So let

$$j : \langle V_{\xi'}, \in, \lambda', \alpha' \rangle \to \langle V_\xi, \in, \lambda, \kappa \rangle$$

with $\langle V_{\xi'}, \in, \lambda', \alpha' \rangle$, $\langle V_\xi, \in, \lambda, \kappa \rangle \in \mathcal{C}$ and $\xi' < \kappa$.

Let $\alpha = crit(j)$. We claim that $\alpha \in C^{(n)}$. Otherwise, let $\gamma := sup(C^{(n)} \cap \alpha)$, so $\gamma < \alpha$. Let $\delta \in C^{(n)}$ be the least such that $\gamma < \delta < \xi'$. Since $\delta$ is definable from $\gamma$ in $\langle V_{\xi'}, \in, \lambda', \alpha', C^{(n)} \cap \xi' \rangle$ and $j(\gamma) = \gamma$, also $j(\delta) = \delta$. Hence $j \restriction V_{\delta+s} \to V_{\delta+2}$, comtradicting Kunen's Theorem.

If $j^m(\alpha) < \xi'$ for all $m$, then $\{j^m(\alpha)\}_{m \in \omega} \in V_{\xi'}$, because $\xi'$ has uncountable cofinality, contradicting Kunen's Theorem. So suppose for some $m$ we have $j^m(\alpha) < \xi' \leq j^{m+1}(\alpha)$.

We claim that there exists an elementary embedding $k : V_{\lambda'} \to V_\mu$, some $\mu$, with $crit(k) = \alpha$ and $k(\alpha) = j^{m+1}(\alpha)$. We prove this by induction on $i \leq m$. For $i = 0$ take $k = j \restriction V_{\lambda'}$. So suppose it true for $i < m$. Since $j^{i+1}(\alpha) < \xi'$, by (3) above, there exist $j'$ and $\mu'$ such that $j' : V_{\lambda'} \to V_{\mu'}$ is elementary, $\mu' < \xi'$, $crit(j') = \alpha$, and $j'(\alpha) = j^{i+1}(\alpha)$. Applying $j$ to $j'$ we have $j(j') : V_{j(\lambda')} \to V_{j(\mu')}$ has critical point $j(\alpha)$ and $j(j')(\alpha) = j^{i+2}(\alpha)$. Let

$$k = j(j') \restriction V_{\lambda'} : V_{\lambda'} \to V_{j(j')(\lambda')}.$$

Thus $k$ has critical point $\alpha$ and $k(\alpha) = j^{i+2}(\alpha)$, which proves our claim.

Note that since $\alpha, \xi', \xi \in C^{(n)}$, we have $j^{m+1}(\alpha) \in C^{(n)}$. Thus, $k$ witnesses that $\alpha$ is $\lambda'$-$C^{(n)}$-extendible, contradicting (4) above. □

**Corollary 3.17.** *A cardinal $\kappa$ reflects all $\Pi_{n+1}$-classes of structures of the same type (where $n \geq 1$) if and only if it is greater or equal than the first $C^{(n)}$-extendible cardinal.*

The parameterized version also follows.

**Theorem 3.18.** *A cardinal $\kappa$ reflects all $\mathbf{\Pi}_{n+1}$ proper classes of structures of the same type if and only if either $\kappa$ is a $C^{(n)}$-extendible cardinal or a limit of $C^{(n)}$-extendible cardinals.*

## 4. $C^{(n)}$-huge cardinals

Recall that a cardinal $\kappa$ is *$m$-huge* if it is the critical point of an elementary embedding $j : V \to M$ with $M$ transitive and closed under $j^m(\kappa)$-sequences, where $j^m$ is the $m$-th iterate of $j$ (i.e., $j^1 = j$ and $j^{m+1} = j \circ j^m$). A cardinal is called *huge* if it is 1-huge. Let us say that $\kappa$ is *$C^{(n)}$-$m$-huge* if it is $m$-huge and $j(\kappa) \in C^{(n)}$. And let us say that $\kappa$ is *$C^{(n)}$-huge* if it is huge and $j(\kappa) \in C^{(n)}$.

As with $m$-huge cardinals, $C^{(n)}$-$m$-huge cardinals can be characterized in terms of normal measures. To wit: $\kappa$ is $C^{(n)}$-$m$-huge if and only if it is uncountable and there is a $\kappa$-complete normal ultrafilter $\mathcal{U}$ over some $\mathcal{P}(\lambda)$

and cardinals $\kappa = \lambda_0 < \lambda_1 < \ldots < \lambda_m = \lambda$, with $\lambda_1 \in C^{(n)}$, and such that for each $i < m$,

$$\{x \in \mathcal{P}(\lambda) : ot(x \cap \lambda_{i+1}) = \lambda_i\} \in \mathcal{U}.$$

(See [3] 24.8 for a proof of the case $n = 1$, which also works for arbitrary $n$.) It follows that "$\kappa$ is $C^{(n)}$-$m$-huge" is $\Sigma_{n+1}$ expressible.

Clearly, every huge cardinal is $C^{(1)}$-huge. But the first huge cardinal is not $C^{(2)}$-huge. For suppose $\kappa$ is the least huge cardinal and $j : V \to M$ witnesses that $\kappa$ is $C^{(2)}$-huge. Then since "$x$ is huge" is $\Sigma_2$ expressible, we have

$$V_{j(\kappa)} \models \text{"}\kappa \text{ is huge"}.$$

Hence, since $(V_{j(\kappa)})^M = V_{j(\kappa)}$,

$$M \models \text{"}\exists \delta < j(\kappa)(V_{j(\kappa)} \models \text{"}\delta \text{ is huge"})\text{"}.$$

By elementarity, there is a huge cardinal less than $\kappa$ in $V$, which is absurd.

A similar argument, using that "$\kappa$ is $C^{(n)}$-$m$-huge" is $\Sigma_{n+1}$ expressible, for all $m$, shows that the first $C^{(n)}$-$m$-huge cardinal is not $C^{(n+1)}$-$m$-huge, for all $m$ and $n$ greater or equal than 1.

## 5. On elementary embeddings of a rank into itself

Consider the large cardinal principle, known as I3, which asserts the existence of a non-trivial elementary embedding $j : V_\delta \to V_\delta$ (see [3], 24). Let us call the critical point of such an embedding an *I3 cardinal*.

If $j : V_\delta \to V_\delta$ witnesses that $\kappa$ is I3, then Kunen's Theorem implies that either $\delta$ is a limit ordinal or is of the form $\gamma + 1$ with $\gamma$ a limit ordinal (see [3], 24). And if $\delta$ is a limit ordinal, then $\delta = sup\{j^m(\kappa) : m \in \omega\}$, where $j^m$ is the m-th iterate of $j$. It follows that $\delta \in C^{(1)}$, because all the $j^m(\kappa)$ are measurable cardinals and therefore they all belong to $C^{(1)}$.

Now suppose $j : V_\delta \to V_\delta$ witnesses that $\kappa$ is I3, with $\delta$ a limit ordinal. Then $V_\kappa$ and $V_{j^m(\kappa)}$, all $m$, are elementary substructures of $V_\delta$. In particular, $V_\delta$ is a model of ZFC. Moreover, it is easily seen that in $V_\delta$ the cardinal $\kappa$, and all the $j^m(\kappa)$, are $C^{(n)}$-supercompact, $C^{(n)}$-extendible, and $C^{(n)}$-$m$-huge, for all $n$ and $m$. In particular, $V_\delta$ satisfies VP. Thus the consistency of the existence of an I3 cardinal implies the consistency of the existence of all $C^{(n)}$-cardinals considered in previous sections.

Let us now say that $\kappa$ is a *$C^{(n)}$-I3 cardinal* if it is an I3 cardinal, witnessed by some embedding $j : V_\delta \to V_\delta$, with $j(\kappa) \in C^{(n)}$.

Observe that if $\kappa$ is $C^{(n)}$-I3, then $\kappa \in C^{(n)}$. For if $\varphi$ is a $\Sigma_n$ sentence with parameters in $V_\kappa$ that holds in $V$, then it must hold in $V_{j(\kappa)}$ as well, for some $j : V_\delta \to V_\delta$ that witnesses the fact that $\kappa$ is $C^{(n)}$-I3. But then, since $j$ fixes the parameters of $\varphi$, by elementarity $\varphi$ also holds in $V_\kappa$.

If $\kappa = crit(j)$, then $\kappa$ and $j(\kappa)$ are measurable cardinals. Thus every I3-cardinal is $C^{(1)}$-I3. However, a simple reflection argument shows that the least $C^{(n)}$-I3 cardinal is not $C^{(n+1)}$-I3, for $n \geq 1$. For suppose $\kappa$ is the least $C^{(n)}$-I3 cardinal ($n \geq 1$), and suppose, towards a contradiction, that there is some $j : V_\delta \to V_\delta$ with $crit(j) = \kappa$, and $j(\kappa) \in C^{(n+1)}$. Then $V_{j(\kappa)}$ satisfies the following $\Sigma_{n+1}$ statement

$$\exists \beta \exists k \exists \lambda (k : V_\beta \to V_\beta \text{ is elementary} \wedge crit(k) = \lambda \wedge k(\lambda) \in C^{(n)})$$

because $\delta$, $j$, and $\kappa$ are such $\beta$, $k$, and $\lambda$, respectively. By elementarity the same holds in $V_\kappa$. Hence, since $\kappa \in C^{(n)}$, it also holds in $V$, contradicting our assumption that $\kappa$ was the least $C^{(n)}$-I3 cardinal.

**Proposition 5.1.** *Suppose $j : V_\delta \to V_\delta$ witnesses that $\kappa$ is I3, and $\delta$ is a limit ordinal. Then the following are equivalent for every $n \geq 1$:*

(1) *$j^m(\kappa) \in C^{(n)}$, all $m < \omega$.*
(2) *$\delta \in C^{(n)}$.*

*Proof.* For $n = 1$, (1) and (2) are true.

(1) implies (2) is immediate, since $\delta = sup\{j^m(\kappa) : m < \omega\}$.

Let's prove (2) implies (1) for n+1, assuming it holds for $n$. Fix $m < \omega$. Let $\exists x\varphi(x)$ be a $\Sigma_{n+1}$ formula whose parameters, if any, are in $V_{j^m(\kappa)}$, and suppose the formula is true in $V$. Then, by (2), it is also true in $V_\delta$. Let $\ell \geq m$ be big enough so that $V_{j^\ell(\kappa)}$ contains a witness to the formula. Then $V_\delta$ satisfies:

$$\exists x \in V_{j^\ell(\kappa)}\, \varphi(x).$$

Hence by elementarity of $j^{\ell-m}$, $V_\delta$ satisfies that there exists $x \in V_{j^m(\kappa)}$ such that $\varphi(x)$ holds. If $a$ is such an $x$, then since by inductive hypothesis $V_{j^m(\kappa)} \preceq_n V_\delta$, we have that $\varphi(a)$ holds in $V_{j^m(\kappa)}$. □

So let us say that $\kappa$ is $C^{(n)+}$-I3 if it is I3, witnessed by $j : V_\delta \to V_\delta$, with $\delta \in C^{(n)}$. Clearly, $\kappa$ is I3 if and only if it is $C^{(1)}$-I3 if and only if it is $C^{(1)+}$-I3.

**Proposition 5.2.** *If $\kappa$ is $C^{(n)}$-I3, then it is $C^{(n)}$-m-huge, for all m, and there is a normal ultrafilter $\mathcal{U}$ over $\kappa$ such that*

$$\{\alpha < \kappa : \alpha \textit{ is } C^{(n)}\textit{-m-huge for every } m\} \in \mathcal{U}.$$

*Proof.* Let $j : V_\delta \to V_\delta$ witness that $\kappa$ is $C^{(n)}$-I3, with $\delta$ limit. Then as in [3] 24.8 one can show that the ultrafilter $\mathcal{V}$ over $\mathcal{P}(\lambda)$, where $\lambda = j^m(\kappa)$, defined by

$$X \in \mathcal{V} \text{ if and only if } j''\lambda \in j(X)$$

witnesses that $\kappa$ is $C^{(n)}$-$m$-huge. Now let $\mathcal{U}$ be the usual normal ultrafilter over $\kappa$ obtained from $j$. Since $V_\delta$ satisfies that $\kappa$ is $C^{(n)}$-$m$-huge, by normality of $\mathcal{U}$ the Proposition follows. □

Let us call a cardinal $\kappa$ a *I1 cardinal* if there exists an elementary embedding $j : V_{\delta+1} \to V_{\delta+1}$ with $crit(j) = \kappa$. By Kunen's Theorem, $\delta$ must be a limit ordinal. So let us say that $\kappa$ is *$C^{(n)}$-I1* if it is I1, witnessed by $j : V_{\delta+1} \to V_{\delta+1}$, with $j(\kappa) \in C^{(n)}$. Clearly, if $\kappa$ is $C^{(n)}$-I1, then it is also $C^{(n)}$-I3.

As with $C^{(n)}$-I3 cardinals, a simple reflection argument shows that the first $C^{(n)}$-I1 cardinal is not $C^{(n+1)}$-I1, for all $n \geq 1$.

Let us also observe that if $\kappa$ is $C^{(n)}$-I1, then the least $\delta$ for which there is an elementary embedding $j : V_{\delta+1} \to V_{\delta+1}$ with $crit(j) = \kappa$ and $j(\kappa) \in C^{(n)}$ is smaller than the first ordinal in $C^{(n+1)}$ greater than $\kappa$. Moreover, the least $C^{(n)}$-I1 cardinal, if it exists, is smaller than the first ordinal in $C^{(n+1)}$, for all $n \geq 1$.

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