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Local exclusion for intermediate and fractional statistics

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A local exclusion principle is observed for identical particles obeying intermediate/fractional exchange statistics in one and two dimensions, leading to bounds for the kinetic energy in terms of the density. This has implications for models of Lieb-Liniger and Calogero-Sutherland type, and implies a non-trivial lower bound for the energy of the anyon gas whenever the statistics parameter is an odd numerator fraction. We discuss whether this is actually a necessary requirement.

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I. INTRODUCTION

The majority of interesting phenomena in many-body quantum mechanics are in some way associated to the fundamental concept of identical particles and statistics. Elementary identical particles in three spatial dimensions are either bosons, obeying Bose-Einstein statistics, or fermions, obeying Fermi-Dirac statistics. The former are usually represented using wave functions which are symmetric under particle permutations, while the latter implement Pauli’s exclusion principle by exhibiting total anti-symmetry under particle interchange. On the other hand, for point particles living in one and two dimensions there are logical possibilities different from bosons and fermions, so-called intermediate or fractional statistics [1–3]. Although first regarded as of purely academic interest — filling the loopholes in the arguments leading to the two standard permutation symmetries — these have recently become a reality in the laboratory, with the advent of trapped bosonic condensates [4] and quantum Hall physics [5], and thereby the discoveries of effective models of particles (or quasi-particles) that seem to obey these generalized rules for identical particles and statistics. We refer to [6–8] for extensive reviews on these topics.

Although non-interacting bosons and fermions are well understood in terms of single-particle Hilbert spaces and operators, the same cannot be said about particles obeying these generalized interchange statistics. Namely, despite some effort in this direction [9, 10], many-particle quantum states for intermediate and fractional exchange statistics have in general not admitted a simple description in terms of single-particle states restricted by some exclusion principle. The reason for this difficulty is that the general symmetry of the wave function under particle interchange is naturally modeled using pairwise or many-body interactions, hence leaving the much simpler realm of single-particle Hamiltonians (and also introducing other mathematical difficulties as well, already at the formulation of these models).

As a different approach, we would like to stress in the following that the effects of exclusion are also encoded in the Lieb-Thirring inequality [11] for many-particle energy forms. For the case of identical spinless fermions in an external potential V in d-dimensional space, it states that there is a uniform bound for the energy of a normalized N-particle state ψ:

\[ \langle \psi, \hat{H} \psi \rangle \geq - \sum_{k=0}^{N-1} |\lambda_k| \geq - C_d \int |V_-(x)|^{1+d/2} \, d^d x, \]

with the N-particle Hamiltonian operator

\[ \hat{H} = \hat{T}_0 + \hat{V} = \sum_{j=1}^{N} \left( -\frac{1}{2} \nabla_j^2 + V(x_j) \right), \]

the conventions \( \hbar = m = 1 \), \( V_\pm := (V \pm |V|)/2 \), and a positive constant \( C_d \). The inequality (1) incorporates Pauli’s exclusion principle via the intermediate sum over the negative energy levels \( \lambda_k \) of the one-particle Hamiltonian \( \hat{h} = -\frac{1}{2} \nabla^2 + V(x) \). It furthermore incorporates the uncertainty principle, and is in fact equivalent to the kinetic energy inequality

\[ \langle \psi, \hat{T}_0 \psi \rangle \geq \frac{d (2/C_d)^{2/d}}{d + 2} \int \rho(x)^{1+2/d} \, d^d x, \]

involving the one-particle density \( \rho \) of \( \psi \); normalized \( \int \rho(x) \, d^d x = N \). In dimension \( d = 3 \), the expression on the r.h.s. of (2) may be recognized as the kinetic energy approximation from Thomas-Fermi theory. It is in this case conjectured [11] that (2) holds with exactly the Thomas-Fermi expression on the right. The best known result is, however, a factor \((3/\pi^{1/3})^{1/3}\) smaller [12]. The bounds (1) and (2) need to be weakened in the case of weaker exclusion. In the case that each single-particle state can be filled \( q \) times (e.g. in models with \( q \) spin states, or cp. Gentile intermediate statistics [13]) the r.h.s. of the inequalities (1) resp. (2) are to be multiplied by \( q \) resp. \( q^{-2/d} \). Bosons can then be accommodated by \( q = N \), yielding trivial bounds as \( N \to \infty \).

In this Letter we wish to report on a new set of Lieb-Thirring-type inequalities for intermediate and fractional...
statistics, which follow from a corresponding local version of the exclusion principle, applicable to such interacting systems. Our approach is very much inspired by the work [14] of Dyson and Lenard (see also [15]), who used only such a local form of the Pauli principle to rigorously prove the stability of ordinary fermionic matter in the bulk (the inequalities (1) and (2) were subsequently invented by Lieb and Thirring to simplify their proof). Although the numerical constants resulting from our method are comparatively weak, we believe the forms of our bounds to be conceptually very useful, and as a result we also learn something non-trivial about the elusive anyon gas. The majority of the results presented here were derived in full mathematical detail in [16, 17], to which we refer the interested reader for detailed reference.

II. IDENTICAL PARTICLES IN ONE AND TWO DIMENSIONS

We recall the conventional models for intermediate and fractional exchange statistics for scalar non-relativistic quantum mechanical particles in one and two spatial dimensions. As mentioned in the introduction, there are by now a number of standard references for their background and derivations, which we will accordingly skip here. We will mainly follow the notation in [6], with technical details addressed in [17].

Identical particles in 2D, anyons, have the possibility to pick up an arbitrary but fixed phase $e^{i\alpha \pi}$ upon continuous simple interchange of two particles [2, 3]. A standard way to model such (abelian) anyons, in the so-called magnetic gauge, is by means of bosons in $\mathbb{R}^2$ together with a statistical magnetic interaction given by the vector potential

$$A_j = \alpha \sum_{k \neq j} \frac{(x_j - x_k)^I}{|x_j - x_k|^2}, \quad \alpha \in \mathbb{R} \quad (\text{mod } 2),$$

where $x^I$ denotes a 90° counter-clockwise rotation of the vector $x$. This attaches to every particle an Aharonov-Bohm point flux of strength $2\pi \alpha$, felt by all the other particles. The kinetic energy for $N$ such particles is thus given by $T_\Lambda^Q := \langle \psi, T_\Lambda \psi \rangle$,

$$T_\Lambda := \frac{1}{2} \sum_{j=1}^N D_j^2,$$

(3)

where $D_j = -i \nabla_j + A_j$, and $\psi$ is represented as a completely symmetric square-integrable function on $(\mathbb{R}^2)^N$. The case $\alpha = 0$ then corresponds to bosons, and $\alpha = 1$ to fermions.

The case of identical particles confined to move in only one spatial dimension is special and in some sense degenerate, since particles cannot be interchanged continuously without colliding. In quantum mechanics this necessitates some choice of boundary conditions for the wave function at the collision points. Depending on which approach one takes to quantization [2, 6, 18], identical particles in 1D can again be modelled as bosons, i.e. wave functions symmetric under the flip $r \leftrightarrow -r$ of any two relative particle coordinates $r := x_j - x_k$, together with a local interaction potential, singular at $r = 0$ and either of the form $\delta(r)/r^2$. We write

$$V_S(r) := 2\eta \delta(r), \quad V_H(r) := \frac{\alpha(\alpha - 1)}{r^2},$$

(4)

with statistics parameters $\eta, \alpha \in \mathbb{R}$, for the corresponding cases of Schrödinger- resp. Heisenberg-type quantization. These correspond to the choices of boundary conditions for the wave function $\psi$ at the boundary $r = 0$ of the configuration space

$$\frac{\partial \psi}{\partial r} = \eta \psi, \quad \text{at } r = 0^+, \quad \text{resp. } \psi(r) \sim r^\alpha, \quad \text{as } r \to 0^+.$$

Here $\eta = 0$ resp. $\alpha = 0$ represent bosons (Neumann b.c.) while $\eta = +\infty$ resp. $\alpha = 1$ represent fermions (Dirichlet or analytically vanishing b.c.; see [17]). Suggested by such pairwise boundary conditions, one may define [19] the total kinetic energy for a normalized completely symmetric wave function $\psi$ describing $N$ identical particles on the full real line $\mathbb{R}$ to be $T_{S/H}^Q := \langle \psi, T_{S/H}^Q \psi \rangle$ where

$$T_{S/H}^Q := -\frac{1}{2} \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + \sum_{1 \leq j < k \leq N} V_S(x_j - x_k).$$

(5)

In other words, the Schrödinger case is nothing but the Lieb-Liniger model for one-dimensional bosons with pairwise Dirac delta interactions [20], while the Heisenberg case corresponds to the homogeneous part of the Calogero-Sutherland model with inverse-square interactions [21]. For the following results we will restrict to $\eta \geq 0$ (Schrödinger type intermediate statistics) and $\alpha \geq 1$ (Heisenberg type ‘superfermions’) for which the statistics potentials (4) are nonnegative, i.e. repulsive.

It is useful to be able to speak about the expected kinetic energy of a wave function on some local region $Q$ of space (typically a cube in the following). For fermions or bosons we naturally define this quantity to be

$$T_0^Q := \frac{1}{2} \sum_{j=1}^N \int_{Q^N} |\nabla_j \psi|^2 \chi_Q(x_j) \, dx,$$

where $\chi_Q \equiv 1$ on $Q$ and $\chi_Q \equiv 0$ on the complement $Q^c$. Analogously for anyons,

$$T_A^Q := \frac{1}{2} \sum_{j=1}^N \int_{Q^N} |D_j \psi|^2 \chi_Q(x_j) \, dx,$$

and for Schrödinger and Heisenberg type intermediate statistics, $T_{S/H}^Q :=

$$\frac{1}{2} \sum_{j=1}^N \int_{Q^N} \left( |\partial_j \psi|^2 + \sum_{k \neq j} V_{S/H}(x_j - x_k) |\psi|^2 \right) \chi_Q(x_j) \, dx.$$
Note that if the full space $\mathbb{R}^d$ has been partitioned into a family of non-overlapping regions $\{Q_k\}$ then the total kinetic energy decomposes as $T_{0/S/H/A} = \sum_k T_{0/Q_k}$. We will furthermore write $T_F = T_0$ to denote the free kinetic energy for the particular case of fermions in $\mathbb{R}^3$.

### III. LOCAL EXCLUSION

The starting point for our energy bounds will be the following local consequence of the Pauli exclusion principle for fermions, which was used by Dyson and Lenard in their proof of stability of matter [14]: Let $\psi$ be a wave function of $n$ spinless fermions in $\mathbb{R}^3$, i.e. anti-symmetric w.r.t. every pair of particle coordinates, and let $Q$ be a cube of side length $l$. Then, for the contribution to the free kinetic energy with all particles in $Q$, we have

$$\frac{1}{2} \int_{Q^n} \sum_{j=1}^n |\nabla_j \psi|^2 \, dx \geq (n-1) \frac{\xi^2_F}{l^2} \int_{Q^n} |\psi|^2 \, dx,$$

where $\xi_F = \pi/\sqrt{2}$. In other words, the energy is nonzero for $n \geq 2$ and grows at least linearly with $n$. In [14], $Q$ was replaced by a ball of radius $l$ and $\sqrt{2} \xi_F$ by the smallest positive root of the equation $(d^2/d\xi^2)(\sin \xi/\xi) = 0$. The inequality (6) follows by expanding $\psi$ in the eigenfunctions of the Neumann Laplacian on $Q$, or by the method below at the cost of a slightly weaker constant $\xi_F$.

Now, for the 1D case we introduce $\xi_S(\eta l)$ resp. $\xi_H(\alpha)$ to be the smallest positive solutions of $\xi \tan \xi = \eta l$, resp. $(d/d\xi)(\xi^{1/2} J(\xi)) = 0$, where $J$ is the Bessel function of order $\alpha - 1/2$. These $\xi_S/H$ arise as quantization conditions for the wave function upon considering the Neumann problems

$$(-\partial^2_r + V_S/H(r))\psi = \lambda \psi, \quad \partial_r \psi|_{r=\pm l} = 0 \quad (7)$$

in the pairwise relative coordinate $r$ on an interval $[-l, l]$, yielding a lowest bound for the energy $\lambda = \xi^{2}_S/H/l^2$.

In the case of anyons we define

$$C_{\alpha,n} := \min_{p \in \{0,1,\ldots,n-2\}} \min_{q \in \mathbb{Z}} |(2p + 1)\alpha - 2q|, \quad (8)$$

which measures the fractionality of the parameter $\alpha$ and arises from a local pairwise relative magnetic Hardy inequality in 2D [16]. In (8), the minimization over even integers $2q$ has to do with the underlying bosonic symmetry of the wave function, while the odd integer $2p + 1$ depends on the number $p$ of other anyons that can appear inside a two-anyon interchange loop (with the additional +1 stemming from the statistics flux of the interchanging pair itself). Note that for bosons $C_{\alpha=0,n} \equiv 0$ while for fermions $C_{\alpha=1,n} \equiv 1$.

We call the following observation a local exclusion principle for generalized exchange statistics since it implies that the local kinetic energy is nonzero whenever we have more than one particle, and hence that the particles cannot occupy the same single-particle state (which on a local region would be the zero-energy ground state).

**Lemma 1 (Local exclusion principle)** Given any finite interval $Q \subset \mathbb{R}$ of length $|Q|$, we have for $\eta \geq 0$

$$\int_{Q^n} \tilde{\psi} \tilde{T}_S \psi \, dx \geq (n-1) \frac{\xi_S(\eta|Q|)^2}{|Q|^2} \int_{Q^n} |\psi|^2 \, dx,$$

and for $\alpha \geq 1$

$$\int_{Q^n} \tilde{\psi} \tilde{T}_H \psi \, dx \geq (n-1) \frac{\xi_H(\alpha)^2}{|Q|^2} \int_{Q^n} |\psi|^2 \, dx,$$

while for a square $Q \subset \mathbb{R}^2$ with area $|Q|$ and any $\alpha \in \mathbb{R}$

$$\frac{1}{2} \int_{Q^n} \sum_{j=1}^n |D_j \psi|^2 \, dx \geq (n-1) \frac{c C_{\alpha,n}^2}{|Q|} \int_{Q^n} |\psi|^2 \, dx,$$

with $c = 0.179$. It then follows for the expected kinetic energy on a d-dimensional cube $Q$ with volume $|Q|$ that

$$T_{0/S/H/A/F}^Q \geq \frac{\xi^2_S/H/A/F}{|Q|^{2/d}} \left( \int_Q \rho(x) \, dx - 1 \right), \quad (12)$$

where $\xi^2_S/H/A/F$ stands for $\xi_S(\eta|Q|)^2$, $\xi_H(\alpha)^2$, $c C_{\alpha,n}^2$, resp. $\xi^2_F$, with corresponding dimension $d = 1, 2, 3$.

Let us consider the proof for the Heisenberg case. Using the separation of the center-of-mass $n \sum_j \partial^2_j = (\sum_j \partial_j)^2 + \sum_{j<k} (\partial_j - \partial_k)^2$, the (Neumann) kinetic energy for $n \geq 2$ particles on an interval $Q = [a, b]$ is

$$\int_{Q^n} \tilde{\psi} \tilde{T}_H \psi \, dx$$

$$\geq \int_{Q^n} \sum_{j<k} \tilde{\psi} \left( -\frac{1}{2n} (\partial_j - \partial_k)^2 + V_H(x_j - x_k) \right) \psi \, dx$$

$$\geq \frac{2}{n} \sum_{j<k} \int_{Q^{n-2}} \int_{Q \setminus [\delta(R), \delta(R)]} \tilde{\psi} (-\partial^2_r + V_H(r)) \psi \, dr \, dx'$$

$$\geq \frac{2}{n} \int_{Q^{n-2}} \int_{Q \setminus [\delta(R), \delta(R)]} |\psi|^2 \, dr \, dx', \quad (13)$$

where $R := (x_j + x_k)/2$, $r := x_j - x_k$, $x' = (x_1, \ldots, x_j, \ldots, x_N)$, and $\delta(R) := \min\{|R|, |R - a|, |R - b|\}$. We then use (13) and $\delta(R)^2 \geq |Q|^2$ to obtain (10). Inserting the partition of unity $1 = \sum_{A \subseteq \{1,\ldots,n\}} \prod_{\lambda \in A} \chi_A(x) \prod_{\lambda \in A^c} \chi_A^c(x)$ into the expression for $T_H^Q$ we then obtain (cp. [14–17]).
\[ T^Q_{\text{H}} = \sum_A \int_{\mathbb{R}^N} \sum_{j \in A} \frac{1}{2} \left( |\partial_j \psi|^2 + \sum_{(j \neq) k=1}^N V_H(x_j - x_k)|\psi|^2 \right) \prod_{l \in A} \chi_Q(x_l) \prod_{l \notin A} \chi_Q(x_l) \, dx \]

\[ \geq \sum_A \int_{(Q^\text{C})^N+\{-A\}} \int_{Q^\text{A}} \frac{1}{2} \left( \sum_{j \in A} |\partial_j \psi|^2 + \sum_{j \neq k \in A} V_H(x_j - x_k)|\psi|^2 \right) \prod_{l \in A} dx_l \prod_{l \notin A} dx_l \]

\[ \geq \sum_A (|A| - 1) \left( \frac{\xi_H(\alpha)^2}{|Q|^2} \right) \int_{(Q^\text{C})^N+\{-A\}} \int_{Q^\text{A}} |\psi|^2 \prod_{l \in A} dx_l \prod_{l \notin A} dx_l = \frac{\xi_H(\alpha)^2}{|Q|^2} \int_{\mathbb{R}^N} \left( \sum_{j=1}^N \chi_Q(x_j) - 1 \right) |\psi|^2 \, dx, \]

where in the last step we again used the partition of unity. This proves (12) in the \( \alpha \geq 1 \) Heisenberg case. The Schrödinger case follows similarly, while the anyon case requires the application of a magnetic Hardy inequality (also giving rise to a local repulsive potential whose strength is measured by (8)), and we refer to [16, 17] for the proofs.

The constants \( \xi^2_{S/H/A/F} \) of proportionality in (12) appear as lower bounds on the strength of local exclusion, and could e.g. be compared with the global constant of proportionality in Haldane’s generalized exclusion statistics [9]. For the case of anyons, the constant \( \xi^2_{A} \propto C^2_{a,N} \) is actually \( N \)-dependent, and it is clear from the definition (8) that this constant can become identically zero for sufficiently large \( N \) if \( \alpha \) is an even numerator (reduced) fraction. However, we have shown in [16] that for \( \alpha = \mu/\nu \) an odd numerator fraction, the limiting constant is non-zero and equal to \( \lim_{N \to \infty} C_{a,N} = 1/\nu \). It hence also becomes weaker with a bigger denominator \( \nu \) in the statistics parameter. For irrational \( \alpha \) the constant is non-zero for all finite \( N \), but the limit is again zero.

IV. LIEB-THIRRING-TYPE INEQUALITIES

The inequalities (1) and (2) for fermions combine the Pauli exclusion principle with the uncertainty principle to produce non-trivial and useful bounds for the energy as the number of particles \( N \) becomes large. We shall complement the local form of the exclusion principle above with the following local form of the uncertainty principle on a \( d \)-dimensional cube \( Q \), valid for the free kinetic energy of any bosonic wave function \( \psi \), and hence applicable in our cases of intermediate statistics after discarding the positive statistics potentials or, in the case of anyons, using the diamagnetic inequality \( |D_j \psi| \geq |\nabla_j |\psi||\):

\[ T^Q_{0/S/H/A} \geq c_1 \int_Q \rho^{1+2/d} \, dx \int_Q \rho \, dx / |Q|^{2/d}. \]

The constants \( c_2 > c_1 > 0 \) only depend on \( d \). Mathematically, (14) is a form of Poincaré-Sobolev inequality, and we refer to [16, 17] for details and proofs. Note that the r.h.s. is bigger for less constant density, but scales with the number of particles only as \( N \) (in contrast to the Lieb-Thirring inequality).

Combining local uncertainty with local exclusion, and cleverly splitting the space into smaller cubes depending on the density, one can then prove the following energy bounds:

**Theorem 2 (L-T inequalities for 1D Schrödinger)**

*For \( \eta \geq 0 \)*

\[ T_S \geq C_S \int_{\mathbb{R}} \xi_s(2\eta/\rho^*(x))^2 \rho(x)^3 \, dx, \]

(15)

for some constant \( 3 \cdot 10^{-5} \leq C_S \leq 2/3 \), where \( \rho^*(x) \) is defined as the supremum of the averaged density \( \int_Q \rho/|Q| \) over all intervals \( Q \) containing \( x \). In particular, if \( \rho \) is homogeneous, e.g. \( \rho^* \leq \gamma \rho \) for some \( \gamma > 0 \), then

\[ T_S \geq C_S \xi_s(2\eta/\gamma \rho)^2 \int_{\mathbb{R}} \rho(x)^3 \, dx, \]

(16)

and if \( \rho \) is supported on an interval of length \( L \)

\[ T_S/L \geq C_S \xi_s(2\eta/\gamma \rho)^2 \rho^3, \quad \rho := N/L. \]

(17)

![FIG. 1. Plot of \( \xi_s(t) \) (solid) and \( \arctan \sqrt{t + 4t^2/\pi^2} \) (dashed) as a function of \( t \geq 0 \).](image.png)
FIG. 2. Plot of $\xi_H(\alpha)$ as a function of $\alpha \geq 0$.

Compare with Lieb-Liniger [20], where for a free system in the thermodynamic limit $N,L \to \infty$ with fixed density $\bar{\rho}$, $T_H/L \to \frac{1}{2} e(2\eta/\bar{\rho})\bar{\rho}^2$ with $e(t) \sim t, t \ll 1$, $e(t) \to \frac{\pi^2}{2}, t \to \infty$ (see also [22]). A good numerical approximation to $\xi_H$ is given by $\xi_H(t) \approx \arctan(\sqrt{t+4t^2/\pi^2}$ for all $t \geq 0$ (see Fig. 1).

Theorem 3 (L-T inequalities for 1D Heisenberg)
For $\alpha \geq 1$ and arbitrary intervals $Q$ such that the expected number of particles $\int_Q \rho \geq 2$

$$T_H^Q \geq C_H \xi_H(\alpha)^2 \left( \frac{\int_Q \rho dx}{|Q|^2} \right)^3,$$  \hspace{1cm} (18)

with a constant $1/32 \leq C_H \leq 2/3$. It follows in particular that if $\rho$ is confined to a length $L$ and $N \geq 2$ then

$$T_H/L \geq C_H \xi_H(\alpha)^2 \bar{\rho}^3, \quad \bar{\rho} := N/L.$$  \hspace{1cm} (19)

We have $\xi_H(1) = \pi/2$ and, asymptotically, $\xi_H(\alpha) \sim \alpha$ as $\alpha \to \infty$ (see Fig. 2). Compare with Calogero and Sutherland [21], where one finds $T_H/L \to \frac{2}{\pi} \alpha^2 \bar{\rho}^3$ in the thermodynamic limit $N,L \to \infty$, and with applications of Thomas-Fermi theory [23]. The local bound (18) can e.g. be applied with the addition of an external potential of Thomas-Fermi theory [23]. We sketch a proof only for the simpler anyonic case, and refer to [16, 17] for details. For an application of the same method to fermions in 3D and the generalization to $q$ spin states we refer to [24] where a model allowing for point interactions was considered.

Let us for simplicity assume $\rho$ to be supported on some square $Q_0$ in the plane which we proceed to split into four smaller squares iteratively, organizing the resulting subquares $Q$ in a tree $T$ (see Fig. 4). The procedure can be arranged so that $Q_0$ is finally covered by a collection $Q \in T_B$ of non-overlapping squares marked B s.t. $2 \leq \int_Q \rho < 8$, and $Q \in T_A$ marked A s.t. $0 \leq \int_Q \rho < 2$, and at least one B-square is at the topmost level of every branch of the tree $T$. On the B-squares we use (12) together with (14) to obtain (with $c_1' > 0$ some numerical constants)

$$T_A^Q \geq C_{A,N}^2 \left( c_1' \int_Q \rho^2 + \frac{c_1}{|Q|} \right), \quad Q \in T_B.$$  \hspace{1cm} (22)

The A-squares are further grouped into a subclass $A_2$ on which the density is sufficiently non-constant, $\int_Q \rho^2 > \frac{2\rho_0}{c_1} (\int_Q \rho^2)/|Q|$ for $Q \in T_{A_2} \subseteq T_A$, so that by (14)

$$T_A^Q \geq \frac{c_1}{4} \int_Q \rho^2, \quad Q \in T_{A_2},$$  \hspace{1cm} (23)

and a subclass $A_1$ on which $\int_Q \rho^2 \leq \frac{2\rho_0}{c_1} (\int_Q \rho^2)/|Q|$.

One can then use the structure of the splitting of squares to prove that, for the set $A_1(Q_B)$ of all such $A_1$-squares which can be found by stepping backwards in the tree $T$ from a fixed B-square $Q_B$ and then one step forward,

$$\sum_{Q \in A_1(Q_B)} \int_Q \rho^2 \leq \sum_{k=0}^{\infty} \frac{3}{c_1} \frac{2}{4^k |Q_B|} = \frac{32c_2}{c_1} \frac{1}{|Q_B|}.$$  \hspace{1cm} (24)
In other words the energy on all subsquares with constant low density is dominated by that from exclusion on the B-squares. We therefore find from (22) that

\[ T^Q_{\alpha} \geq C_{\alpha,N}^2 \left( c_1 \int_{Q_0} \rho^2 + c_3 \sum_{Q \in A(Q_0)} \int_Q \rho^2 \right), \]

and hence, since all A₁-squares are covered in this way,

\[ T_A = \sum_{Q \subseteq \mathbb{R}^2} T^Q_A \geq \sum_{Q \subseteq \mathbb{R}^2 \cup T_2} T^Q_A \geq C_{\alpha,N} \int_{Q_0} \rho^2 \text{ for some numerical constant } C_{\alpha} > 0. \]

For (17) and (21) we use \( \int_{Q_0} \rho^2 dx \geq N^\mu|Q_0|^{1-\mu} \), and for (21) we used the fact that \( \lim_{N \to \infty} C_{\alpha,N} = 1/\nu \) for such odd numerator fractions and zero otherwise.

V. APPLICATION AND DISCUSSION

The bound (21) provides a non-trivial lower bound for the energy per unit area for an ideal gas of anyons with odd-fractional statistics parameter \( \alpha \). The numerical constant \( C_{\alpha} \geq 10^{-4} \) in this bound has in [17] been improved to \( \geq 0.059 \), which is still quite far from the exact semiclassical constant \( \pi \) for the two-dimensional spinless fermion gas. In any case, this non-trivial bound raises the very interesting question whether such Lieb-Thirring inequalities are in fact not valid for even numerator and irrational \( \alpha \). We give some motivation for why this could be the case by considering the following observations.

By minimizing the lower bound for the energy following from (20) in terms of \( \rho \) it is straightforward [17] to obtain a bound

\[ T_A + \langle \hat{V} \rangle \psi \geq E_0 \geq \frac{1}{3} \sqrt{8C_{\alpha,N} \omega N^{3/2}} \]

for the ground state energy \( E_0 \) of \( N \) anyons placed in a harmonic oscillator potential \( V(x) = \frac{\omega^2}{2} |x|^2 \). For odd numerator \( \alpha \) this improves the bound given in [25] (which is also valid for arbitrary \( \alpha \)):

\[ E_0 \geq \omega \left( N + \left| L + \alpha \frac{N(N-1)}{2} \right| \right), \]

where \( L \) denotes the total angular momentum of the state \( \psi \). Note that if \( L = -\alpha \binom{N}{2} \) (which could occur for certain \( N \) and fractional \( \alpha \) as long as the r.h.s. is an even integer) then this bound reduces to the bosonic bound for the energy.

It is well-known that energy levels and degeneracies in this model depend very non-trivially on both \( N \) and \( \alpha \), and we can point out certain similarities in the limiting graph of \( C_{\alpha,N} \) (see Fig. 3) with known features in spectra for \( N = 2, 3, 4 \) and corresponding extrapolations to large \( N \) [2, 7, 25, 26]. It is e.g. intriguing to compare this graph — which can be obtained by cutting out a wedge of slope \( \nu \) from the upper half-plane at every even numerator rational point \( \mu/\nu \) on the horizontal axis — with the general structure indicated in Fig. 1 in [25].

It was in [25] argued by perturbation theory that the behavior for the exact ground state energy as \( N \to \infty \) is approximately \( E_0 \sim \sqrt{\omega} N^{3/2} \) for \( \alpha \sim 0 \) and \( E_0 \sim \sqrt{\omega} N^{3/2}/3 \) for \( \alpha \sim 1 \), requiring \( L = -\alpha \binom{N}{2} + O(N^{3/2}) \) by (26). In any case, as long as \( E_0 \) grows slower than quadratically with \( N \), we are led to look for sequences of ground states \( \psi \) with \( L = -\alpha \binom{N}{2} \) to leading order in \( N \). Note that there are also constraints on \( \psi \) coming from the bosonic symmetry.

Motivated by the Laughlin states in the fractional quantum Hall effect [27], we could for particular \( N = \nu K \) consider trial wave functions of the form \( \psi = \Phi \psi_\alpha \), with

\[ \psi_\alpha := \prod_{j<k} |z_{jk}|^{-\alpha} \prod_{\sigma \in S_N} \prod_{q=1}^{\nu} (\xi_{jk})^\mu \prod_{j \in \sigma \mathcal{V}_q} \varphi_\mu(x_j) \]

for even numerator fractions \( \alpha = \mu/\nu \in [0,1] \), and

\[ \psi_\alpha := \prod_{j<k} |z_{jk}|^{-\alpha} \prod_{\sigma \in S_N} \prod_{q=1}^{\nu} (\xi_{jk})^\mu \prod_{j \in \sigma \mathcal{V}_q} K^{-1}_{k=0} \varphi_\mu(x_j \in \sigma \mathcal{V}_q) \]

for odd numerators \( \mu \), where \( \Phi \) should regularize the short-scale behavior (due to the singular Jastrow factor). We write \( z_{jk} := z_j - z_k \) for the complex relative particle coordinates, \( \varphi_\mu \) denote the eigenstates of the one-particle Hamiltonian \( \hat{h} = -\frac{1}{2} \Delta + V \) and of which we may form a Slater determinant \( \prod_k \varphi_\mu \), while \( \mathcal{E}_q \) and \( \mathcal{V}_q \) are sets of edges and vertices of \( \nu \) disjoint complete graphs involving \( K \) particles each, and which we act on by permutations \( \sigma \) to symmetrize the states. Two possible choices of regularizing symmetric functions \( \Phi \), giving rise to the expected pairwise short-scale behavior \( \sim |z_{jk}|^{\alpha} \) in \( \psi \), could be

\[ \Phi_{\nu_0} = \prod_{j<k} |z_{jk}|^{2\alpha} (r_0^2 + |z_{jk}|^2)^{-\alpha}, \]

with a parameter \( r_0 > 0 \), or the parameter-free (but less smooth)

\[ \Phi = \prod_{j} \prod_{k=1}^{N} |z_{jk}|^{\alpha}. \]
with $k(j)$ denoting the $k$th nearest neighbor of particle $j$. These states $\psi$ have $L = -\alpha \left( \frac{N}{2} \right) + \alpha \nu \frac{\omega + 1}{2} N$ (for (27) and for certain magic numbers $K$ in (28)) and the property that only up to $\nu$ particles can be selected in each term without involving a repulsive factor $(\varepsilon_j \mu)^{\nu}$ from an edge in $\mathcal{E}_q$ for some $q$, allowing for the formation of groups of $\nu$ anyons with integer statistics flux $\mu$. Namely, while the Jastrow factor acts to attract all particles, this attraction is on large scales exactly balanced whenever a group of $\nu$ non-repelling anyons has formed, since an anyon $x_j$ far outside the group, seeing the total attractive factor $\sim (r^{-\nu})^\nu = r^{-\mu}$ where $r$ is the distance from the group, is also repelled by at least one anyon $x_k$ in the group, with a factor $|\varepsilon_{jk}|^\mu \sim r^{\mu}$ from that corresponding edge in $\mathcal{E}_q$. This balance could act to distribute the anyons, on the average, in such groups of $\nu$. Furthermore, the total contribution from such a group to the statistics potential $A_j$ seen by the distant particle $x_j$ would be $\sim \omega r^I / r^2 = \mu r^I / r^2$, while the particle also has an orbital angular momentum $-\mu$ around the group (due to that same edge to $x_k$ and phase of $(\varepsilon_{jk})^\mu$) with velocity $\sim -\mu r^I / r^2$, again leading to an exact cancellation.

The forms (27) and (28) bring out a structural difference between even and odd numerators $\mu$. The limit $\alpha = 1$ of (28) is the fermionic ground state in the bosonic representation, while the states in (27) are (modulo the Jastrow factor actually found to be exactly the Read-Rezayi states [28] for fractional Quantum Hall liquids in their bosonic form. The state (27) is an exact but singular (requiring the regularization by $\Phi$) eigenstate of the Hamiltonian with energy $E = \omega(N + \text{deg} \psi_0)$, where $\text{deg} \psi_0 = -\alpha \frac{N}{2} N$ is the total degree of the non-Gaussian part of the wave function (cp. [29]). In all known exact eigenstates there is this simple correspondence between the degree and the energy. It is an interesting fact that adding the degree of $\Phi$ in the nearest-neighbor form (30) produces $\omega(1 + \alpha \frac{x^2}{2}) N$, i.e. exactly the r.h.s. of (26) for the above value of $L$, speaking for a low energy for even numerator fractions. On the other hand, the degree of the odd numerator states (28) necessarily grows with $K$ as $\sim K^{3/2}$ due to the Slater determinant. While the resulting energy $E = \omega(N + \text{deg} \psi_0) \sim \omega(N/N\nu)^{3/2}$ satisfies but does not match the bound (25) exactly w.r.t. $\alpha$, a corresponding picture of ideal anyons forming essentially free $\nu$-anyon groups with fermionic type statistics would actually match the form of the bound (21), involving the reduced density $\bar{\rho}/\nu = K/L^2$.

We finally remark that there are also many interesting connections between the forms of the fractions appearing here and those of fractionally charged quantum Hall quasiparticles [30]. Another question concerns possible relations with $q$-commutation relations, with $q = e^{i\pi \alpha}$ [31].

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[19] As emphasized in [6], this definition is the natural one for all $N$ in the Schrödinger case, however, the Heisenberg case arises naturally only for $N = 2$, but is extended in this way to all $N$ (keeping this terminology for brevity). The resulting model is relevant for anyons in the lowest Landau level; see e.g. T. H. Hansson, J. M. Leinaas, J. Myrheim, Nucl. Phys. B 384, 559 (1992), and S. Ouvry, Phys. Lett. B 510, 335 (2001).
[28] See N. Read, E. Rezayi, Phys. Rev. B 59, 8084 (1999), and the bosonic version given in A. Cappelli et al., Nucl. Phys. B 599, 499 (2001). We were not aware of this interesting coincidence at the time we first discovered these anyonic trial states. It is in this context amusing to speculate whether non-abelian anyons could arise as quasiparticle excitations of such abelian anyon states.