$C^{(n)}$-Cardinals

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C\(^{(n)}\)-CARDINALS

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ABSTRACT. For each natural number \( n \), let \( C^{(n)} \) be the closed and unbounded proper class of ordinals \( \alpha \) such that \( V_{\alpha} \) is a \( \Sigma_n \)-elementary substructure of \( V \). A \( C^{(n)} \)-cardinal is a cardinal \( \kappa \) that is the critical point of an elementary embedding \( j: V \rightarrow M, M \) transitive, with \( j(\kappa) \in C^{(n)} \). We show that the \( C^{(n)} \)-cardinals form a hierarchy that refines the usual large cardinal hierarchy, and we give characterizations of various natural reflection principles in terms of such cardinals. In particular, building on results of [1], we give sharper formulations of Vopěnka’s Principle in terms of \( C^{(n)} \)-extendible cardinals.

For each natural number \( n \), let \( C^{(n)} \) denote the club (i.e., closed and unbounded) proper class of ordinals \( \alpha \) that are \( \Sigma_n \)-correct in the universe \( V \) of all sets, meaning that \( V_{\alpha} \) is a \( \Sigma_n \)-elementary substructure of \( V \), and written \( V_{\alpha} \preceq_n V \). Thus, \( C^{(0)} \) is the class of all ordinals. But if \( V_{\alpha} \preceq_1 V \), then \( \alpha \) is already an uncountable strong limit cardinal, for if \( \beta < \alpha \), then the sentence
\[
\exists \gamma \exists f(\gamma \text{ an ordinal } \land f : \gamma \rightarrow V_\beta \text{ is onto})
\]
is \( \Sigma_1 \) in the parameter \( V_\beta \), and therefore it must hold in \( V_{\alpha} \). Further, if \( \alpha \in C^{(1)} \), then \( V_{\alpha} = H(\alpha) \). Thus \( C^{(1)} \) is precisely the class of all uncountable cardinals \( \alpha \) such that \( V_{\alpha} = H(\alpha) \). It follows that \( C^{(1)} \) is \( \Pi_1 \) definable, for \( \alpha \in C^{(1)} \) iff \( \alpha \) is a cardinal and
\[
\forall M (M \text{ a transitive model of } ZFC^* \land \alpha \in M \rightarrow M \vDash V_{\alpha} = H(\alpha)).
\]
(Here \( ZFC^* \) denotes a sufficiently large finite fragment of \( ZFC \).) The point is that if \( \alpha \in C^{(1)} \) and \( M \) is a transitive model of \( ZFC^* \) that contains \( \alpha \), then if in \( M \) we could find some transitive \( x \in V_{\alpha} \setminus H(\alpha) \), we would have that \( |x| \geq \alpha \). But since in \( V \) the cardinality of \( x \) is less than \( \alpha \), because \( V_{\alpha} = H(\alpha) \), this would mean that, in \( V \), \( \alpha \) is not a cardinal, which is absurd.

More generally, since the truth predicate \( \vDash_n \) for \( \Sigma_n \) sentences (for \( n \geq 1 \)) is \( \Sigma_n \) definable (see [3], section 0.2), and since the relation \( x = V_{\gamma} \) is \( \Pi_1 \), the class \( C^{(n)} \) is \( \Pi_n \) definable: \( \alpha \in C^{(n+1)} \) iff
\[
\alpha \in C^{(n)} \land \forall \varphi(x) \in \Sigma_{n+1} \forall \alpha \in V_{\alpha} (\vDash_n \varphi(\alpha) \rightarrow V_{\alpha} \vDash \varphi(\alpha)).
\]
Let us remark that \( C^{(n)} \), for \( n \geq 1 \), cannot be \( \Sigma_n \) definable. Otherwise, if \( \alpha \) is the least ordinal in \( C^{(n)} \), then the sentence “there is some ordinal in \( C^{(n)} \)” would be \( \Sigma_n \) and so it would hold in \( V_{\alpha} \), yielding an ordinal in \( C^{(n)} \) smaller than \( \alpha \), which is impossible.

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The classes $C(n)$, $n \geq 1$, form a basis for definable club proper classes of ordinals, in the sense that every $\Sigma_n$ club proper class of ordinals contains $C(n)$. More generally, every club proper class $C$ of ordinals that is $\Sigma_n$-definable (i.e., $\Sigma_n$-definable with parameters) contains all $\alpha \in C(n)$ that are greater than the rank of the parameters involved in any $\Sigma_n$ definition of $C$.

Finally, note that since the least ordinal in $C(n)$ does not belong to $C(n+1)$, $C(n+1) \subset C(n)$, all $n$.

When considering non-trivial elementary embeddings $j : V \to M$, with $M$ transitive, one would like to have some control over where the image $j(\kappa)$ of the critical point $\kappa$ goes. A especially interesting case is when one wants $V_{j(\kappa)}$ to reflect some specific property of $V$ or, more generally, when one wants $j(\kappa)$ to belong to a particular definable club proper class of ordinals.

Now, since the $C(n)$, $n \in \omega$, form a basis for such classes, the problem can be reformulated as follows: when can we have $j(\kappa) \in C(n)$, for a given $n \in \omega$?

This prompts the following definition.

Let us say that a cardinal $\kappa$ is $C(n)$-measurable if there is an elementary embedding $j : V \to M$, some transitive class $M$, with critical point $\text{crit}(j) = \kappa$ and with $j(\kappa) \in C(n)$.

Observe that if $j : V \to M \cong \text{Ult}(V, U)$, $M$ transitive, is the ultrapower elementary embedding obtained from a non-principal $\kappa$-complete ultrafilter $U$ on $\kappa$, then $2^\kappa < j(\kappa) < (2^\kappa)^+$ (see [3]). Hence, since $V_{j(\kappa)} \preceq V$ implies that $j(\kappa)$ is a (strong limit) cardinal, $j$ cannot witness the $C(1)$-measurability of $\kappa$. Nonetheless, by using iterated ultrapowers (see [2], 19.15), one has that for every cardinal $\alpha > 2^\kappa$, the $\alpha$-th iterated ultrapower embedding $j_\alpha : V \to M_\alpha \cong \text{Ult}(V, \mathcal{U}_\alpha)$, where $\mathcal{U}_\alpha$ is the $\alpha$-th iterate of $U$, has critical point $\kappa$ and $j_\alpha(\kappa) = \alpha$. So, if $\kappa$ is measurable, then for each $n$ we can always find an elementary embedding $j : V \to M$, $M$ transitive, with $j(\kappa) \in C(n)$.

We have thus shown the following.

**Proposition 0.1.** Every measurable cardinal is $C(n)$-measurable, for all $n$.

A similar situation occurs in the case of strong cardinals.

Let us say that a cardinal $\kappa$ is $C(n)$-strong if for every $\lambda > \kappa$, $\kappa$ is $\lambda$-$C(n)$-strong, that is, there exists an elementary embedding $j : V \to M$, $M$ transitive, with critical point $\kappa$, and such that $j(\kappa) > \lambda$, $V_\lambda \subseteq M$, and $j(\kappa) \in C(n)$. Equivalently (see [3] 26.7), $\kappa$ is $\lambda$-$C(n)$-strong iff there exists a $(\kappa, \beta)$-extender $E$, for some $\beta > |V_\lambda|$, with $V_\lambda \subseteq M_E$ and $\lambda < j_E(\kappa) \in C(n)$.

Suppose now that $j : V \to M$ witnesses the $\lambda$-strongness of $\kappa$, with $j(\kappa)$ not necessarily in $C(n)$. Let $E$ be the $(\kappa, j(\kappa))$-extender obtained from $j$, and let $j_E : V \to M_E$ be the corresponding $\lambda$-strong embedding (see [3]). Then in $M_E$, $E' := j_E(E)$ is a $(j_E(\kappa), j_E(j(\kappa)))$-extender, which gives rise to an elementary embedding $j_{E'} : M_E \to M_{E'}$ with critical point $j_E(\kappa)$. Still in $M_E$, let $U$ be the $j_E(\kappa)$-complete ultrafilter on $j_E(\kappa)$ derived from $j_{E'}$, i.e.,

$$U = \{ X \subseteq j_E(\kappa) : j_E(\kappa) \in j_{E'}(X) \}$$

and let $j_U : M_E \to M$ be the corresponding elementary embedding. Then we can iterate $j_U$ $\alpha$-times, for some $\alpha \in C(n)$ greater than $2^{j_E(\kappa)}$, so that if $j_\alpha : M_E \to M_\alpha$ is the resulting elementary embedding, then $j_\alpha(j_E(\kappa)) = \alpha$. 


Letting $k := j_\alpha \circ j_E$, we have that $k : V \rightarrow M_\alpha$ is a $\lambda$-strong elementary embedding with critical point $\kappa$ and with $k(\kappa) \in C^{(n)}$. Thus we have shown the following.

**Proposition 0.2.** Every $\lambda$-strong cardinal is $\lambda$-$C^{(n)}$-strong, for all $n$. Hence, every strong cardinal is $C^{(n)}$-strong, for every $n$.

Thus for measurable or strong cardinals $\kappa$, the requirement that $j(\kappa) \in C^{(n)}$ for the corresponding elementary embeddings $j : V \rightarrow M$ does not yield stronger large cardinal notions. But, as we shall see next, the situation changes completely in the case of superstrong embeddings, that is, when $j$ is such that $V_j(\kappa) \subseteq M$.

## 1. $C^{(n)}$-Superstrong Cardinals

In the case of superstrong cardinals $\kappa$, the requirement that $j(\kappa) \in C^{(n)}$, for $n > 1$, produces a hierarchy of ever stronger large cardinal principles.

**Definition 1.1.** A cardinal $\kappa$ is $C^{(n)}$-superstrong if there exists an elementary embedding $j : V \rightarrow M$, $M$ transitive, with critical point $\kappa$, $V_j(\kappa) \subseteq M$, and $j(\kappa) \in C^{(n)}$.

Note that in the definition above, from the fact that $j(\kappa) \in C^{(1)}$ and $V_j(\kappa) \subseteq M$ it already follows that $\kappa \in C^{(n)}$. Thus, every $C^{(n)}$-superstrong cardinal belongs to $C^{(n)}$.

**Proposition 1.2.** If $j : V \rightarrow M$ is an elementary embedding such that $V_j(\kappa) \subseteq M$, where $\kappa = \text{crit}(j)$, then $j(\kappa) \in C^{(1)}$. Hence, every superstrong cardinal is $C^{(1)}$-superstrong.

**Proof.** Since $\kappa \in C^{(1)}$, $M$ thinks that $j(\kappa) \in C^{(1)}$, i.e., $M$ thinks $j(\kappa)$ is a strong limit cardinal and $V_j(\kappa) = H(j(\kappa))$. But then, since $(V_j(\kappa))^M = V_j(\kappa)$, $j(\kappa)$ is, in $V$, a strong limit cardinal with $V_j(\kappa) = H(j(\kappa))$, and therefore $j(\kappa) \in C^{(1)}$.

Observe that for $n \geq 1$, the sentence \("\kappa \text{ is } C^{(n)}\)-superstrong" \(\text{is } \Sigma_{n+1}\), for $\kappa$ is $C^{(n)}$-superstrong iff

\[
\exists \beta \exists \mu \exists E (\beta < \mu \land \beta, \mu \in C^{(n)} \land E \text{ is a } (\kappa, \beta)-\text{extender} \land E \in V_\mu \land V_\mu \models "V_{j_E(\kappa)} \subseteq M_E".)
\]

**Proposition 1.3.** For every $n \geq 1$, if $\kappa$ is $C^{(n+1)}$-superstrong, then there is a $\kappa$-complete normal ultrafilter $U$ over $\kappa$ such that

\[
\{ \alpha < \kappa : \alpha \text{ is } C^{(n)}\text{-superstrong} \} \in U.
\]

Hence, the first $C^{(n)}$-superstrong cardinal $\kappa$, if it exists, is not $C^{(n+1)}$-superstrong.

**Proof.** Suppose $\kappa$ is $C^{(n+1)}$-superstrong, witnessed by a normal $(\kappa, \beta)$-extender $E$ with associated elementary embedding $j_E = j : V \rightarrow M$ such that $\beta = j(\kappa)$ and $V_j(\kappa) \subseteq M$. Since $j(\kappa) \in C^{(n+1)}$,

\[
V_{j(\kappa)} \models "\kappa \text{ is } C^{(n)}\text{-superstrong}."
\]
And since \( \kappa \in C(n) \), \( M \models "j(\kappa) \in C(n)" \). Hence, since \( V_{j(\kappa)} = (V_{j(\kappa)})^M \), and since "\( \kappa \) is \( C(n) \)-superstrong" is a \( \Sigma_n \) statement, we have:
\[
M \models "\kappa \text{ is } C(n)\text{-superstrong}".
\]
Therefore, by normality of \( E \), the set \( \{ \{ \alpha \} : \alpha < \kappa \land \alpha \text{ is } C(n)\text{-superstrong} \} \) belongs to \( E_{j(\kappa)} \). Hence, \( \{ \alpha < \kappa : \alpha \text{ is } C(n)\text{-superstrong} \} \) belongs to the standard \( \kappa \)-complete normal ultrafilter \( \mathcal{U} := \{ X \subseteq \kappa : \kappa \in j(X) \} \).

The following Proposition gives an upper bound on the relative position of \( C(n) \)-superstrong cardinals in the large cardinal hierarchy.

**Proposition 1.4.** If \( \kappa \) is \( 2^\kappa \)-supercompact and belongs to \( C(n) \), then there is a \( \kappa \)-complete normal ultrafilter \( \mathcal{U} \) over \( \kappa \) such that the set of \( C(n) \)-superstrong cardinals smaller than \( \kappa \) belongs to \( \mathcal{U} \).

**Proof.** Let \( j : V \rightarrow M \) be an elementary embedding coming from a normal ultrafilter \( \mathcal{V} \) on \( \mathcal{P}_\kappa(2^\kappa) \). Let \( j := j \upharpoonright V_{\kappa+1} \). So, \( j : V_{\kappa+1} \rightarrow M_{j(\kappa)+1} \) is elementary and \( j \in M \). Hence \( M \models "j : V_{\kappa+1} \rightarrow V_{j(\kappa)+1} \text{ is elementary}" \). Since \( \kappa \in C(n) \), also \( M \models "j(\kappa) \in C(n)" \). Thus, \( M \models "\kappa \text{ is } \kappa+1-C(n)\text{-extendible}" \). Hence, \( \{ x \in \mathcal{P}_\kappa(2^\kappa) : \text{ot}(x \cap \kappa) \text{ is } \text{ot}(x \cap \kappa) + 1 \text{-} C(n)\text{-extendible} \} \in \mathcal{V} \). Letting \( \mathcal{U} \) be the projection of \( \mathcal{V} \) on \( \kappa \), we have
\[
\{ \alpha < \kappa : \alpha \text{ is } \alpha + 1-C(n)\text{-extendible} \} \in \mathcal{U}.
\]
Now as in [3], Proposition 26.11 (a), one can show that if \( \alpha \) is \( \alpha + 1-C(n) \)-extendible, then \( \alpha \) is \( C(n) \)-superstrong. \( \square \)

2. \( C(n) \)-extendible cardinals

Recall that a cardinal \( \kappa \) is \( \lambda \)-extendible if there is an elementary embedding \( j : V_\lambda \rightarrow V_\mu \), some \( \mu \), with critical point \( \kappa \) and such that \( j(\kappa) > \lambda \). And \( \kappa \) is extendible if it is \( \lambda \)-extendible for all \( \lambda > \kappa \).

The following lemma implies that every extendible cardinal is supercompact (see also [3], 23.6).

**Lemma 2.1** (M. Magidor [4]). Suppose \( j : V_\lambda \rightarrow V_\mu \) is elementary, \( \lambda \) is a limit ordinal, and \( \kappa \) is the critical point of \( j \). Then \( \kappa \) is \( < \lambda \)-supercompact.

**Definition 2.2.** For a cardinal \( \kappa \) and \( \lambda > \kappa \), we say that \( \kappa \) is \( \lambda \)-\( C(n) \)-extendible if there is an elementary embedding \( j : V_\lambda \rightarrow V_\mu \), some \( \mu \), with critical point \( \kappa \), and such that \( j(\kappa) > \lambda \) and \( j(\kappa) \in C(n) \).

We say that \( \kappa \) is \( C(n) \)-extendible if it is \( \lambda \)-\( C(n) \)-extendible for all \( \lambda > \kappa \).

It follows from the next Proposition that a cardinal is extendible if and only if it is \( C(1) \)-extendible.

**Proposition 2.3.** Every extendible cardinal is \( C(1) \)-extendible.

**Proof.** Suppose \( \kappa \) is extendible and \( \lambda \) is greater than \( \kappa \). Pick \( \lambda' \geq \lambda \) such that \( \lambda' \in C(1) \). Let \( j : V_{\lambda'} \rightarrow V_\mu \) be an elementary embedding with \( \text{crit}(j) = \kappa \) and \( j(\kappa) > \lambda' \). Since \( \lambda' \) is a cardinal and \( V_{\lambda'} = H(\lambda') \), by elementarity we must also have that \( \mu \) is a cardinal and \( V_\mu = H(\mu) \). Hence \( \mu \in C(1) \). And since, again by elementarity, \( V_\mu \models j(\kappa) \in C(1) \), we have that \( j(\kappa) \in C(1) \). \( \square \)
In the definition above, notice that if $\kappa, \lambda, \mu \in C^{(n)}$, then the requirement $j(\kappa) \in C^{(n)}$ follows automatically. Also, if $\kappa$ is $C^{(n)}$-extendible, then it follows easily from the fact that $j(\kappa) \in C^{(n)}$ and $j$ is elementary that $\kappa \in C^{(n)}$. But more is true.

**Proposition 2.4.** If $\kappa$ is $C^{(n)}$-extendible, then $\kappa \in C^{(n+2)}$.

*Proof.* By induction on $n$. For $n = 0$, since $\kappa \in C^{(1)}$, we only need to show that if $\exists x \varphi(x)$ is a $\Sigma_2$ sentence, where $\varphi$ is $\Pi_1$ and has parameters in $V_\kappa$, that holds in $V$, then it holds in $V_\kappa$. So suppose $a$ is such that $\varphi(a)$ holds in $V$. Let $\lambda > \kappa$ be such that $a \in V_\lambda$, and let $j : V_\lambda \rightarrow V_\mu$ be elementary, with critical point $\kappa$ and with $j(\kappa) > \lambda$. Then $V_{j(\kappa)} \models \varphi(a)$, and so by elementarity $V_\kappa \models \exists x \varphi(x)$. □

Now suppose $\kappa$ is $C^{(n)}$-extendible and $\exists x \varphi(x)$ is a $\Sigma_{n+2}$ sentence, where $\varphi$ is $\Pi_{n+1}$ and has parameters in $V_\kappa$. If $\exists x \varphi(x)$ holds in $V_\kappa$, then since by induction hypothesis $\kappa \in C^{(n+1)}$, we have that $\exists x \varphi(x)$ holds in $V$. Now suppose $a$ is such that $\varphi(a)$ holds in $V$. Let $\lambda > \kappa$ be such that $a \in V_\lambda$, and let $j : V_\lambda \rightarrow V_\mu$ be elementary, with critical point $\kappa$ and with $j(\kappa) > \lambda$. Then since $j(\kappa) \in C^{(n)}$, we have $V_{j(\kappa)} \models \varphi(a)$, and so by elementarity $V_\kappa \models \exists x \varphi(x)$. □

Let us observe that for any $\alpha < \lambda$, the relation “$\alpha$ is $\lambda$-$C^{(n)}$-extendible” is $\Sigma_{n+1}$ (for $n \geq 1$), for it holds if and only if

$$\exists \mu \exists j : V_\lambda \rightarrow V_\mu \land j \text{ elementary } \land \text{crit}(j) = \alpha \land j(\alpha) > \lambda \land j(\alpha) \in C^{(n)}.$$ 

Hence, “$x$ is a $C^{(n)}$-extendible cardinal” is a $\Pi_{n+2}$ property of $x$.

**Proposition 2.5.** For every $n \geq 1$, if $\kappa$ is $C^{(n)}$-extendible and $\kappa + 1$-$C^{(n+1)}$-extendible, then the set of $C^{(n)}$-extendible cardinals is unbounded below $\kappa$. Hence, the first $C^{(n)}$-extendible cardinal $\kappa$, if it exists, is not $\kappa + 1$-$C^{(n+1)}$-extendible. In particular, the first extendible cardinal $\kappa$ is not $\kappa + 1$-$C^{(2)}$-extendible.

*Proof.* Suppose $\kappa$ is $C^{(n)}$-extendible and $\kappa + 1$-$C^{(n+1)}$-extendible, witnessed by $j : V_{\kappa+1} \rightarrow V_{j(\kappa)+1}$. Since $j(\kappa) \in C^{(n+1)}$,

$$V_{j(\kappa)} \models \text{“} \kappa \text{ is } C^{(n)}\text{-extendible} \text{”}.$$ 

Hence, for every $\alpha < \kappa$,

$$V_{j(\kappa)} \models \text{“} \exists \beta > \alpha (\beta \text{ is } C^{(n)}\text{-extendible}) \text{”},$$ 

since this is witnessed by $\kappa$. By elementarity of $j$, for every fixed $\alpha < \kappa$, there is $\beta > \alpha$ such that,

$$V_\kappa \models \text{“} \beta > \alpha \land \beta \text{ is } C^{(n)}\text{-extendible} \text{”}.$$ 

And since, by Proposition 2.4, $\kappa \in C^{(n+2)}$, $\beta$ is $C^{(n)}$-extendible in $V$. □

**Proposition 2.6.** For every $\kappa \in C^{(n+2)}$, $\beta$ is $C^{(n)}$-extendible in $V$.

*Proof.* By the last Proposition, if $\kappa$ is $C^{(n+2)}$-extendible, then the set of $C^{(n)}$-extendible cardinals is unbounded below $\kappa$. Now the Proposition follows easily from the fact that if $\kappa \in C^{(n+2)}$-extendible, then $\kappa \in C^{(n+4)}$ (Proposition 2.4), and the fact that being $C^{(n)}$-extendible is a $\Pi_{n+2}$-property. □
Note that the existence of a $C^{(n+1)}$-extendible cardinal $\kappa$ (for $n \geq 1$) does not imply the existence of a $C^{(n)}$-extendible cardinal greater than $\kappa$. For if $\lambda$ is such a cardinal, then $V_\lambda \models “\kappa \text{ is } C^{(n+1)}\text{-extendible}”.$

The following Proposition shows that $C^{(n)}$-extendible cardinals are much larger than $C^{(n)}$-superstrong cardinals.

**Proposition 2.7.** If $\kappa$ is $\kappa + 1$-$C^{(n)}$-extendible and belongs to $C^{(n)}$, then $\kappa$ is $C^{(n)}$-superstrong, and there is a $\kappa$-complete normal ultrafilter $U$ over $\kappa$ such that the set of $C^{(n)}$-superstrong cardinals smaller than $\kappa$ belongs to $U$.

**Proof.** As in [3], Proposition 26.11 (a). \qed

3. $C^{(n)}$-extendible cardinals and Vopěnka’s Principle

This section builds on results from [1], giving sharper characterizations of Vopěnka’s Principle in terms of $C^{(n)}$-extendible cardinals.

Recall that Vopěnka’s Principle (VP) states that for every proper class $\mathcal{C}$ of structures of the same type, there exist $A \neq B$ in $\mathcal{C}$ such that $A$ is elementarily embeddable into $B$.

VP can be formulated in the first-order language of set theory as an axiom schema, i.e., as an infinite set of axioms, one for each formula with two free variables. Formally, for each such formula $\varphi(x, y)$ one has the axiom:

$$\forall x [(\forall y \forall z (\varphi(x, y) \land \varphi(x, z) \rightarrow y \text{ and } z \text{ are structures of the same type}) \land$$

$$\forall \alpha \in OR \exists y (\text{rank}(y) > \alpha \land \varphi(x, y)) \rightarrow$$

$$\exists z (\varphi(x, y) \land \varphi(x, z) \land y \neq z \land \exists e : y \rightarrow z \text{ is elementary})].$$

Henceforth, VP will be understood as this axiom schema.

The theory $ZFC + VP$ implies, for instance, that the class of extendible cardinals is stationary, i.e., every definable club proper class contains an extendible cardinal ([4]). And its consistency is known to follow from the consistency of $ZFC$ plus the existence of an almost-huge cardinal (see [3], or [2]). We will give the exact consistency strength below.

Let us consider the following variants of VP, the first apparently much stronger than the second.

**Definition 3.1.** If $\Gamma$ is one of $\Sigma_n, \Pi_n, \Delta_n$, for $n \in \omega$, and $\kappa$ is an infinite cardinal, then we write $VP(\kappa, \Gamma)$ for the following assertion:

For every $\Gamma$ proper class $\mathcal{C}$ of structures of the same type $\tau$ such that both $\tau$ and the parameters of some $\Gamma$-definition of $\mathcal{C}$, if any, belong to $H(\kappa)$, $\mathcal{C}$ reflects below $\kappa$, i.e., for every $B \in \mathcal{C}$, there exists $A \in \mathcal{C} \cap H(\kappa)$ that is elementarily embeddable into $B$.

If $\Gamma$ is one of $\Sigma_n, \Pi_n, \Delta_n$, or $\Sigma_n, \Pi_n, \Delta_n$, for $n \in \omega$, we write $VP(\Gamma)$ for the following statement:

For every $\Gamma$ proper class $\mathcal{C}$ of structures of the language of set theory with one (equivalently, finitely-many) additional 1-ary relation symbol, there exist distinct $A$ and $B$ in $\mathcal{C}$ with an elementary embedding of $A$ into $B$.

Clearly, for every $\Gamma$, $VP(\kappa, \Gamma)$ for some $\kappa$ implies $VP(\Gamma)$.

VP for $\Sigma_1$ classes follows from ZFC. In fact, the following holds.

**Theorem 3.2.** $VP(\kappa, \Sigma_1)$ holds for every uncountable cardinal $\kappa$. 
Proof. Fix an uncountable cardinal $\kappa$ and a class $\mathcal{C}$ of structures of the same type $\tau \in H(\kappa)$, definable by a $\Sigma_1$ formula with parameters in $H(\kappa)$.

Given $B \in \mathcal{C}$, let $\lambda$ be a regular cardinal greater than $\kappa$, with $B \in H(\lambda)$, and let $N$ be an elementary substructure of $H(\lambda)$, of cardinality less than $\kappa$, which contains $B$ and the transitive closure of $\{\tau\}$ together with the parameters involved in some $\Sigma_1$ definition of $\mathcal{C}$.

Let $A$ and $M$ be the transitive collapses of $B$ and $N$, respectively, and let $j : M \to N$ be the collapsing isomorphism. Then $A \in H(\kappa)$, and $j \upharpoonright A : A \to B$ is an elementary embedding. Observe that $j(\tau) = \tau$. So, since $\Sigma^1_1$ formulas are upwards absolute for transitive models, and since $M \models A \in \mathcal{C}$, we have that $A \in \mathcal{C}$. \qed

In contrast to the last theorem, Vopênka’s Principle for $\Pi_1$ proper classes implies the existence of very large cardinals.

Theorem 3.3.

1. If $VP(\Pi_1)$ holds, then there exists a supercompact cardinal.

1. If $VP(\Pi_1)$ holds, then there is a proper class of supercompact cardinals.

Proof. (1). Let $\mathcal{C}$ be the class of structures of the form $\langle V_{\lambda+2}, \in, \alpha, \lambda \rangle$, where $\lambda$ is the least limit ordinal greater than $\alpha$ such that no $\kappa \leq \alpha$ is $<\lambda$-supercompact.

We claim that $\mathcal{C}$ is $\Pi_1$ definable without parameters. For $X \in \mathcal{C}$ if and only if $X = \langle X_0, X_1, X_2, X_3 \rangle$, where

1. $X_2$ is an ordinal.

1. $X_3$ is a limit ordinal greater than $X_2$.

1. $X_0 = V_{X_3+2}$

1. $X_1 = \in | X_0$

1. And the following holds in $\langle X_0, X_1 \rangle$:

(a) $\forall \kappa \leq X_2(\kappa$ is not $< X_3$-supercompact)

(b) $\forall \mu (\mu$ limit $\land X_2 < \mu < X_3 \rightarrow \exists \kappa \leq X_2 (\kappa$ is $< \mu$-supercompact)).

If there is no supercompact cardinal, then $\mathcal{C}$ is a proper class. So by $VP(\Pi_0)$, there exist $\langle V_{\lambda+2}, \in, \alpha, \lambda \rangle \neq \langle V_{\mu+2}, \in, \beta, \mu \rangle$ and an elementary embedding

$$j : \langle V_{\lambda+2}, \in, \alpha, \lambda \rangle \to \langle V_{\mu+2}, \in, \beta, \mu \rangle.$$ 

Since $j$ must send $\alpha$ to $\beta$ and $\lambda$ to $\mu$, $j$ is not the identity. Hence by Kunen’s theorem (see [3]) we must have $\lambda < \mu$, and therefore also $\alpha < \beta$. So, $j$ has critical point some $\kappa \leq \alpha$. It now follows by Lemma 2.1 that $\kappa$ is $<\lambda$-supercompact. But this is impossible because $\langle V_{\lambda+2}, \in, \alpha, \lambda \rangle \in \mathcal{C}$.

(2). Fixing an ordinal $\xi$, to show that there is a supercompact cardinal greater than $\xi$, we argue as above. The only difficulty now is to ensure that $\kappa > \xi$. But this can be achieved by letting $\mathcal{C}$ be the class of structures of the form $\langle V_{\lambda+2}, \in, \alpha, \lambda, \{\gamma\}_{\gamma \leq \xi} \rangle$, where $\alpha > \xi$ and $\lambda$ is the least limit ordinal greater than $\alpha$ such that no $\kappa \leq \alpha$ is $<\lambda$-supercompact. The class $\mathcal{C}$ is now $\Pi_1$ definable with $\xi$ as an additional parameter. \qed

Let us say that a limit ordinal $\lambda$ captures a proper class $\mathcal{C}$ if the ranks of elements of $\mathcal{C}$ are unbounded below $\lambda$. 


Note that if $C$ is $\Pi_n$, then every $\lambda$ in $C^{(n+1)}$ greater than the rank of the parameters involved in a $\Pi_n$ definition of $C$ captures $C$. Notice also that every cardinal in $C^{(2)}$ belongs to $\text{Lim}(C^{(1)})$ - i.e., is a limit point of $C^{(1)}$ - and captures all $\Pi_1$ proper classes. However, the least ordinal $\lambda$ in $\text{Lim}(C^{(1)})$ that captures all $\Pi_1$ proper classes is strictly less than the least ordinal in $C^{(2)}$. The point is that, fixing an enumeration $\langle \varphi_n(x) : n < \omega \rangle$ of all $\Pi_1$ formulas that define proper classes, the sentence
\[ \exists \lambda \exists x (\lambda \in \text{Lim}(C^{(1)}) \land x = V_\lambda \land \forall n (V_\lambda \models \forall \alpha \exists \beta > \alpha \exists a (\text{rk}(a) > \beta \land \varphi_n(a))) \]
is $\Sigma_2$ in the parameter $\langle \varphi_n(x) : n < \omega \rangle$.

**Proposition 3.4.** Suppose $C$ is a $\Pi_1$ proper class of structures of the same type, and $\kappa$ is $< \lambda$-supercompact, for some $\lambda \in \text{Lim}(C^{(1)})$ greater than $\kappa$ that captures $C$. Then VP holds for $C$.

**Proof.** Since $\lambda$ captures $C$, in $V_\lambda$ there exist elements of $C$ of arbitrarily high rank. So, since $\lambda \in \text{Lim}(C^{(1)})$, we can find $\delta < \lambda$ such that $V_\delta = H(\delta)$, and $B \in C \cap V_\delta$ of rank greater than $\kappa$. Let $j : V \rightarrow M$ be an elementary embedding with critical point $\kappa$, with $j(\kappa) > \delta$, and $M$ closed under $\delta$-sequences. Since $B \in M$ and $C$ is $\Pi_1$-definable, $M \models "B \in C"$. And since $M$ is closed under $\delta$-sequences, the elementary embedding $j \upharpoonright B : B \rightarrow j(B)$ belongs to $M$. Thus,
\[ M \models "\exists A \in C \exists e (\text{rank}(A) < j(\kappa) \land e : A \rightarrow j(B) \text{ is elementary}), \]

since this is witnessed by $B$ and $j \upharpoonright B$.

By elementarity, the same must hold in $V$, namely,
\[ \exists A \in C \exists e (\text{rank}(A) < \kappa \land e : A \rightarrow B \text{ is elementary}), \]

which is what we wanted. \hfill $\Box$

**Theorem 3.5 ([1]).** Suppose that $C$ is a $\Sigma_2$ class of structures of the same type, and suppose that there exists a supercompact cardinal $\kappa$ larger than the rank of the parameters that appear in some $\Sigma_2$ definition of $C$. Then for every $B \in C$ there exists $A \in C \cap V_\kappa$ that is elementarily embeddable into $B$.

**Proof.** Fix a $\Sigma_2$ formula $\varphi(x, y)$ and a set $b$ such that $C = \{ a : \varphi(a, b) \}$, and suppose that $\kappa$ is a supercompact cardinal with $b \in V_\kappa$. Fix $B \in C$, and let $\lambda \in C^{(2)}$ be greater than $\text{rank}(B)$. Let $j : V \rightarrow M$ be an elementary embedding with $M$ transitive and critical point $\kappa$, such that $j(\kappa) > \lambda$ and $M$ is closed under $\lambda$-sequences. Thus, $B$ and $j \upharpoonright B : B \rightarrow j(B)$ are in $M$, and also $V_\lambda \in M$. Hence $V_\lambda \preceq_1 M$.

Since $V_\lambda \preceq_2 V$, $V_\lambda \models \varphi(B, b)$. And since $\Sigma_2$ formulas are upwards absolute between $V_\lambda$ and $M$, $M \models \varphi(B, b)$.

Thus, in $M$ it is true that there exists $X \in M_{j(\kappa)}$ such that $\varphi(X, b)$, namely $B$, and there exists an elementary embedding $e : X \rightarrow j(B)$, namely $j \upharpoonright B$. Therefore, by elementarity, the same holds in $V$; that is, there exists $X \in V_\kappa$ such that $\varphi(X, b)$, and there exists an elementary embedding $e : X \rightarrow B$. \hfill $\Box$

The following corollaries give characterizations of Vopěnka’s principle for $\Pi_1$ and $\Sigma_2$ classes (and therefore also for $\Delta_2$ classes) in terms of supercompactness. The equivalence of (2) and (3) was already proved in [1].
Corollary 3.6. The following are equivalent:

1. $VP(\Pi_1)$.
2. $VP(\kappa, \Sigma_2)$, for some $\kappa$.
3. There exists a supercompact cardinal.

Corollary 3.7. The following are equivalent:

1. $VP(\Pi_1)$.
2. $VP(\kappa, \Sigma_2)$, for a proper class of cardinals $\kappa$.
3. There exists a proper class of supercompact cardinals.

Recall from Definition 3.1 that a cardinal $\kappa$ reflects a class of structures $\mathcal{C}$ if for every $B \in \mathcal{C}$ there exists $A \in \mathcal{C} \cap V_\kappa$ which is elementary embeddable into $B$. The following theorem is due to Magidor [4], which follows from his characterization of supercompact cardinals, namely: $\kappa$ is supercompact if and only if for a proper class of $\lambda$ greater than $\kappa$ there exists $\alpha < \kappa$ and an elementary embedding $j : V_\alpha \to V_\lambda$ such that $\kappa$ is the image under $j$ of the critical point (see [4, 3]).

Theorem 3.8 (Magidor [4]). If $\kappa$ is the least cardinal that reflects the $\Pi_1$ proper class $\mathcal{C}$ of structures of the form $\langle V_\lambda, \in \rangle$, then $\kappa$ is supercompact.

Proof. For each $\lambda$ greater than $\kappa$ there is $\alpha < \kappa$ and an elementary embedding

$$j_\lambda : \langle V_\alpha, \in \rangle \to \langle V_\lambda, \in \rangle.$$ 

Let $\alpha$ be the least ordinal for which there is such an embedding for a proper class of $\lambda$s. We may assume that the $j_\lambda$ are not the identity (otherwise, $V_\alpha$ would be an elementary substructure of $V$) and that the critical point of all these embeddings is the same, say $\beta$. Moreover, we may assume that the image of $\beta$ is always the same, for otherwise for a proper class of $\lambda$s, $j_\lambda | \beta$ would witness that $V_\beta$ is an elementary substructure of $V_{j_\lambda(\beta)}$, with the $j_\lambda(\beta)$s forming a proper class, which in turn would imply that $V_\beta$ is an elementary substructure of $V$.

So let $\delta$ be least such that for a proper class of $\lambda$s the $\alpha$ is the same, $j_\lambda$ is not the identity, and the image of the critical point is $\delta$. By Magidor’s characterization of supercompactness, $\delta$ is supercompact. Thus $\delta \geq \kappa$, because $\delta$ reflects $\mathcal{C}$, by Theorem 3.5, and $\kappa$ is the least cardinal that does this. So suppose, aiming for a contradiction, that $\delta > \kappa$. Then by Theorem 3.5, $\delta$ reflects the proper class of structures of the form $\langle V_\lambda, \in, \gamma \rangle$, where $\gamma$ is some ordinal less than $\lambda$, which is $\Pi_1$. Arguing similarly as above we have that for a proper class of $\lambda$s there are fixed $\gamma < \alpha < \kappa$ with an embedding from $\langle V_\alpha, \in, \gamma \rangle$ into $\langle V_\lambda, \in, \kappa \rangle$ whose image of the critical point is less or equal than $\kappa$, contradicting the minimality of $\delta$. □

From the last two Theorems we obtain a characterization of the first supercompact cardinal in terms of a natural form of reflection.

Corollary 3.9. A cardinal $\kappa$ reflects all $\Pi_1$ proper classes of structures of the same type if and only if it is greater or equal than the first supercompact cardinal.

David Asperó pointed out the following parameterized form of the Corollary above. One direction follows from Theorem 3.5 and by observing that
the property of reflecting $\Pi_1$ classes is closed under limits. The other direction can be proved similarly as in the Theorem above, but working, for any fixed $\xi < \kappa$, with the class of structures of the form $\langle V_\alpha, \in, \{\eta\}_{\eta < \xi}\rangle$, where $\beta$ is some ordinal less than $\lambda$, which is $\Pi_1$ definable in the parameter $\alpha$.

**Corollary 3.10.** A cardinal $\kappa$ reflects all $\Pi_1$ proper classes of structures of the same type if and only if either $\kappa$ is a supercompact cardinal or a limit of supercompact cardinals.

We will prove similar results for classes of higher complexity, for which we shall need $C^{(n)}$-extendible cardinals.

**Theorem 3.11.** For every $n \geq 1$, if $\kappa$ is a $C^{(n)}$-extendible cardinal, then $VP(\kappa, \Sigma_{n+2})$ holds.

**Proof.** Fix a $\Sigma_{n+2}$ formula $\exists x \varphi(x, y, z)$, where $\varphi$ is $\Pi_{n+1}$, such that for some set $b \in V_\kappa$, $C := \{B : \exists x \varphi(x, B, b)\}$ is a proper class of structures of the same type.

Fix $B \in C$ and let $\lambda \in C^{(n+2)}$ be greater than $\kappa$ and the ranks of $b$ and $B$. Then $V_\lambda \models \exists x \varphi(x, B, b)$.

Let $j : V_\lambda \rightarrow V_\mu$ be an elementary embedding with critical point $\kappa$, with $j(\kappa) > \lambda$, and $j(\kappa) \in C^{(n)}$. Note that $B$ and $j \upharpoonright B : B \rightarrow j(B)$ are in $V_\mu$.

As $\kappa, \lambda \in C^{(n+2)}$ (see Proposition 2.4), it follows that $V_\kappa \preceq_{n+2} V_\lambda$. And this implies that $V_{j(\kappa)} \preceq_{n+2} V_\mu$. Indeed, we have that $V_\lambda \models "x \in V_\kappa \forall \theta \in \Sigma_{n+2}(V_\kappa \models \theta(x) \leftrightarrow \models_{n+2} \theta(x))"$.

Hence, by elementarity, $V_\mu \models "x \in V_{j(\kappa)} \forall \theta \in \Sigma_{n+2}(V_{j(\kappa)} \models \theta(x) \leftrightarrow \models_{n+2} \theta(x))"$,

which implies $V_{j(\kappa)} \preceq_{n+2} V_\mu$.

Since $j(\kappa) \in C^{(n)}$, we also have $V_\lambda \preceq_{n+1} V_{j(\kappa)}$, and therefore $V_\lambda \preceq_{n+1} V_\mu$.

It follows that $V_\mu \models \exists x \varphi(x, B, b)$.

Thus, in $V_\mu$ it is true that there exists $X \in V_{j(\kappa)}$ such that $X \in C$, namely $B$, and there exists an elementary embedding $e : X \rightarrow j(B)$, namely $j \upharpoonright B$. Therefore, by elementarity of $j$, the same is true in $V_\lambda$, that is, there exists $X \in V_\kappa$ such that $X \in C$, and there exists an elementary embedding $e : X \rightarrow B$. Let $A \in V_\kappa$ be such an $X$, and let $e : A \rightarrow B$ be an elementary embedding. Since $\lambda \in C^{(n+2)}$, $A \in C$, and we are done. \qed

The next Theorem will yield a strong converse to Theorem 3.11.

The notion of $C^{(n)}$-extendibility used in [1] has the following (apparently) stronger form – let us call it $C^{(n)+}$-extendibility: For $\lambda \in C^{(n)}$, a cardinal $\kappa$ is $\lambda$-$C^{(n)+}$-extendible if it is $\lambda$-$C^{(n)}$-extendible, witnessed by some $j : V_\lambda \rightarrow V_\mu$, which, in addition to satisfying $j(\kappa) > \lambda$ and $j(\kappa) \in C^{(n)}$, it also satisfies that $\mu \in C^{(n)}$.

$\kappa$ is $C^{(n)+}$-extendible if it is $\lambda$-$C^{(n)+}$-extendible for every $\lambda > \kappa$ with $\lambda \in C^{(n)}$.

Every extendible cardinal is $C^{(1)+}$-extendible (see [1], or the proof of Proposition 2.3 above).
Theorem 3.12. Suppose \( n \geq 1 \). If \( V P(\Pi_{n+1}) \) holds, then there exists a \( C^{(n)+}-\)extendible cardinal.

Proof. Suppose there are no \( C^{(n)+}\)-extendible cardinals. Then the class function \( F \) on the ordinals given by:

\[
F(\alpha) = \text{the least } \lambda \in C^{(n+1)} \text{ greater than } \alpha \text{ such that } \alpha \text{ is not } \lambda-C^{(n)+}-\text{extendible},
\]

is defined for all ordinals \( \alpha \).

Let \( C = \{ \eta > 0 : \forall \alpha < \eta \ F(\alpha) < \eta \} \). So \( C \) is a closed unbounded proper class of ordinals contained in \( C^{(n+1)} \).

We claim that \( C \) is \( \Pi_{n+1} \) definable, without parameters. First note that \( F \) is \( \Pi_{n+1} \) definable, for \( \lambda = F(\alpha) \) iff

\[
(1) \ \lambda \in C^{(n+1)} \\
(2) \ \alpha < \lambda \\
(3) \ \forall \beta > \lambda (\beta \in C^{(n)} \rightarrow V_\beta \models (\alpha \text{ is not } \lambda-C^{(n)+}-\text{extendible})), \text{ and} \\
(4) \ V_\lambda \models \forall \lambda' > \alpha (\lambda' \in C^{(n+1)} \rightarrow (\alpha \text{ is } \lambda'-C^{(n)+}-\text{extendible})).
\]

The point is that, for any \( \alpha < \lambda' \), the relation “\( \alpha \) is \( \lambda'-C^{(n)+}-\text{extendible} \)” is \( \Sigma_{n+1} \), for it holds iff

\[
\exists \mu \exists j(j: V_\mu \rightarrow V_\mu \wedge j \text{ elementary } \wedge \text{crit}(j) = \alpha \wedge j(\alpha) > \lambda' \wedge j(\alpha), \mu \in C^{(n)}).
\]

So it holds in \( V \) if and only if it holds in \( V_\lambda \), for any \( \lambda \in C^{(n+1)} \) greater than \( \lambda' \). And if it holds in \( V_\beta \), with \( \beta \in C^{(n)} \), then it holds in \( V \). Moreover, since \( \lambda \in C^{(n+1)} \), for every \( \lambda' < \lambda \) we have \( \lambda' \in C^{(n+1)} \) if and only if \( V_\lambda \models \lambda' \in C^{(n+1)} \).

It follows that \( C \) is also \( \Pi_{n+1} \) definable.

For each ordinal \( \alpha \), let \( \lambda_\alpha \) be the least limit point of \( C \) greater than \( \alpha \). We have that \( x = \lambda_\alpha \) iff \( x \) is an ordinal greater than \( \alpha \) that belongs to \( C \) and such that

\[
(1) \ V_x \models \forall \beta \exists \gamma (\gamma > \beta \wedge \gamma \in C) \\
(2) \ V_x \models \forall \beta (\beta > \alpha \rightarrow \exists \gamma < \beta \forall \eta (\gamma < \eta < \beta \rightarrow \eta \notin C)),
\]

which shows that the function \( \alpha \mapsto \lambda_\alpha \) is \( \Pi_{n+1} \) definable.

Consider now the proper class \( \mathcal{C} \) of structures \( \mathcal{A}_\alpha \) of the form

\[
(V_{\lambda_\alpha}, \in, \alpha, \lambda_\alpha, C \cap \alpha + 1),
\]

where \( \alpha \in C \).

We claim that \( \mathcal{C} \) is \( \Pi_{n+1} \) definable. We have: \( X \in \mathcal{C} \) if and only if \( X = \langle X_0, X_1, X_2, X_3, X_4 \rangle \), where

\[
(1) \ X_2 \in C \\
(2) \ X_3 = \lambda X_2 \\
(3) \ X_0 = V_{X_3} \\
(4) \ X_1 = \in|X_0 \\
(5) \ X_4 = C \cap X_2 + 1
\]

We have already seen that (1) and (2) are \( \Pi_{n+1} \) expressible. And the same is true for (3) and (4), as one can easily see. As for (5), note that \( X_4 = C \cap \alpha + 1 \) holds in \( V \) if it holds in \( V_{X_1} \).

So by \( V P(\Pi_{n+1}) \) there exist \( \alpha \neq \beta \) and an elementary embedding

\[
j : \mathcal{A}_\alpha \rightarrow \mathcal{A}_\beta.
\]
Since $j$ must send $\alpha$ to $\beta$, $j$ is not the identity. So $j$ has critical point some $\kappa \leq \alpha$.

We claim that $\kappa \in C$. Otherwise, $\gamma := \sup(C \cap \kappa) < \kappa$. Let $\delta$ be the least ordinal in $C$ greater than $\gamma$ such that $\delta < \lambda_\alpha$. Since $\delta$ is definable from $\gamma$ in $\mathcal{A}_\alpha$, and since $j(\gamma) = \gamma$, we must also have $j(\delta) = \delta$.

But then $j \upharpoonright V_{\delta+2} : V_{\delta+2} \to V_{\delta+2}$ is a nontrivial elementary embedding, contradicting Kunen’s Theorem (see [3]).

Hence by elementarity, $j(\kappa) \in C$.

Since $\alpha \in C$, we have $\kappa < F(\kappa) < \alpha$. Thus,

$$j \upharpoonright V_{F(\kappa)} : V_{F(\kappa)} \to V_{j(F(\kappa))}$$

is elementary, with critical point $\kappa$.

And since $j(\kappa) \in C$, $F(\kappa) < j(\kappa)$. Moreover, by elementarity of $j$, $V_{\lambda_\beta}$ satisfies that $j(F(\kappa))$ belongs to $C(\alpha)$, so since $\lambda_\beta \in C(\alpha+1)$ this is true in $V$.

This shows that $j \upharpoonright V_{F(\kappa)}$ witnesses that $\kappa$ is $F(\kappa)$-$C(\alpha)$-extendible. But this is impossible, by definition of $F$. □

The proof of the last theorem can easily be adapted to prove the parameterized version: if $VP(\Pi_{n+1})$ holds, then there is a proper class of $C(\alpha)$-extendible cardinals. (See the proof of Proposition 3.12 (2).)

Note that from the last two theorems it follows that there exists a $C(\alpha)$-extendible cardinal if and only if there exists a $C(\alpha)$-extendible cardinal.

The following corollaries summarize the results above (one may replace $C(\alpha)$-extendible by $C(\alpha)$-extendible).

**Corollary 3.13.** The following are equivalent:

1. $VP(\Pi_2)$.
2. $VP(\kappa, \Sigma_3)$, for some $\kappa$.
3. There exists an extendible cardinal.

**Corollary 3.14.** The following are equivalent for $n \geq 1$:

1. $VP(\Pi_{n+1})$.
2. $VP(\kappa, \Sigma_{n+2})$, for some $\kappa$.
3. There exists a $C(\alpha)$-extendible cardinal.

**Corollary 3.15.** The following are equivalent:

1. $VP(\Pi_n)$, for every $n$.
2. $VP(\kappa, \Sigma_n)$, for a proper class of cardinals $\kappa$, and for every $n$.
3. $VP$.
4. For every $n$, there exists a $C(\alpha)$-extendible cardinal.

We shall next obtain characterizations of $C(\alpha)$-extendible cardinals in terms of natural reflection principles.

**Theorem 3.16.** If $\kappa$ is the least cardinal that reflects all $\Pi_{n+1}$-classes of structures of the same type, then $\kappa$ is $C(\alpha)$-extendible.

**Proof.** Suppose otherwise. Then by 3.11 there is no $C(\alpha)$-extendible cardinal less or equal than $\kappa$.

Consider the class $C$ of structures of the form $\langle V_\xi, \in, \lambda, \alpha, C(\alpha) \cap \xi \rangle$, where $\alpha < \lambda < \xi$, and
(1) $\lambda \in C^{(n)}$,
(2) $\xi \in \text{Lim}(C^{(n)})$,
(3) the cofinality of $\xi$ is uncountable,
(4) $\forall \beta < \xi \forall \mu (\exists j : V_\lambda \rightarrow V_\mu \land \text{crit}(j) = \alpha \land j(\alpha) = \beta) \rightarrow \exists j' : V_\lambda \rightarrow V'_\mu \land \mu' < \xi \land \text{crit}(j') = \alpha \land j'(\alpha) = \beta)$, and
(5) $\lambda$ witnesses that no ordinal less or equal than $\alpha$ is $\lambda$-$C^{(n)}$-extendible.

Clearly, $C$ is a $\Pi_{n+1}$ definable proper class. So let

$$j : (V_{\xi'}, \in, \lambda', \alpha') \rightarrow (V_{\xi}, \in, \lambda, \kappa)$$

with $(V_{\xi'}, \in, \lambda', \alpha')$, $(V_{\xi}, \in, \lambda, \kappa) \in C$ and $\xi' < \kappa$.

Let $\alpha = \text{crit}(j)$. We claim that $\alpha \in C^{(n)}$. Otherwise, let $\gamma := \sup(C^{(n)} \cap \alpha)$, so $\gamma < \alpha$. Let $\delta \in C^{(n)}$ be the least such that $\gamma < \delta < \xi'$. Since $\delta$ is definable from $\gamma$ in $(V_{\xi'}, \in, \lambda', C^{(n)} \cap \xi')$ and $j(\gamma) = \gamma$, also $j(\delta) = \delta$. Hence $j \upharpoonright V_{\delta+2} \rightarrow V_{\delta+2}$, contradicting Kunen’s Theorem.

If $j''(\alpha) < \xi'$ for all $m$, then $(j''(\alpha))_{m \in \omega} \in V_{\xi'}$, because $\xi'$ has uncountable cofinality, contradicting Kunen’s Theorem. So suppose for some $m$ we have $j''(\alpha) < \xi' \leq j''(\alpha)$.

We claim that there exists an elementary embedding $k : V_\lambda \rightarrow V_\mu$, some $\mu$, with $\text{crit}(k) = \alpha$ and $k(\alpha) = j''(\alpha)$. We prove this by induction on $i \leq m$. For $i = 0$ take $k = j \upharpoonright V_\lambda$. So suppose it true for $i < m$. Since $j''(\alpha) < \xi'$, by (3) above, there exist $j'$ and $\mu'$ such that $j' : V_\lambda \rightarrow V_{\mu'}$ is elementary, $\mu' < \xi'$, $\text{crit}(j') = \alpha$, and $j''(\alpha) = j''(\alpha)$. Applying $j$ to $j'$ we have $j(j') : V_{j(\lambda')} \rightarrow V_{j(\mu')}$ has critical point $j(\alpha)$ and $j(j')(\alpha) = j''(\alpha)$. Let

$$k = j(j') \upharpoonright V_\mu : V_\lambda \rightarrow V_{j(j')}(\lambda').$$

Thus $k$ has critical point $\alpha$ and $k(\alpha) = j''(\alpha)$, which proves our claim.

Note that since $\alpha, \xi', \xi \in C^{(n)}$, we have $j''(\alpha) \in C^{(n)}$. Thus, $k$ witnesses that $\alpha$ is $\lambda'$-$C^{(n)}$-extendible, contradicting (4) above.

**Corollary 3.17.** A cardinal $\kappa$ reflects all $\Pi_{n+1}$-classes of structures of the same type (where $n \geq 1$) if and only if it is greater or equal than the first $C^{(n)}$-extendible cardinal.

The parameterized version also follows.

**Theorem 3.18.** A cardinal $\kappa$ reflects all $\Pi_{n+1}$ proper classes of structures of the same type if and only if either $\kappa$ is a $C^{(n)}$-extendible cardinal or a limit of $C^{(n)}$-extendible cardinals.

4. $C^{(n)}$-Huge Cardinals

Recall that a cardinal $\kappa$ is $m$-huge if it is the critical point of an elementary embedding $j : V \rightarrow M$ with $M$ transitive and closed under $j^m(\kappa)$-sequences, where $j^m$ is the $m$-th iterate of $j$ (i.e., $j^1 = j$ and $j^{m+1} = j \circ j^m$). A cardinal is called huge if it is 1-huge. Let us say that $\kappa$ is $C^{(n)}$-huge if it is $m$-huge and $j(\kappa) \in C^{(n)}$. And let us say that $\kappa$ is $C^{(n)}$-huge if it is huge and $j(\kappa) \in C^{(n)}$.

As with $m$-huge cardinals, $C^{(n)}$-huge cardinals can be characterized in terms of normal measures. To wit: $\kappa$ is $C^{(n)}$-huge if and only if it is uncountable and there is a $\kappa$-complete normal ultrafilter $\mathcal{U}$ over some $\mathcal{P}(\lambda)$.
and cardinals $\kappa = \lambda_0 < \lambda_1 < \ldots < \lambda_m = \lambda$, with $\lambda_1 \in C(n)$, and such that for each $i < m$, 
$$\{x \in P(\lambda) : ot(x \cap \lambda_{i+1}) = \lambda_1\} \in U.$$ 
(See [3] 24.8 for a proof of the case $n = 1$, which also works for arbitrary $n$.) It follows that “$\kappa$ is $C(n)\text{-}m$-huge” is $\Sigma_{n+1}$ expressible.

Clearly, every huge cardinal is $C^{(1)}$-huge. But the first huge cardinal is not $C^{(2)}$-huge. For suppose $\kappa$ is the least huge cardinal and $j : V \to M$ witnesses that $\kappa$ is $C^{(2)}$-huge. Then since “$x$ is huge” is $\Sigma_2$ expressible, we have 
$$V_{j(\kappa)} \models \text{“$\kappa$ is huge”}.$$ 
Hence, since $(V_{j(\kappa)})^M = V_{j(\kappa)}$, 
$$M \models \exists \delta < j(\kappa)(V_{j(\kappa)} \models \text{“$\delta$ is huge”}).$$
By elementarity, there is a huge cardinal less than $\kappa$ in $V$, which is absurd.

A similar argument, using that “$\kappa$ is $C^{(n)}\text{-}m$-huge” is $\Sigma_{n+1}$ expressible, for all $m$, shows that the first $C^{(n)}\text{-}m$-huge cardinal is not $C^{(n+1)}\text{-}m$-huge, for all $m$ and $n$ greater or equal than 1.

5. ON ELEMENTARY EMBEDDINGS OF A RANK INTO ITSELF

Consider the large cardinal principle, known as I3, which asserts the existence of a non-trivial elementary embedding $j : V_\delta \to V_\delta$ (see [3], 24). Let us call the critical point of such an embedding an I3 cardinal.

If $j : V_\delta \to V_\delta$ witnesses that $\kappa$ is I3, then Kunen’s Theorem implies that either $\delta$ is a limit ordinal or is of the form $\gamma + 1$ with $\gamma$ a limit ordinal (see [3], 24). And if $\delta$ is a limit ordinal, then $\delta = \sup \{j^m(\kappa) : m \in \omega\}$, where $j^m$ is the $m$-th iterate of $j$. It follows that $\delta \in C^{(1)}$, because all the $j^m(\kappa)$ are measurable cardinals and therefore they all belong to $C^{(1)}$.

Now suppose $j : V_\delta \to V_\delta$ witnesses that $\kappa$ is I3, with $\delta$ a limit ordinal. Then $V_\kappa$ and $V_{j^m(\kappa)}$, all $m$, are elementary substructures of $V_\delta$. In particular, $V_\delta$ is a model of ZFC. Moreover, it is easily seen that in $V_\delta$ the cardinal $\kappa$, and all the $j^m(\kappa)$, are $C^{(n)}$-supercompact, $C^{(n)}$-extendible, and $C^{(n)}$-huge, for all $n$ and $m$. In particular, $V_\delta$ satisfies VP. Thus the consistency of the existence of an I3 cardinal implies the consistency of the existence of all $C^{(n)}$-cardinals considered in previous sections.

Let us now say that $\kappa$ is a $C^{(n)}\text{-}I3$ cardinal if it is an I3 cardinal, witnessed by some embedding $j : V_\delta \to V_\delta$, with $j(\kappa) \in C^{(n)}$.

Observe that if $\kappa$ is $C^{(n)}$-I3, then $\kappa \in C^{(n)}$. For if $\varphi$ is a $\Sigma_n$ sentence with parameters in $V_\kappa$ that holds in $V$, then it must hold in $V_{j(\kappa)}$ as well, for some $j : V_\delta \to V_\delta$ that witnesses the fact that $\kappa$ is $C^{(n)}$-I3. But then, since $j$ fixes the parameters of $\varphi$, by elementarity $\varphi$ also holds in $V_\kappa$.

If $\kappa = \text{crit}(j)$, then $\kappa$ and $j(\kappa)$ are measurable cardinals. Thus every I3-cardinal is $C^{(1)}$-I3. However, a simple reflection argument shows that the least $C^{(n)}$-I3 cardinal is not $C^{(n+1)}$-I3, for $n \geq 1$. For suppose $\kappa$ is the least $C^{(n)}$-I3 cardinal ($n \geq 1$), and suppose, towards a contradiction, that there is some $j : V_\delta \to V_\delta$ with $\text{crit}(j) = \kappa$, and $j(\kappa) \in C^{(n+1)}$. Then $V_{j(\kappa)}$ satisfies the following $\Sigma_{n+1}$ statement 
$$\exists \beta \exists k \exists \lambda (k : V_\beta \to V_\beta \text{ is elementary} \land \text{crit}(k) = \lambda \land k(\lambda) \in C^{(n)})$$
because $\delta$, $j$, and $\kappa$ are such $\beta$, $k$, and $\lambda$, respectively. By elementarity the same holds in $V_\kappa$. Hence, since $\kappa \in C^{(n)}$, it also holds in $V$, contradicting our assumption that $\kappa$ was the least $C^{(n)}$-$I_3$ cardinal.

**Proposition 5.1.** Suppose $j : V_\delta \rightarrow V_\delta$ witnesses that $\kappa$ is $I_3$, and $\delta$ is a limit ordinal. Then the following are equivalent for every $n \geq 1$:

1. $j^m(\kappa) \in C^{(n)}$, all $m < \omega$.
2. $\delta \in C^{(n)}$.

**Proof.** For $n = 1$, (1) and (2) are true.

(1) implies (2) is immediate, since $\delta = sup\{j^m(\kappa) : m < \omega\}$.

Let’s prove (2) implies (1) for $n+1$, assuming it holds for $n$. Fix $m < \omega$. Let $\exists x.\varphi(x)$ be a $\Sigma_{n+1}$ formula whose parameters, if any, are in $V_{j^m(\kappa)}$, and suppose the formula is true in $V$. Then, by (2), it is also true in $V_\delta$. Let $\ell \geq m$ be big enough so that $V_{j^\ell(\kappa)}$ contains a witness to the formula. Then $V_\delta$ satisfies:

$$\exists x \in V_{j^\ell(\kappa)} \varphi(x).$$

Hence by elementarity of $j^{\ell-m}$, $V_\delta$ satisfies that there exists $x \in V_{j^m(\kappa)}$ such that $\varphi(x)$ holds. If $a$ is such an $x$, then since by inductive hypothesis $V_{j^m(\kappa)} \subseteq V_\delta$, we have that $\varphi(a)$ holds in $V_{j^m(\kappa)}$.

So let us say that $\kappa$ is $C^{(n)+}I_3$ if it is $I_3$, witnessed by $j : V_\delta \rightarrow V_\delta$, with $\delta \in C^{(n)}$. Clearly, $\kappa$ is $I_3$ if and only if it is $C^{(1)+}I_3$ if and only if it is $C^{(1)+}I_3$.

**Proposition 5.2.** If $\kappa$ is $C^{(n)}$-$I_3$, then it is $C^{(n)}$-$m$-huge, for all $m$, and there is a normal ultrafilter $U$ over $\kappa$ such that

$$\{\alpha < \kappa : \alpha \text{ is } C^{(n)}$-$m$-huge for every $m\} \in U.$$

**Proof.** Let $j : V_\delta \rightarrow V_\delta$ witness that $\kappa$ is $C^{(n)}$-$I_3$, with $\delta$ limit. Then as in [3] 24.8 one can show that the ultrafilter $V$ over $P(\lambda)$, where $\lambda = j^m(\kappa)$, defined by

$$X \in V \text{ if and only if } j^n \lambda \in j(X)$$

witnesses that $\kappa$ is $C^{(n)}$-$m$-huge. Now let $U$ be the usual normal ultrafilter over $\kappa$ obtained from $j$. Since $V_\delta$ satisfies that $\kappa$ is $C^{(n)}$-$m$-huge, by normality of $U$ the Proposition follows.

Let us call a cardinal $\kappa$ a $II$ cardinal if there exists an elementary embedding $j : V_{\delta+1} \rightarrow V_{\delta+1}$ with $crit(j) = \kappa$. By Kunen’s Theorem, $\delta$ must be a limit ordinal. So let us say that $\kappa$ is $C^{(n)}$-$II$ if it is $II$, witnessed by $j : V_{\delta+1} \rightarrow V_{\delta+1}$, with $j(\kappa) \in C^{(n)}$. Clearly, if $\kappa$ is $C^{(n)}$-$II$, then it is also $C^{(n)}$-$I_3$.

As with $C^{(n)}$-$I_3$ cardinals, a simple reflection argument shows that the first $C^{(n)}$-$II$ cardinal is not $C^{(n+1)}$-$II$, for all $n \geq 1$.

Let us also observe that if $\kappa$ is $C^{(n)}$-$II$, then the least $\delta$ for which there is an elementary embedding $j : V_{\delta+1} \rightarrow V_{\delta+1}$ with $crit(j) = \kappa$ and $j(\kappa) \in C^{(n)}$ is smaller than the first ordinal in $C^{(n+1)}$ greater than $\kappa$. Moreover, the least $C^{(n)}$-$II$ cardinal, if it exists, is smaller than the first ordinal in $C^{(n+1)}$, for all $n \geq 1$. 
C(n).CARDINALS

REFERENCES


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