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A collocation method for a hypersingular boundary integral equation via trigonometric differentiation

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Abstract

Revisiting earlier work [8] on this topic, we propose a fully discrete collocation method for the hypersingular integral equation arising from the double-layer approach for the solution of Neumann boundary value problems in two dimensions which is based on trigonometric differentiation to discretize the principal part of the hypersingular operator. Convergence in a Sobolev space setting is proven and the spectral convergence of the method is exhibited by an example.

1 Introduction

The scattering of time-harmonic acoustic or electromagnetic waves from infinitely long cylindrical obstacles is modeled by exterior boundary value problems for the Helmholtz equation in two dimensions. In this paper we will be concerned with the Neumann boundary condition, i.e., scattering from sound-hard or non-conducting obstacles. Let $\Omega \subset \mathbb{R}^2$ be a simply connected bounded domain with infinitely differentiable boundary $\partial\Omega$ and unit normal vector ν directed into the exterior of Ω. Given $g \in H^{-1/2}(\partial\Omega)$ the exterior Neumann problem for the Helmholtz equation consists of finding a solution $u \in H^1_{\rm loc}(\mathbb{R}^2 \setminus \bar{\Omega})$ to

$$\Delta u + \kappa^2 u = 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{\Omega}$$

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with wave number $\kappa > 0$ that satisfies

$$\frac{\partial u}{\partial \nu} = g \quad \text{on } \partial\Omega$$

in the weak sense and the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \quad r = |x|,$$

uniformly for all directions. By

$$\Phi(x,y) = \frac{i}{4}\, H_0^{(1)}(\kappa|x-y|), \quad x \neq y,$$

we denote the fundamental solution to the two-dimensional Helmholtz equation in terms of the first kind Hankel function of order zero. Trying to find the solution in the form of a double-layer potential

$$u(x) = \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\, \psi(y)\, ds(y), \quad x \in \mathbb{R}^2 \setminus \bar{\Omega} \tag{1.1}$$

with density $\psi \in H^{1/2}(\partial\Omega)$ leads to the hypersingular integral equation

$$T\psi = g \tag{1.2}$$

with the hypersingular operator $T : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ given by

$$(T\psi)(x) := \frac{1}{\partial\nu(x)} \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial\nu(y)}\, \psi(y)\, ds(y), \quad x \in \partial\Omega.$$

Provided $-\kappa^2$ is not an eigenvalue for the Helmholtz equation in Ω with homogeneous Neumann condition on $\partial\Omega$, for each $g \in H^{-1/2}(\partial\Omega)$ the equation (1.2) has a unique solution $\psi \in H^{-1/2}(\partial\Omega)$ (for details in the three-dimensional case see [6]).

The transformation

$$\begin{aligned}(T\psi)(x) = {} & \frac{\partial}{\partial s(x)} \int_{\partial\Omega} \Phi(x,y)\, \frac{\partial\psi}{\partial s}(y)\, ds(y) \\ & + \kappa^2 \nu(x) \cdot \int_{\partial\Omega} \Phi(x,y)\nu(y)\psi(y)\, ds(y), \quad x \in \partial\Omega,\end{aligned} \tag{1.3}$$

of the hypersingular operator T is known as *Maue's formula* [10] and reflects the hypersingularity of T. Here, $\partial/\partial s$ denotes the derivative on $\partial\Omega$ with respect to arc length and the dot indicates the bilinear inner product between vectors in $\mathbb{R}^2$ or $\mathbb{C}^2$. For a proof of (1.3) for the limiting potential theoretic case $\kappa = 0$ we refer to [9] and the proof for the three-dimensional Helmholtz case can be found in [6]. Maue's formula suggests to discretize (1.2) via numerical differentiation of a discretization of the single-layer operator in order to take advantage of the simpler structure of the latter. Observing Atkinson's [1] remark *...the most efficient numerical methods for solving boundary integral equations on smooth planar boundaries are those based on trigonometric polynomial approximations, and such methods are sometimes called spectral methods. When calculations using piecewise polynomial approximations are compared with those using trigonometric polynomial approximations, the latter are almost always the more efficient*, in this paper we propose a collocation method for solving the hypersingular integral equation (1.2) via Maue's formula using trigonometric interpolation and differentiation.

To this end we first need to parameterize the boundary integral equation (1.2). We assume that the boundary curve $\partial\Omega$ is described by a regular infinitely differentiable and 2π–periodic parametric representation of the form

$$\partial\Omega = \{z(t) : 0 \le t \le 2\pi\} \tag{1.4}$$

satisfying $|z'(t)| > 0$ for all t. After introducing the parameterized single-layer potential operator $S : H^{-1/2}[0, 2\pi] \to H^{1/2}[0, 2\pi]$ by

$$(S\varphi)(t) := \frac{i}{4} \int_0^{2\pi} H_0^{(1)}(\kappa|z(t) - z(\tau)|)\varphi(\tau)\, d\tau, \quad 0 \le t \le 2\pi,$$

and the differentiation operator $D : H^{1/2}[0, 2\pi] \to H^{-1/2}[0, 2\pi]$ by

$$D\varphi := \varphi',$$

the parameterized version of Maue's formula (1.3) reads

$$|z'|\,(T\psi) \circ z = DSD\varphi + \kappa^2 z' \cdot S(z'\varphi) \tag{1.5}$$

for $\varphi = \psi \circ z$.

Due to the logarithmic behavior of the Hankel function $H_0^{(1)}(t)$ as $t \to 0$, the kernel

$$H(t, \tau) := \frac{i}{4}\, H_0^{(1)}(\kappa|x(t) - x(\tau)|)$$

of the operator S has a logarithmic singularity. Therefore, following the presentation in [6] we split KH into

$$H(t,\tau) = H_1(t,\tau) \ln\left(4\sin^2\frac{t-\tau}{2}\right) + H_2(t,\tau)$$

where

$$H_1(t,\tau) := -\frac{1}{4\pi}\, J_0(\kappa|x(t)-x(\tau)|)$$

$$H_2(t,\tau) := H(t,\tau) - H_1(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right)$$

and J_0 denotes the Bessel function of order zero. The kernels H_1 and H_2 turn out to be infinite differentiable. In particular, using the Taylor expansions for the Bessel and Neumann functions J_0 and Y_0 one can deduce the diagonal term

$$H_2(t,t) = \frac{i}{4} - \frac{C}{2\pi} - \frac{1}{2\pi}\ln\left(\frac{\kappa}{2}\,|x'(t)|\right), \quad 0 \le t \le 2\pi,$$

with Euler's constant C.

In order to write the equation (1.4) in a form that exhibits the logarithmic singularities we introduce the following operators. We begin with the leading part of the single-layer operator S given by

$$(S_0\varphi)(t) := \frac{1}{2\pi}\int_0^{2\pi} \ln\left(4\sin^2\frac{t-\tau}{2}\right)\varphi(\tau)\,d\tau, \quad 0 \le t \le 2\pi. \tag{1.6}$$

Its basic properties follow from the fact that the trigonometric basis functions $f_m(t) := e^{imt}$ for $m \in \mathbb{Z}$ are eigenfunctions, i.e.,

$$S_0 f_m = \beta_m f_m \tag{1.7}$$

with the eigenvalues $\beta_m = -1/|m|$ for $m \neq 0$ and $\beta_0 = 0$. This, in particular, implies that $S_0 : H^p[0,2\pi] \to H^{p+1}[0,2\pi]$ is bounded for all $p \in \mathbb{R}$ (see [9, Theorem 8.22]). The principal part of the operator on the right-hand side of (1.5) can now be introduced as

$$T_0 := DS_0D + M \tag{1.8}$$

where M is the mean value operator given by

$$M : g \mapsto \frac{1}{2\pi}\int_0^{2\pi} g(t)\,dt.$$

From

$$T_0 f_m = \gamma_m f_m \tag{1.9}$$

for $m \in \mathbb{Z}$ with $\gamma_m = |m|$ for $m \neq 0$ and $\gamma_0 = 1$ we observe that $T_0 : H^p[0, 2\pi] \to H^{p-1}[0, 2\pi]$ is an isomorphism. Finally, the compact perturbations of T_0 in (1.5) are of the form

$$\begin{aligned}(A\varphi)(t) &:= \frac{1}{2\pi}\int_0^{2\pi} \left\{K_A(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + L_A(t,\tau)\right\}\varphi(\tau)\,d\tau,\\ (B\varphi)(t) &:= \frac{1}{2\pi}\int_0^{2\pi} \left\{K_B(t,\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + L_B(t.\tau)\right\}\varphi(\tau)\,d\tau,\end{aligned} \tag{1.10}$$

for $0 \leq t \leq 2\pi$ with infinitely differentiable kernels K_A, L_A, K_B, and L_B such that

$$K_A(t,t) = 0, \quad 0 \leq t \leq 2\pi. \tag{1.11}$$

For all $p \geq 0$, the latter assumption implies that $A : H^p[0, 2\pi] \to H^{p+2}[0, 2\pi]$ is bounded [9, Theorem 13.20]) whereas, without the vanishing diagonal values for K_B we only have that $B : H^p[0, 2\pi] \to H^{p+1}[0, 2\pi]$ is bounded [9, Theorem 12.15].

Now we can summarize our task into designing and analyzing a numerical method for the approximate numerical solution of a hypersingular integral equation of the form

$$T_0\varphi - DAD\varphi - B\varphi = f \tag{1.12}$$

with the above operators T_0, A and B. Given the fact that as consequence of its positive eigenvalues according to (1.9) the operator T_0 is strictly coercive, a Galerkin method offers the advantage of a straightforward convergence analysis [2, 4, 7, 11]. However, designing, analyzing and implementing a fully discrete variant is more involved than the corresponding task for a collocation method. Therefore the latter seems to be preferable.

The plan for developing a collocation method in this paper is as follows. In Section 2 we will introduce a semi-discrete collocation method via trigonometric interpolation with an even number of equidistant collocation points and establish a convergence result in the Sobolev space setting. This is followed in Section 3 by a fully discrete variant where the integral operators A and B are approximated by quadrature operators A_n and B_n via trigonometric interpolation quadratures that take proper care of the logarithmic singularities of A and B. The differentiation operator D will be approximated by trigonometric differentiation, i.e., by differentiation of the trigonometric interpolation polynomial. Again we will establish a convergence result for the fully discrete version and conclude with a numerical example exhibiting the rapid convergence for smooth data.

The collocation method as described in [8] also exploits Maue's formula. However, only for the inner derivative trigonometric differentiation is employed. The derivative of the single-layer potential is reduced to a Hilbert transform and this results in a more complicated operation for splitting off the logarithmic singularities. As a result the method proposed in this paper is much easier to implement than the method in [8].

A similar method, without convergence analysis, has been recently employed by Cakoni and Kress [3] for a closely related hypersingular integral equation arising from the solution of a generalized impedance boundary value problem.

2 Semi-discrete collocation

We begin by describing a semi-discrete method by collocation via trigonometric interpolation. Let X_n be the space of trigonometric polynomials of degree less than or equal to n of the form

$$\varphi(t) = \sum_{m=0}^{n} \alpha_m \cos mt + \sum_{m=1}^{n-1} \beta_m \sin mt \tag{2.1}$$

and denote by P_n the interpolation operator that maps 2π-periodic functions g into the unique trigonometric polynomial $P_n g$ that interpolates $(P_n g)(t_j) = g(t_j)$ at the equidistant interpolation points $t_j := 2\pi j/n$ for $j = 0, \ldots, 2n-1$. For the interpolation error we note that

$$\|P_n g - g\|_q \leq \frac{C}{n^{p-q}} \, \|g\|_p, \quad 0 \leq q \leq p, \ \frac{1}{2} < p, \tag{2.2}$$

for all $g \in H^p[0, 2\pi]$ and some constant C depending on p and q (see [9, Theorem 11.8]).

From (1.9) we observe that T_0 maps X_n into itself. Hence, we can use the set $t_j = j\pi/n$, $j = 0, \ldots, 2n-1$, as collocation points. We assume unique solvability of (1.12) and approximate the solution φ by a trigonometric polynomial $\varphi_n \in X_n$ satisying the projected equation

$$T_0\varphi_n - P_n DAD\varphi_n - P_n\varphi_n = P_n f. \tag{2.3}$$

For this semi-discrete collocation method we can state the following result.

Theorem 2.1 *Under the assumption that the operator $T_0 - DAD - B$ is bijective, the semi-discrete collocation method given by (2.3) converges in $H^p[0, 2\pi]$ for each $p \geq 1$.*

Proof. All constants occurring in this proof depend on p. For $p \geq 1$, using (2.2) and the boundedness of $A : H^p[0, 2\pi] \to H^{p+2}[0, 2\pi]$ we obtain

$$\|DP_nAD\varphi - DAD\varphi\|_{p-1} \leq \frac{c_1}{n} \|\varphi\|_p \tag{2.4}$$

for all $\varphi \in H^p[0, 2\pi]$ and some constant c_1. With the triangle inequality, again using (2.2) we can estimate

$$\|P_nD\psi - DP_n\psi\|_{p-1} \leq \|P_nD\psi - D\psi\|_{p-1} + \|D(\psi - P_n\psi)\|_{p-1} \leq \frac{c_2}{n} \|\psi\|_{p+1}$$

for all $\psi \in H^{p+1}[0, 2\pi]$ and some constant c_2. From this, setting $\psi = AD\varphi$ and using the boundedness of $A : H^p[0, 2\pi] \to H^{p+2}[0, 2\pi]$ we find that

$$\|P_nDAD\varphi - DP_nAD\varphi\|_{p-1} \leq \frac{c_3}{n} \|\varphi\|_p \tag{2.5}$$

for all $\varphi \in H^p[0, 2\pi]$ and some constant c_3. Now we combine (2.4) and (2.5) to obtain

$$\|P_nDAD\varphi - DAD\varphi\|_{p-1} \leq \frac{c_4}{n} \|\varphi\|_p \tag{2.6}$$

for all $\varphi \in H^p[0, 2\pi]$ and some constant c_4.

Using (2.2) and the boundedness of $B : H^p[0, 2\pi] \to H^{p+1}[0, 2\pi]$ we can estimate

$$\|P_nB\varphi - B\varphi\|_{p-1} \leq \frac{c_5}{n} \|\varphi\|_p$$

for all $\varphi \in H^p[0, 2\pi]$ and some constant c_5. From this estimate and (2.6) it follows that $\|P_n(DAD + B) - (DAD + B)\|_{H^p \to H^{p-1}} \to 0$ and the assertion of the theorem follows from the standard convergence analysis for projection methods for operator equations that are compact perturbations of a principal operator that leaves the approximation space invariant (see [9, Theorem 13.12]). □

3 Fully discrete collocation

We now proceed with describing a fully discrete method for which we have to approximate both the integral operators A and B and the differentiation operator D. To this end for the latter we give a short description of trigonometric differentiation which approximates D by $D_n := DP_n$, i.e., the derivative Dg of a 2π-periodic function g by the derivative $D_n g$ of the trigonometric interpolation polynomial $P_n g \in X_n$. From the Lagrange basis

$$L_k(t) = \frac{1}{n}\left\{1 + 2\sum_{m=1}^{n-1}\cos m(t-t_k) + \cos n(t-t_k)\right\}, \quad k = 0,\dots,2n-1, \tag{3.1}$$

for trigonometric interpolation by summation and straightforward differentiation we obtain that

$$(D_n g)(t_j) = \sum_{k=0}^{2n-1} d_{|j-k|}^{(n)} g(t_k), \quad j = 0,\dots,2n-1,$$

where

$$d_j^{(n)} = \begin{cases} \dfrac{(-1)^j}{2}\cot\dfrac{j\pi}{2n}, & j = 1,\dots,2n-1, \\ 0, & j = 0. \end{cases} \tag{3.2}$$

From (2.2) we immediately have the error estimate

$$\|D_n g - Dg\|_{q-1} \le \frac{C}{n^{p-q}}\,\|g\|_p, \quad 0 \le q \le p,\ \frac{1}{2} < p, \tag{3.3}$$

for all $g \in H^p[0, 2\pi]$ and some constant C depending on p and q.

For the approximations A_n and B_n we use the quadrature operators defined via interpolatory quadratures

$$(A_n\varphi)(t) := \frac{1}{2\pi}\int_0^{2\pi}\left\{[P_n(K_A(t,\cdot)\varphi](\tau)\ln\left(4\sin^2\frac{t-\tau}{2}\right) + [P_n(L_A(t,\cdot)\varphi](\tau)\right\}d\tau$$

and the analogous expression for B_n. From (1.7) and the Lagrange basis (3.1) the discrete form

$$(A_n\varphi)(t_j) := \sum_{k=0}^{2n-1}\left\{R_{|j-k|}^{(n)}K_A(t_j,t_k) + \frac{1}{2n}\,L_A(t_j,t_k)\right\}\varphi(t_k), \quad j = 0,\dots,2n-1,$$

of A_n (and correspondingly of B_n) can be deduced with the weights

$$R_j^{(n)} = -\frac{1}{n}\left\{1 + 2\sum_{m=1}^{n-1}\frac{1}{m}\cos mt_j + \frac{1}{2n}\cos nt_j\right\}, \quad j = 0, \dots, 2n-1, \tag{3.4}$$

(see [9, Chapter 12]). For these approximations, the error estimates

$$\|B_n\varphi - B\varphi\|_{q+1} \le \frac{C}{n^{p-q}}\,\|\varphi\|_p, \quad 0 \le q \le p,\ \frac{1}{2} < p, \tag{3.5}$$

for all $\varphi \in H^p[0, 2\pi]$ and some constant C depending on p and q (and correspondingly for A) are available (see [9, Theorem 12.18]). Further, due to $K_A(t, t) = 0$, the estimate on $A_n - A$ can be strengthened into

$$\|(P_nA_n - P_nA)\varphi\|_{q+1} \le \frac{C}{n^{p-q+1}}\,\|\varphi\|_p, \quad 0 \le q \le p,\ \frac{1}{2} < p, \tag{3.6}$$

for all trigonometric polynomials φ of degree less than or equal to n and some constant C depending on p and q.

The latter estimates now can be used to prove convergence for the fully discrete method.

Theorem 3.1 *Under the assumption that the operator $T_0 - DAD - B$ is bijective, the fully discrete collocation method converges in $H^p[0, 2\pi]$ for each $p \ge 1$.*

Proof. For all trigonometric polynomials φ of the form (3.1), in view of $D_n\varphi = D\varphi$, we can transform

$$P_n(D_nA_nD_n - DAD)\varphi = P_nD(P_nA_n - A)D\varphi$$

$$= P_nDP_n(A_n - A)D\varphi + P_nD(P_nA - A)D\varphi.$$

Note that the interpolation operators $P_n : H^p[0, 2\pi] \to H^p[0, 2\pi]$ are uniformly bounded for $p > 1/2$ as consequence of (2.2). Then for $p \ge 1$, with the aid of (3.6) for the first term on the right-hand side we can estimate

$$\|P_nDP_n(A_n - A)D\varphi\|_{p-1} \le \frac{c_1}{n}\,\|\varphi\|_p$$

for all trigonometric polynomials φ of the form (3.1) and some constant c_1. For the second term, from (2.2) and the boundedness of $A : H^p[0, 2\pi] \to H^{p+2}[0, 2\pi]$ we conclude that

$$\|P_n D(P_n A - A) D\varphi\|_{p-1} \le \frac{c_2}{n} \, \|\varphi\|_p$$

for all trigonometric polynomials φ of the form (3.1) and some constant c_2. Combining both estimates we find that

$$\|P_n(D_n A_n D_n - DAD)\varphi\|_{p-1} \le \frac{c_3}{n} \, \|\varphi\|_p \tag{3.7}$$

for all trigonometric polynomials φ of the form (3.1) and some constant c_3. The boundedness of P_n and the estimate (3.5) imply that

$$\|P_n(B_n - B)\varphi\|_{p-1} \le \frac{c_4}{n} \, \|\varphi\|_p \tag{3.8}$$

all $\varphi \in H^p[0, 2\pi]$ and some constant c_4. Combining (3.7) and (3.8) we arrive at the estimate

$$\|P_n(D_n A_n D_n - DAD)\varphi + \|P_n(B_n - B)\|_{p-1} \le \frac{c}{n} \, \|\varphi\|_p \tag{3.9}$$

for all trigonometric polynomials φ of the form (3.1) and some constant c.

From (3.3) and (3.5), applied to $A_n - A$, we conclude uniform boundedness of the operators $D_n : H^p[0, 2\pi] \to H^{p-1}[0, 2\pi]$ and $A_n : H^{p-1}[0, 2\pi] \to H^p[0, 2\pi]$. Together with the uniform boundedness of $P_n : H^p[0, 2\pi] \to H^p[0, 2\pi]$ this implies that the operators $P_n(D_n A_n D_n - DAD) : H^p[0, 2\pi] \to H^{p-1}[0, 2\pi]$ are uniformly bounded. For a fixed trigonometric polynomial φ we have $D_n\varphi = D\varphi$ for all sufficiently large n. For these n we can write

$$(D_n A_n D_n - DAD)\varphi = D_n(A_n - A)D\varphi + (D_n - D)AD\varphi$$

and consequently the estimates (3.3) and (3.5) imply that

$$\|P_n(D_n A_n D_n - DAD)\varphi\|_{p-1} \to 0, \quad n \to \infty,$$

that is, we have pointwise convergence for all trigonometric polynomials. By the Banach–Steinhaus theorem this implies pointwise convergence for all $\varphi \in H^p[0, 2\pi]$. Together with the estimate (3.5) for $B_n - B$ we now have pointwise convergence of the operators $P_n(D_n A_n D_n + B_n - A - B) : H^p[0, 2\pi] \to$

$H^{p-1}[0,2\pi]$ to zero. From this and the estimate (3.9) the statement of the theorem again follows by applying a standard convergence results for collocation methods (see [9, Corollary 13.13]). $\square$

We note that from the general error analysis of, for example, Theorem 13.12 and Corollary 13.13 in [9] the approximation error is determined by the interpolation error for the exact solution and how well the quadrature operators A_n and B_n approximate A and B for the exact solution. Therefore, in particular, we have super algebraic convergence of order $O(n^{-m})$ for all $m \in \mathbb{N}$ if the exact solution is infinitely differentiable.

From the Lagrange basis (3.1) and the eigenvalues (1.9) we obtain that

$$(P_n T_0 \varphi)(t_j) = \sum_{k=0}^{2n-1} b^{(n)}_{|j-k|} \varphi(t_k), \quad j = 0, \ldots, 2n-1,$$

for all trigonometric polynomials φ of degree n the form (3.1) where

$$b^{(n)}_j = \begin{cases} \dfrac{1}{2}\, [(-1)^j - 1] \sin^{-2} \dfrac{j\pi}{2n}\,, & j = 1, \ldots, 2n-1, \\[2ex] \dfrac{n}{2}\,, & j = 0. \end{cases} \tag{3.10}$$

With this our fully discrete collocation method leads to the linear system for the nodal values of the approximating trigonometric polynomial $\varphi^{(n)}$ in the form

$$\sum_{k=0}^{2n-1} (U_{jk} - V_{jk})\, \varphi^{(n)}(t_k) = f(t_j), \quad j = 0, \ldots, 2n-1.$$

Here the matrix

$$U_{jk} := \frac{1}{2n} + b^{(n)}_{|j-k|} - \sum_{\ell=0}^{2n-1} \sum_{m=0}^{2n-1} d^{(n)}_{|j-\ell|} \left\{ R^{(n)}_{|\ell-m|} K_A(t_\ell, t_m) + \frac{1}{2n}\, L_A(t_\ell, t_m) \right\} d^{(n)}_{|m-k|}$$

corresponds to the operator $T_0 - A$ and the matrix

$$V_{jk} := R^{(n)}_{|j-k|} K_B(t_j, t_k)] + \frac{1}{2n}\, L_B(t_j, t_k)$$

corresponds to the operator B. Note that the two summations in the expression for U_{jk} represent matrix multiplications.

We explicitly note that the extra weights $b_j^{(n)}$ are required since the approximation of T_0 by $D_n S_0 D_n$ is not exact for $c_n(t) := \cos nt$. On one hand we have $T_0 c_n = n c_n$ but on the other hand $D_n S_0 D_n c_n = 0$. This effect would not occur if we would work with an odd number of interpolation and collocation points. However this would be at the expense of making the implementation of a multi-grid method, for example, more involved.

We also note that, in practice, the extra term $1/2n$ occurring in the expression for the weights U_{jk} can be avoided by incorporating the mean value operator M into the smooth kernel L_B.

4 Numerical example

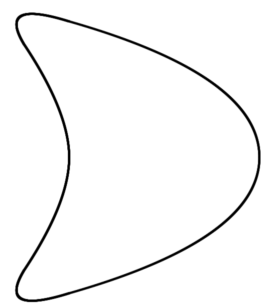

Figure 4.1: Kite-shaped domain for numerical example

For a numerical example, we consider the scattering of a plane wave $u^i(x) = e^{i\kappa\, x\cdot d}$ with incident direction $d \in S^1$ by a sound-hard or non-conducting cylinder with a non-convex kite-shaped cross section with boundary $\partial\Omega$ illustrated in Fig. 4.1 and described by the parametric representation

$$z(t) = (\cos t + 0.65\cos 2t - 0.65, 1.5\sin t), \quad 0 \le t \le 2\pi.$$

The coefficient u_∞ in the asymptotics

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}} \left\{ u_\infty\left(\frac{x}{|x|}\right) + O\left(\frac{1}{|x|}\right) \right\}, \quad |x| \to \infty,$$

for the scattered wave u^s is known as the far field pattern. From the asymptotics for the Hankel functions it can be deduced that the far field pattern of the double-layer potential (1.1) is given by

$$u_\infty(\hat{x}) = \frac{e^{-i\frac{\pi}{4}}\sqrt{\kappa}}{\sqrt{8\pi}} \int_{\partial\Omega} \nu(y) \cdot \hat{x} e^{-i\kappa\,\hat{x}\cdot y} \psi(y)\, ds(y), \quad \hat{x} \in S^1,$$

which can be evaluated by the composite trapezoidal rule after solving the integral equation for ψ. Table 4.1 gives some approximate values for the far field pattern $u_\infty(d)$ and $u_\infty(-d)$ in the forward direction d and the backward direction $-d$. The direction d of the incident wave is $d = (1, 0)$. Note that the super algebraic convergence is clearly exhibited. Of course, the correct digits coincide with those obtained by the collocation method described in [8].

Table 4.1: Numerical results

	n	$\operatorname{Re} u_\infty(d)$	$\operatorname{Im} u_\infty(d)$	$\operatorname{Re} u_\infty(-d)$	$\operatorname{Im} u_\infty(-d)$
$k = 1$	8	-0.50940996	1.10242896	-1.40934553	0.14206653
	16	-0.50918682	1.10234222	-1.40900870	0.14232619
	32	-0.50918720	1.10234229	-1.40900896	0.14232585
	64	-0.50918720	1.10234229	-1.40900896	0.14232585
$k = 5$	8	-1.45194510	1.51807492	0.10327808	0.14165130
	16	-1.38345660	2.00138757	0.48933620	0.39856262
	32	-1.27590705	1.94749252	0.45026329	0.56340590
	64	-1.27590705	1.94749251	0.45026329	0.56340590

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