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Hadwiger’s Conjecture for inflations of 3-chromatic graphs

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Abstract. Hadwiger’s Conjecture states that every $k$-chromatic graph has a complete minor of order $k$. A graph $G'$ is an inflation of a graph $G$ if $G'$ is obtained from $G$ by replacing each vertex $v$ of $G$ by a clique $C_v$ and joining two vertices of distinct cliques by an edge if and only if the corresponding vertices of $G$ are adjacent. We present an algorithm for computing an upper bound on the chromatic number $\chi(G')$ of any inflation $G'$ of any 3-chromatic graph $G$. As a consequence, we deduce that Hadwiger’s Conjecture holds for any inflation of any 3-colorable graph.

Keywords: Hadwiger’s Conjecture, graph coloring, inflation, 3-chromatic graph, complete minor

1 Introduction

A proper $k$-coloring of a graph $G$ is a function $f : V(G) \to \{1, \ldots, k\}$ such that $f(v) \neq f(u)$ whenever $u$ and $v$ are adjacent. The chromatic number $\chi(G)$ of $G$ is the smallest $k$ such that there is a proper $k$-coloring of $G$. A graph $G$ is $k$-chromatic if $\chi(G) = k$.

Hadwiger’s Conjecture is one of the fundamental open questions in graph coloring. It dates back to 1943, when Hadwiger [7] suggested that every $k$-chromatic graph $G$ contains a complete minor of order $k$, i.e. a complete graph of order $k$ can be obtained from $G$ by deleting and/or contracting edges.

The conjecture is a far-reaching generalization of the well-known Four Color Problem, which asks if every planar graph has chromatic number at most 4, and it remains open for all $k$ greater than 6. (See [15] for a survey on Hadwiger’s Conjecture.) The case $k \leq 4$ was proved by Hadwiger in his original paper [7]. Wagner [16] proved that the case $k = 5$ is equivalent to the Four Color
Problem. The latter problem was solved in the affirmative by Apel and Haken [1, 2] in 1977, and in 1993 Robertson et al. [12] proved Hadwiger’s Conjecture for \( k = 6 \).

Hadwiger’s Conjecture has also been proved to hold for some special families of graphs, e.g. line graphs [11] and quasi-line graphs [13]. Bollobás et al. [5] proved that Hadwiger’s Conjecture is true for almost every graph.

In this paper we study Hadwiger’s Conjecture for inflations of graphs: given a graph \( G \) with vertex set \( V(G) = \{v_1, \ldots, v_n\} \) and non-negative integers \( k_1, \ldots, k_n \), we define the inflation \( G' = G(k_1, \ldots, k_n) \) of \( G \) to be the graph obtained from \( G \) by replacing vertices \( v_1, \ldots, v_n \) by disjoint cliques \( A_1, \ldots, A_n \) of size \( k_1, \ldots, k_n \), respectively, such that vertices \( x, y \) are adjacent in \( G' \) if \( x \in V(A_s) \) and \( y \in V(A_t) \), \( s \neq t \), are adjacent in \( G \). The cliques \( A_1, \ldots, A_n \) are referred to as the inflated vertices, and the numbers \( k_1, \ldots, k_n \) are referred to as inflation sizes of \( G' \). If \( k_1 = \cdots = k_n \), then \( G' \) is a uniform inflation. We also say that \( G' \) is obtained by inflating \( G \).

One motivation for studying Hadwiger’s Conjecture for inflations of graphs stems from Hajós’ Conjecture which states that every \( k \)-chromatic graph contains a subdivision of the complete graph on \( k \) vertices. In 1979, Catlin [6] showed that this latter conjecture is false for all values of \( k \) greater than 6. Catlin’s counterexamples are surprisingly simple: they are just uniform inflations of the 5-cycle. Catlin’s counterexamples to Hajós’ Conjecture are not counterexamples to Hadwiger’s Conjecture, but perhaps a similar construction might yield a counterexample to Hadwiger’s Conjecture. Thomassen [14] proved that a graph \( G \) is perfect if and only if every inflation of \( G \) satisfies Hajós’ Conjecture. In particular, this means that any non-perfect graph can be inflated to a counterexample to Hajós’ Conjecture. We prove that no counterexample to Hadwiger’s Conjecture can be constructed by inflating a 3-colorable graph.

There are some other results on Hadwiger’s Conjecture for inflations of graphs in the literature: Plummer et al. [10] proved that no counterexample to Hadwiger’s Conjecture can be obtained by inflating a graph with independence number at most 2 (complements of triangle-free graphs) and order at most 11. Kawarabayashi conjectured that Hadwiger’s Conjecture holds for any inflation of an outerplanar graph [private communication to Pedersen, 2012]. Since every outerplanar graph is 3-colorable, the main result of this paper settles that conjecture in the affirmative. Pedersen [9] proved that Hadwiger’s Conjecture holds for any inflation of the Petersen graph. Here we prove the following stronger proposition.

**Theorem 1.** Hadwiger’s Conjecture is true for any inflation of any 3-colorable graph.

## 2 Proof of Theorem 1

Let \( \eta(G) \) denote the Hadwiger number of \( G \), i.e., the order of the largest complete minor of \( G \). Hadwiger’s Conjecture then states that \( \eta(G) \geq \chi(G) \) for every graph \( G \). In this section we will prove that for any inflation \( G' \) of any 3-colorable graph \( G \), we have \( \eta(G') \geq \chi(G') \).

Inflations of graphs are studied in e.g. [3, 4]. Therein the authors were, among other things, interested in determining the chromatic number of (uniform) inflations. Here we do not attempt to calculate the chromatic number of such graphs explicitly; rather we obtain an upper bound on the chromatic number of any (possibly non-uniform) inflation \( G' \) of any 3-colorable graph \( G \) and give a lower bound on the Hadwiger number of \( G' \).

Suppose that \( G' \) is an inflation of \( G \) with inflation sizes \( k_1, k_2, \ldots, k_s \). We denote by \( G_{k_1,k_2,\ldots,k_t} \) the subgraph of \( G \) induced by the vertices which are replaced by cliques with sizes in the set \( \{k_1, k_2, \ldots, k_t\} \) in \( G' \). Similarly, \( G'_{k_1,k_2,\ldots,k_t} \) denotes the subgraph of \( G' \) induced by all cliques with sizes in \( \{k_1, k_2, \ldots, k_t\} \) that correspond to vertices of \( G \).
Given two graphs $G_1$ and $G_2$ such that $V(G_1) \cap V(G_2) \neq \emptyset$, we define the intersection of $G_1$ and $G_2$, denoted by $G_1 \cap G_2$, as the graph with vertex set $V(G_1) \cap V(G_2)$ and edge set $E(G_1) \cap E(G_2)$. Similarly, we define the union of $G_1$ and $G_2$, denoted by $G_1 \cup G_2$, as the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$.

In the following we will present an algorithm for computing an upper bound on the chromatic number $\chi(G')$ of any inflation $G'$ of any 3-chromatic graph $G$. By analyzing this algorithm we will then be able to prove that Hadwiger’s Conjecture is true for any inflation of any 3-colorable graph. We shall need some preliminary results. The following was noted by Albertson et al. [3].

**Lemma 1.** Let $G$ be a graph, and $G'$ the inflation obtained from $G$ by replacing each vertex by a clique of size $k$. Then, $\chi(G') \leq k\chi(G)$.

If $G$ is a graph and $G'$ an inflation of $G$, then an edge $e = uv$ of $G$ is called an $\alpha\beta$-edge if in $G'$ $u$ and $v$ are replaced by cliques of size $\alpha$ and $\beta$, respectively. Similarly, a vertex in $G$ which is replaced by a clique of size $\alpha$ in $G'$ is called an $\alpha$-vertex. We will use the following observation, which easily follows from the well-known fact that the chromatic number of a graph equals the maximum of the chromatic numbers of its blocks, and so we leave the proof to the reader.

**Lemma 2.** Let $G$ be a graph and let $E_c$ denote a set of cut-edges in $G$. Suppose that $G'$ is some inflation of $G$. Denote by $H$ the graph $G - E_c$, and let $H'$ denote the subgraph of $G'$ obtained by removing all edges corresponding to edges of $E_c$. Then

$$\chi(G') \leq \max \left( \{\chi(H')\} \cup \{\alpha + \beta \mid e \in E_c \text{ is an } \alpha\beta\text{-edge} \} \right)$$

We shall repeatedly apply the following consequence of Lovasz’ Perfect Graph Theorem [8].

**Theorem 2.** Every inflation of a perfect graph is perfect.

**Proof of Theorem 1.** Suppose the result is false. Let $G$ be a vertex-minimal graph with chromatic number at most 3 such that there is an inflation $G'$ of $G$ that is a counterexample to Hadwiger’s Conjecture. Moreover, let $G'$ be vertex-minimal with respect to the property of being an inflation of $G$ that is a counterexample to Hadwiger’s Conjecture. It is straightforward to see that $G$ must be 2-connected. Suppose that $G$ is 2-colorable. By Theorem 2, any inflation of a perfect graph is perfect and so $\chi(G') = \omega(G') \leq \eta(G')$, a contradiction to the assumption that $G'$ is a counterexample to Hadwiger’s Conjecture. Hence, we may assume that $G$ is 3-chromatic.

Let $a_1$ be the largest inflation size of $G'$. If $\chi(G_{a_1}) = 3$, then it follows from Lemma 1 that $\chi(G') \leq 3a_1$. Furthermore, $\eta(G') \geq 3a_1$, because $G_{a_1}$ contains a cycle. Hence, $\eta(G') \geq \chi(G')$ which contradicts that $G'$ is a counterexample to Hadwiger’s Conjecture. Thus, we conclude that $\chi(G_{a_1}) \leq 2$. Since $\chi(G) = 3$, this means that $a_1$ is not the only inflation size of $G'$.

Let $a_1, \ldots, a_m, b_1, \ldots, b_n$ denote the inflation sizes in $G'$, where

$$a_1 > \cdots > a_m > b_1 > \cdots > b_n,$$

and $\chi(G_{a_1,\ldots,a_m}) \leq 2$ while $\chi(G_{a_1,\ldots,a_m,b_1}) = 3$.

Let $A$ denote the set $\{a_1, \ldots, a_m\}$, and let $S$ be the set of all ordered pairs $(a_i, a_j)$ of $A$ with $a_i \geq a_j$ for which there is an $a_i a_j$-edge in $G$. Since $\chi(G_{a_1,\ldots,a_m}) \leq 2$, Theorem 2 yields that

$$\chi(G'_{a_1,\ldots,a_m}) = \max (\{a_i + a_j \mid (a_i, a_j) \in S\} \cup \{a_1, \ldots, a_m\}).$$

(1)

We define the graph $G''_{a_1,\ldots,a_m}$ to be the graph obtained from $G'_{a_1,\ldots,a_m}$ by removing $b_1$ vertices from each of the inflated vertices of $G_{a_1,\ldots,a_m}$. Similarly, we set $b_{n+1} = 0$, and, for each $i \in [n]$, we
let \( G'' \) denote the graph obtained from \( G'_1, \ldots, a_m, b_1, \ldots, b_i \) by removing \( b_{i+1} \) vertices from each of the inflated vertices of \( G'_1, \ldots, a_m, b_1, \ldots, b_i \) in such a way that \( G'' \) is a subgraph of \( G'_1, \ldots, a_m, b_1, \ldots, b_j \) whenever \( i < j \). (This is possible since \( G'_1, \ldots, a_m, b_1, \ldots, b_j \subseteq G'' \) and \( b_i > b_j \) if \( i < j \).) As a shorthand, we will often write

\[
G_{a_{b_1}}, G'_{a_{b_1}}, \text{ and } G''_{a_{b_1}}
\]

for the graphs

\[
G'_1, a_m, b_1, \ldots, b_i, G'_1, a_m, b_1, \ldots, b_i, \text{ and } G''_1, a_m, b_1, \ldots, b_i,
\]

respectively. The analogue of (1) for \( G'' \) then reads

\[
\chi(G''_{a_{b_1}}) = \max\{a_i + a_j - 2b_1 \mid (a_i, a_j) \in S \} \cup \{a_1 - b_1, \ldots, a_m - b_1\}.
\]

Below we shall give our algorithm for computing a useful upper bound on the chromatic number of \( G' \). First we discuss it informally:

The algorithm proceeds by steps and at Step \( i \) of the algorithm \((1 \leq i \leq n)\) it considers the graph \( G_{a_{b_1}} \), and defines the sets \( A_{i+1} \) from \( A_i \), \( S_{i+1} \) from \( S_i \), the set \( T_{i+1} \) from \( T_i \), and the auxiliary sets \( S'_i, A'_i, A''_i \) and \( T'_i \). Each step consists of the three parts (a), (b) and (c), and at each such part certain sets are defined.

At the beginning of Step \( 1 \) we have \( A_1 = A, S_1 = S, \) and \( T_1 = \emptyset \). Then at Step \( i \) \((1 \leq i \leq n)\) the set \( S_{i+1} \) is constructed from \( S_i \) by adding a new element \((\alpha, b_i)\) if

- \( \alpha \in A_i \),
- there is no \( \alpha \)-vertex in a cycle of \( G_{a_{b_1}} \), and
- there is an \( \alpha b_i \)-edge in \( G_{a_{b_1}} \),

and removing any element \((\alpha, \beta)\) such that there is an \( \alpha \beta \)-edge on a cycle in \( G_{a_{b_1}} \).

The set \( A_{i+1} \) is constructed from \( A_i \) at Step \( i \) by removing any element \( \alpha \) such that there is an \( \alpha \)-vertex on a cycle in \( G_{a_{b_1}} \).

Finally, the set \( T_{i+1} \) is constructed from \( T_i \) at Step \( i \) by adding any element \((\alpha, \beta, b_i)\) such that there is an \( \alpha \beta \)-edge in a cycle of \( G_{a_{b_1}} \), and adding every element \((\alpha, b_i, b_i)\) such that there is an \( \alpha \)-vertex in a cycle of \( G_{a_{b_1}} \) and there is no \( \beta > b_i \), such that there is an \( \alpha \beta \)-edge in a cycle of \( G_{a_{b_1}} \).

Note that if \((\alpha, \beta) \in S_j \setminus S_{j+1}\), then \( j \) is the minimum integer \( q \) such that there is an \( \alpha \beta \)-edge in a cycle of \( G_{a_{b_1}} \), and one might think of the set \( S_{i+1} \) as “the set of pairs \((\alpha, \beta)\) such that \( \alpha \geq \beta \) and \( \alpha \geq a_m \), and for which there is an \( \alpha \beta \)-edge in \( G_{a_{b_1}} \) but no cycle containing an \( \alpha \beta \)-edge”. Similarly, one might think of the set \( A_{i+1} \) as “the set of all constants \( \alpha \geq a_m \) for which there is an \( \alpha \)-vertex in \( G_{a_{b_1}} \) but no cycle containing an \( \alpha \)-vertex”. Note further that since \( G \) is 2-connected, for each \( \alpha \in A_i \), there is some minimum integer \( j \) such that \( G_{a_{b_j}} \) contains a cycle with an \( \alpha \)-vertex. A similar statement holds for the elements of \( S_i \).

Let us now give a formal description of the algorithm:
Algorithm 1

**Step 0:** Define $A_1 := A$, $S_1 := S$, and $T_1 := \emptyset$.

**Step 1:**

(a) For each element $(a_{j_1}, a_{j_2})$ of $S_1$, if there is an $a_{j_1}a_{j_2}$-edge on a cycle in $G_{\geq b_1}$, then include $(a_{j_1}, a_{j_2})$ in $S'_1$.

(b) For each element $a_j$ of $A_1$:

- If there is an $a_j$-vertex on a cycle in $G_{\geq b_1}$, then include $a_j$ in $A'_1$.
- If there is an $a_j$-vertex on a cycle in $G_{\geq b_1}$ and no element $a_{j_1} \in \{a_1, \ldots, a_m\}$ such that there is an $a_{j_1}a_j$-edge on a cycle in $G_{\geq b_1}$, then include $(a_j, b_1, b_1)$ in $T'_1$.
- If there is no $a_j$-vertex on a cycle in $G_{\geq b_1}$ but there is an $a_jb_1$-edge in $G_{\geq b_1}$, then include $a_j$ in $A''_1$.

(c) Define

- $S_2 := (S_1 \setminus S'_1) \cup \{(a_j, b_1) \mid a_j \in A''_1\}$,
- $A_2 := A_1 \setminus A'_1$,
- $T'_2 := T_1 \cup \{(a_{j_1}, a_{j_2}, b_1) \mid (a_{j_1}, a_{j_2}) \in S'_1 \cup T'_1 \cup \{(b_1, b_1, b_1)\}$,

and go to Step 2.

**Step i** ($2 \leq i \leq n$):

(a) For each element $(\alpha, \beta)$ of $S_i$, if there is an $\alpha\beta$-edge on a cycle in $G_{\geq b_i}$, then include $(\alpha, \beta)$ in $S'_i$.

(b) For each element $a_j$ of $A_i$:

- If there is an $a_j$-vertex on a cycle in $G_{\geq b_i}$, then include $a_j$ in $A'_i$.
- If there is an $a_j$-vertex on a cycle in $G_{\geq b_i}$ and no element $\alpha \in \{a_1, \ldots, a_m, b_1, \ldots, b_{i-1}\}$ such that there is an $a_j\alpha$-edge on a cycle in $G_{\geq b_i}$, then include $(a_j, b_i, b_i)$ in $T'_i$.
- If there is no $a_j$-vertex on a cycle in $G_{\geq b_i}$ but there is an $a_jb_i$-edge in $G_{\geq b_i}$, then include $a_j$ in the set $A''_i$.

(c) Define

- $S_{i+1} := (S_i \setminus S'_i) \cup \{(a_j, b_i) \mid a_j \in A''_i\}$,
- $A_{i+1} := A_i \setminus A'_i$,
- $T_{i+1} := T_i \cup \{(\alpha, \beta, b_i) \mid (\alpha, \beta) \in S'_i \cup T'_i$, and go to Step $(i + 1)$ if $i \leq n - 1$, otherwise Stop.

We now prove some properties of the algorithm. The algorithm stops after Step $n$ when the sets $S_{n+1}, A_{n+1}$ and $T_{n+1}$ have been defined.

Lemma 3.

(1) $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_{n+1}$.
The sets \( A'_1, \ldots, A'_{n-1}, \) and \( A'_n \) are all disjoint.

\( T_1 \subseteq T_2 \subseteq \cdots \subseteq T_{n+1}. \)

Proof. (1): The inclusions follow directly from the description of the algorithm, in particular, part (c) of Steps 0, 1, \ldots, \( n. \)

(2) Suppose that \( a \) is some element of \( A'_{j_1} \cap A'_{j_2} \) with \( j_1 < j_2. \) By part (b) of Step \( j_2, a \in A_{j_2}. \)

Since also \( a \in A'_{j_1}, \) it follows from part (e) of Step \( j_1 \) that \( a \) is not in \( A_{j_1+1} \) and so, by (1), \( a \)

is not in \( A_{j_2}, \) a contradiction.

(3) The inclusions follow directly from part (c) of Steps 0, 1, \ldots, \( n. \)

Lemma 4. At the end of Step \( i \) of the algorithm, the following holds:

\[
\chi(G'_i) \leq \max\{\alpha + \beta + \gamma \mid (\alpha, \beta, \gamma) \in T_{i+1}\}
\]

\[
\cup \{\alpha + \beta + b_{i+1} \mid (\alpha, \beta) \in S_{i+1}\}
\]

\[
\cup \{a_j + 2b_{i+1} \mid a_j \in A_{i+1}\}.
\]

Proof. For each \( i \in [n], \) let \( E_i \) denote the set of all \( \alpha \beta \)-edges in \( G_{2b_i} \) with \( (\alpha, \beta) \in S_{i+1}. \)

For each \( i \in [n], \) let \( V_i \) denote the set of all \( \alpha \)-vertices of \( G_{2b_i} \) with \( \alpha \in A_{i+1}. \)

Define \( H_i := G_{2b_i} - E_i - V_i. \)

The following three claims are easily deduced from the description of the algorithm.

Claim 1. For each \( i \in [n] \) and each \( \alpha \beta \)-edge \( e \) of \( G_{2b_i} \) with \( (\alpha, \beta) \in S_i, \) the edge \( e \) is in \( H_i \) if and only if there is an \( \alpha \beta \)-edge on a cycle in \( G_{2b_i}. \)

Claim 2. For every \( i \in [n], \) each edge of \( E_i \) is a cut-edge of \( G_{2b_i}. \)

Claim 3. For each \( i \in [n] \) and each \( \alpha \)-vertex \( v \) of \( G_{2b_i} \) with \( \alpha \in A_i, \) the vertex \( v \) is in \( H_i \) if and only if there is an \( \alpha \)-vertex on a cycle in \( G_{2b_i}. \)

Claim 4. For each \( i \in [n], \) each vertex of \( V_i \) is an isolated vertex of \( G_{2b_i} - E_i. \)

Proof of Claim 4. Suppose that there is some edge \( uv \) in \( E(G_{2b_i}) \setminus E_i \) with \( v \in V_i. \) Since \( v \in V_i, v \)

is an \( \alpha \)-vertex for some \( \alpha \in A_{i+1}. \) and, by Claim 3, this means that there is no \( \alpha \)-vertex on a cycle

in \( G_{2b_i}. \)

The edge \( uv \) is an \( \alpha \beta \)-edge where neither \( (\alpha, \beta) \) nor \( (\beta, \alpha) \) is in \( S_{i+1}, \) since otherwise \( uv \)

would be in \( E_i. \) If \( \alpha, \beta \geq a_m, \) then \( (\alpha, \beta) \in S_1 \) or \( (\beta, \alpha) \in S_1, \) by the definition of \( S_1 = S. \) and if \( \beta = b_r \) for

some \( 1 \leq r < j, \) then \( (\alpha, \beta) \in S_{r+1} \) according to part (b) and (c) of Step \( r. \) In both cases we must

have that either \( (\alpha, \beta) \) or \( (\beta, \alpha) \) is in \( S_j \) for some \( j < i+1, \) because otherwise \( (\alpha, \beta) \) or \( (\beta, \alpha) \) is in

\( S_{i+1}. \) However, according to part (a) of Step \( j, \) this happens only if there is an \( \alpha \beta \)-edge on a cycle

in \( G_{2b_j}. \) This clearly contradicts the fact that there is no \( \alpha \)-vertex on a cycle in \( G_{2b_i}. \)

Next, we define \( H''_i \) to be the graph obtained from \( G''_{2b_i} \) by removing all edges corresponding to edges in \( E_i \) and all vertices corresponding to vertices in \( V_i. \) So \( H''_i \) is the subgraph of \( G''_{2b_i} \)

corresponding to the subgraph \( H_i \) of \( G_{2b_i}. \)
Instead of proving (3), we prove, by induction, that the following stronger statement holds for every integer \( i \in [n] \):

\[
\chi(G''_{2b_i}) \leq \max\{\alpha + \beta + \gamma - 3b_{i+1} \mid (\alpha, \beta, \gamma) \in T_{i+1}\}
\cup \{\alpha + \beta - 2b_{i+1} \mid (\alpha, \beta) \in S'_{i+1}\}
\cup \{a_j - b_{i+1} \mid a_j \in A_{i+1}\},
\]

(4)

and

\[
\chi(H''_i) \leq \max\{\alpha + \beta + \gamma - 3b_{i+1} \mid (\alpha, \beta, \gamma) \in T_{i+1}\}. \tag{5}
\]

The subgraph \( G''_{2b_i} - V(G''_{2b_i}) \) of \( G''_{2b_i} \) is a \( b_{i+1} \)-inflation of a 3-colorable graph, and so, by Lemma 1, \( \chi(G''_{2b_i} - V(G''_{2b_i})) \leq 3b_{i+1} \). This along with (4) implies that (3) holds.

We first prove that (4) and (5) hold for \( i = 1 \).

**Claim 5.** The upper bounds (4) and (5) hold for \( i = 1 \).

**Proof of Claim 5.** We shall first give an upper bound on \( \chi(G''_{2a_m} \cap H''_1) \) and then extend this to an upper bound on \( \chi(H''_1) \), thus establishing that (5) hold for \( i = 1 \). Of course, \( G''_{2a_m} \cap H_1 \) is a subgraph of \( G''_{2a_m} \), and \( G''_{2a_m} \) is a 2-colorable graph, in particular, it is a perfect graph. Thus, by Theorem 2, \( G''_{2a_m} \cap H''_1 \) is a perfect graph, and so \( \chi(G''_{2a_m} \cap H''_1) = \omega(G''_{2a_m} \cap H''_1) \). Since \( G''_{2a_m} \cap H''_1 \) is an inflation of the triangle-free graph \( G''_{2a_m} \cap H_1 \) it follows that any largest clique in \( G''_{2a_m} \cap H''_1 \) corresponds to a single vertex or a pair of adjacent vertices in \( G''_{2a_m} \cap H_1 \). Thus, \( \chi(G''_{2a_m} \cap H''_1) \) is at most

\[
\max\{\{\alpha + \beta - 2b_1 \mid (\alpha, \beta) \in S'_1\} \cup \{\alpha - b_1 \mid \alpha \in A'_1\}\}. \tag{6}
\]

By Claim 1, an \( \alpha \beta \)-edge \( e \) of \( G''_{2a_m} \) with \( (\alpha, \beta) \in S_1 \) is in \( H_1 \) if and only if there is an \( \alpha \beta \)-edge on a cycle in \( G_{2b_1} \). According to part (a) of Step 1, an element \( (\alpha, \beta) \in S_1 = S \) is in \( S'_1 \) if and only if there is an \( \alpha \beta \)-edge on a cycle in \( G_{2b_1} \).

Similarly, by Claim 3, an \( \alpha \)-vertex \( v \) of \( G''_{2a_m} \) with \( \alpha \in A_1 \) is in \( H_1 \) if and only if there is an \( \alpha \)-vertex on a cycle in \( G_{2b_1} \). According to part (b) of Step 1, an element \( \alpha \in A_1 = A \) is in \( A'_1 \) if and only if there is an \( \alpha \)-vertex on a cycle in \( G_{2b_1} \). Hence, by (6) we may now conclude that

\[
\chi(G''_{2a_m} \cap H''_1) \text{ is at most}
\max\{\{\alpha + \beta - 2b_1 \mid (\alpha, \beta) \in S'_1\} \cup \{\alpha - b_1 \mid \alpha \in A'_1\}\}. \tag{7}
\]

For any element \( \alpha \) of \( A'_1 \) for which there is an element \( a_j \in \{a_1, \ldots, a_m\} \) such that there is an \( \alpha a_j \)-edge on a cycle in \( G_{2b_1} \), \( (\alpha, a_j) \) or \( (a_j, \alpha) \) is included in \( S'_1 \), and so, since \( \alpha + a_j - 2b_1 \geq \alpha - b_1 \), the value of (7) is unaffected by removing such an element \( \alpha - b_1 \) from the second set in (7). By part (b) of Step 1, for any element \( \alpha \) of \( A'_1 \) for which there is no \( a_j \in \{a_1, \ldots, a_m\} \) such that \( G_{2b_1} \) contains an \( \alpha a_j \)-edge on a cycle, the element \( (\alpha, b_1, 1) \) is included in \( T'_1 \). Thus, \( \chi(G''_{2a_m} \cap H''_1) \) is at most

\[
\max\{\{\alpha + \beta - 2b_1 \mid (\alpha, \beta) \in S'_1\} \cup \{\alpha - b_1 \mid (\alpha, b_1, 1) \in T'_1\}\}. \tag{8}
\]

Since \( G''_{2b_1} - V(G''_{2a_m}) \) is an inflation of a 3-colorable graph with inflation sizes at most \( b_1 - b_2 \), it follows from Lemma 1 that \( \chi(G''_{2b_1} - V(G''_{2a_m})) \leq 3(b_1 - b_2) \). Thus, since \( H''_1 \) is a subgraph of \( G''_{2b_1} \), we also have \( \chi(H''_1 - V(G''_{2a_m})) \leq 3(b_1 - b_2) \). Thus, combining optimal colorings of \( H''_1 - V(G''_{2a_m}) \) and \( G''_{2a_m} \cap H''_1 \) using disjoint sets of colors for a coloring of \( H''_1 \), we deduce that

\[
\chi(H''_1) \leq \max\{\{\alpha + \beta + b_1 - 3b_2 \mid (\alpha, \beta) \in S'_1\} \cup \{\alpha + 2b_1 - 3b_2 \mid (\alpha, b_1, b_1) \in T'_1\} \cup \{3b_1 - 3b_2\}\}. \tag{9}
\]
Since, by part (c) of Step 1,
\[ T_2 = T_1 \cup \{(a_{j1}, a_{j2}, b_1) | (a_{j1}, a_{j2}) \in S'_1 \} \cup T'_1 \cup \{(b_1, b_1, b_1) \in S_i \} \cup \{a_j - b_i | a_j \in A_i\}, \]
it follows that
\[ \chi(H''_1) \leq \max\{\alpha + \beta + \gamma - 3b_2 | (\alpha, \beta, \gamma) \in T_2\}, \] (10)
which means that (5) holds for \( i = 1 \).

Let \( I'' \) denote the subgraph of \( G''_{bz_1} \) corresponding to the edge-induced subgraph \( G_{bz_1}[E_1] \) of \( G_{bz_1} \). By Claim 2, each edge in \( E_1 \) is a cut-edge of \( G_{bz_1} \), and so, \( G_{bz_1}[E_1] \) is a forest, in particular, it is a 2-colorable graph and, hence, a perfect graph. Thus, by Theorem 2, \( I'' \) is a perfect graph, and so the chromatic number of \( I'' \) is equal to the clique number of \( I'' \). This implies that
\[ \chi(I'') \leq \max\{\alpha + \beta - 2b_2 | \alpha\beta\text{-edge in } E_1\}. \] (11)

Recall that \( E_1 \) is the set of all \( \alpha\beta \)-edges in \( G_{bz_1} \) with \( (\alpha, \beta) \in S_2 \). Thus,
\[ \chi(I'') \leq \max\{\alpha + \beta - 2b_2 | (\alpha, \beta) \in S_2\}. \] (12)

Let \( J'' \) denote the subgraph \( G''_{bz_1} - V(H''_1) - V(I'') \) of \( G''_{bz_1} \). Note that any component in \( J'' \) corresponds to an isolated vertex of \( G_{bz_1} \) that is in \( V_1 \). Recall that \( V_1 \) is the set of all \( \alpha \)-vertices in \( G_{bz_1} \) with \( \alpha \in A_2 \). This implies that the chromatic number of \( J'' \) is at most
\[ \max\{a_j - b_2 | a_j \in A_2\}. \] (13)

Putting (10), (12), and (13) together and using Lemma 2, we may now deduce that (4) holds for \( i = 1 \) in the following way:

First we properly color the graph \( H''_1 \) with at most the number of colors in the right hand side of (10). Then by using Lemma 2 for the edges of \( I'' \), which correspond to edges of \( E_1 \), we may properly color the graph \( H''_1 \cup I'' \) using at most
\[ \max\{\alpha + \beta + \gamma - 3b_2 | (\alpha, \beta, \gamma) \in T_2\} \cup \{\alpha + \beta - 2b_2 | (\alpha, \beta) \in S_2\}\}
colors. Finally, we can color the vertices of \( J'' \) using at most
\[ \max\{a_j - b_2 | a_j \in A_2\}\)
colors. This completes the proof of the claim. \( \square \)

We now prove that (4) and (5) hold in the general case.

**Claim 6.** The upper bounds (4) and (5) hold for any \( i \in [n] \).

**Proof of Claim 6.** Our induction hypothesis is that the following holds:
\[ \chi(G''_{bz_{i-1}}) \leq \max\{\alpha + \beta + \gamma - 3b_i | (\alpha, \beta, \gamma) \in T_i\} \]
\[ \cup \{\alpha + \beta - 2b_i | (\alpha, \beta) \in S_i\} \]
\[ \cup \{a_j - b_i | a_j \in A_i\}, \] (14)
and
\[ \chi(H''_{i-1}) \leq \max\{\alpha + \beta + \gamma - 3b_i | (\alpha, \beta, \gamma) \in T_i\}. \] (15)
The basis for the induction was established in Claim 5. We are going to be using much the same approach as in the proof of Claim 5. First we give an upper bound on $\chi(G''_{2b_{i-1}} \cap H''_i)$ and then extend this to an upper bound on $\chi(H''_i)$.

Recall that $E_i$ is the set of all $\alpha \beta$-edges in $G_{2b_{i-1}}$ with

$$(\alpha, \beta) \in S_{i+1} = (S_i \cup S'_i) \cup \{ (\gamma, b_i) \mid \gamma \in A''_i \}$$

and that $V_i$ is the set of all $\alpha$-vertices $v$ of $G_{2b_{i-1}}$ with $\alpha \in A_{i+1} = A_i \setminus A'_i$. Furthermore, we have that $H_i = G_{2b_{i-1}} - E_i - V_i$, and $H''_i$ is the subgraph of $G''_{2b_{i-1}}$ corresponding to $H_i$.

Consider the graph $G_{2b_{i-1}} \cap H_i$. Since $A_{i+1} \subseteq A_i$ and $S_{i+1} \subseteq S_i \cup \{ (\gamma, b_i) \mid \gamma \in A''_i \}$, it holds that $H_{i-1} \subseteq H_i$, and thus $H_{i-1}$ is a subgraph of $G_{2b_{i-1}} \cap H_i$.

Suppose that $e$ is an $\alpha \beta$-edge of $E_{i-1}$. This means that $(\alpha, \beta) \in S_i$; also by Claim 2 there is no $\alpha \beta$-edge on a cycle in $G_{2b_{i-1}}$. Moreover, by part (c) of Step $i$, $(\alpha, \beta) \in S_{i+1}$, and thus $e \in E_i$, unless $(\alpha, \beta) \in S'_i$, which by part (a) means that there is an $\alpha \beta$-edge in a cycle of $G_{2b_i}$. Hence $e \in E(H_i)$ if and only if $(\alpha, \beta) \in S'_i$.

Now consider an $\alpha$-vertex $v \in V_{i-1}$. Clearly, $\alpha \in A_i$; also by Claims 2 and 4, there is no $\alpha$-vertex on a cycle of $G_{2b_{i-1}}$. Moreover, by part (c) of Step $i$, $\alpha \in A_{i+1}$, and thus $v \in V_i$, unless $\alpha \in A'_i$, which by part (b) means that there is an $\alpha$-vertex on a cycle in $G_{2b_i}$. Hence $v \in V(H_i)$ if and only if $\alpha \in A'_i$.

Now, any edge of $E_{i-1}$ is a cut-edge of $G_{2b_{i-1}}$, and any vertex of $V_{i-1}$ is an isolated vertex of $G_{2b_{i-1}} - E_{i-1}$ (by Claims 2 and 4). So for the inflation $G''_{2b_{i-1}} \cap H''_i$ it now follows from (15) and by applying Lemma 2 that

$$\chi(G''_{2b_{i-1}} \cap H''_i) \leq \max(\{ \alpha + \beta + \gamma - 3b_i \mid (\alpha, \beta, \gamma) \in T_i \}$$

$$\cup \{ \alpha + \beta - 2b_i \mid (\alpha, \beta) \in S'_i \}$$

$$\cup \{ a_u - b_i \mid a_u \in A'_i \}).$$

(16)

Let us now prove the following:

**Subclaim 1.** If $a_u \in A'_i$, then either $(a_u, b_i, b_1)$ in $T'_i$, or there is an $\alpha \in \{ a_1, \ldots, a_m, b_1, \ldots, b_{i-1} \}$, such that $(a_u, \alpha) \in S'_i$ or $(\alpha, a_u) \in S'_i$.

**Proof of Subclaim 1.** Suppose that $a_u$ is some element of $A'_i$. By part (b) of Step $i$, $a_u \in A_i$, and so, by Lemma 3 (1), $a_u \in A_q$ for any $q < i$. By Lemma 3 (2), $a_u \notin A'_q$ for any $q < i$. Thus $i$ is the minimum integer $q$ such that there is an $a_u$-vertex on a cycle in $G_{2b_q}$.

Suppose that $(a_u, b_i, b_1) \notin T'_i$. Then it follows from part (b) of Step $i$ that there is an $a_u \alpha$-edge $e$ on a cycle in $G_{2b_q}$ for some $\alpha \in \{ a_1, \ldots, a_m, b_1, \ldots, b_{i-1} \}$. We shall prove that $(a_u, \alpha) \in S'_i$ or $(\alpha, a_u) \in S'_i$. Since there is an $a_u \alpha$-edge $e$ on a cycle in $G_{2b_q}$ for some $\alpha \in \{ a_1, \ldots, a_m, b_1, \ldots, b_{i-1} \}$, the desired result will follow from part (a) of Step $i$ if we can prove that $(a_u, \alpha) \in S_i$ or $(\alpha, a_u) \in S_i$.

Thus in the following we will argue that $(a_u, \alpha)$ or $(\alpha, a_u)$ is in $S_i$. We shall distinguish between two cases: $\alpha \in \{ a_1, \ldots, a_m \}$ and $\alpha \in \{ b_1, \ldots, b_{i-1} \}$.

(i) Suppose $\alpha \in \{ a_1, \ldots, a_m \}$. Then, at least one of the elements $(a_u, \alpha)$ and $(\alpha, a_u)$ must be in $S_1$, since $S_1 = S$ and, by definition, $S$ contains all ordered pairs $(a_i, a_j)$ of $A$ with $a_i \geq a_j$ for which there is an $a_i a_j$-edge in $G$.

Suppose $(a_u, \alpha)$ is in $S_1$. Assume that $(a_u, \alpha)$ is not in $S_i$. Then $(a_u, \alpha)$ is included in $S'_p$ at Step $p$ of the algorithm for some $p < i$. By part (a) of Step $p$, this means that there is an $a_u \alpha$-edge on a cycle in $G_{2b_p}$. This, however, is a contradiction to the fact that $i$ is the minimum integer $q$ for which there is an $a_u$-vertex on a cycle in $G_{2b_q}$. Hence $(a_u, \alpha) \in S_i$. Hence $(a_u, \alpha) \in S_i$.
If \((\alpha, a_u) \in S_1\), then a similar argument shows that \((\alpha, a_u) \in S_i\).

(ii) Suppose \(\alpha \in \{b_1, \ldots, b_{i-1}\}\), say \(\alpha = b_p\) for some \(p \in [i-1]\).

The integer \(i\) is the minimum integer \(q\) such that there is an \(a_u\)-vertex on a cycle in \(G_{2b_q}\) and thus \(G_{2b_q}\) has no cycle with an \(a_u\)-\(b_p\)-edge. Moreover, by part (b) of Step \(p\), \(a_u\) is included in \(A''_p\). Now, by part (c) of Step \(p\), \((a_u, b_p)\) is included in \(S_{p+1}\).

The rest of the argument goes along the same lines as in (i): Assume that \((a_u, b_p)\) is not in \(S_i\). Then \((a_u, b_p)\) is included in \(S'_k\) at some step \(k\) of the algorithm for some integer \(k\) satisfying \(p < k < i\). But this means that there is an \(a_u\)-\(b_p\)-edge on a cycle in \(G_{2b_k}\). This, however, is a contradiction to the fact that \(i\) is the minimum integer \(q\) for which there is an \(a_u\)-vertex on a cycle in \(G_{2b_q}\). Hence \((a_u, b_p) \in S_i\). \(\square\)

Subclaim 1 along with (16) implies

\[
\chi(G''_{2b_{i-1}} \cap H''_i) \leq \max(\{\alpha + \beta + \gamma - 3b_i \mid (\alpha, \beta, \gamma) \in T_i\} \\
\cup \{\alpha + \beta - 2b_i \mid (\alpha, \beta) \in S'_i\} \\
\cup \{a_u - b_i \mid a_u \in T'_i\}),
\]

(17)

It follows from Lemma 1 that any proper coloring of \(G''_{2b_{i-1}} \cap H''_i\) can be extended to a proper coloring of \(H''_i\) by using at most \(3(b_i - b_{i+1})\) new colors, because the graph \(H''_i - V(G''_{2b_{i-1}} \cap H''_i)\) is an inflation of a 3-colorable graph with inflation sizes at most \(3(b_i - b_{i+1})\). That fact along with (17) implies

\[
\chi(H''_i) \leq \max(\{\alpha + \beta + \gamma - 3b_{i+1} \mid (\alpha, \beta, \gamma) \in T_i\} \\
\cup \{\alpha + \beta + b_i - 3b_{i+1} \mid (\alpha, \beta) \in S'_i\} \\
\cup \{a_j + 2b_i - 3b_{i+1} \mid (a_j, b_i, b_i) \in T'_i\}),
\]

(18)

Note that, since \(T_{i+1} = T_i \cup \{(\alpha, \beta, b_i) \mid (\alpha, \beta) \in S'_i \cup T'_i\} \cup T'_i\), (18) implies that (5) holds. By Claim 2, every edge in \(E_i\) is a cut-edge of \(G_{2b_i}\), so the edge-induced subgraph \(G_{2b_i}[E_i]\) is a forest, in particular, it is a perfect graph. Thus, by Theorem 2, the subgraph \(I''_i\) of \(G''_{2b_i}\) corresponding to \(G_{2b_i}[E_i]\) satisfies

\[
\chi(I''_i) = \omega(I''_i) \leq \max(\{\alpha + \beta - 2b_{i+1} \mid (\alpha, \beta) \in S_i \setminus S'_i\} \\
\cup \{\alpha + b_i - 2b_{i+1} \mid \alpha \in A''_i\}).
\]

(19)

Finally, let \(J''_i\) denote the subgraph \(G''_{2b_i} - V(H''_i) - V(I''_i)\). Clearly, any component of \(J''_i\) corresponds to an isolated vertex of \(G_{2b_i}\) that is in \(V_i\). Thus

\[
\chi(J''_i) \leq \max(\alpha - b_{i+1} \mid \alpha \in A_i \setminus A''_i).
\]

(20)

Putting (18)-(20) together and applying Lemma 2 we now deduce that

\[
\chi(G''_{2b_i}) \leq \max(\{\alpha + \beta + \gamma - 3b_{i+1} \mid (\alpha, \beta, \gamma) \in T_{i+1}\} \\
\cup \{\alpha + \beta - 2b_{i+1} \mid (\alpha, \beta) \in S_{i+1}\} \\
\cup \{\alpha - b_{i+1} \mid \alpha \in A_{i+1}\}),
\]

which implies that (4) holds.

It now follows by induction that (4) and (5) hold for every \(i \in [n]\). \(\square\)

The statement of the lemma now follows from (4), since, as pointed out above, for any \(i \in [n]\), the inequality (3) follows from (4).
Lemma 5. At the end of Step $n$, the sets $S_{n+1}$ and $A_{n+1}$ are empty.

Proof. We first consider the sets $S_1, \ldots, S_{n+1}$. According to the description of the algorithm, $S_{i+1}$ is constructed from $S_i$ at Step $i$ by removing any element $(\alpha, \beta)$ from $S_i$ for which there is an $\alpha\beta$-edge on a cycle in $G_{2b_i}$, and adding any element $(\alpha, b_i)$ for which

(i) $\alpha \in A_i$,

(ii) there is an $\alpha b_i$-edge of $G_{2b_i}$, and

(iii) there is no $\alpha b_i$-edge on a cycle in $G_{2b_i}$.

Note that by part (b) and (c) of Step 1, i.e., $\alpha$ is in $A_i$ if and only if $\alpha \in \{a_1, \ldots, a_m\}$ and there is no $\alpha$-vertex in a cycle of $G_{2b_i}$. Since $G$ is 2-connected, every edge (and vertex) of $G$ lies on a cycle in $G$, and since $G = G_{2b_1}$, this means that $S_{n+1}$ is empty.

According to the description of the algorithm, $A_{i+1}$ is constructed from $A_i$ at Step $i$ by removing any element $a_j$ from $A_i$ such that there is an $a_j$-vertex that lies on a cycle in $G_{2b_i}$. Again, since $G$ is 2-connected, any vertex of $G = G_{2b_1}$ lies on a cycle, which implies the desired result.

Lemma 6. For each $(\alpha, \beta, \gamma) \in T_{n+1}$, $G'$ contains a complete minor of size $\alpha + \beta + \gamma$.

Proof. By Lemma 3 (3), $T_1 \subseteq T_2 \subseteq \cdots \subseteq T_{n+1}$. Let $j$ be the minimum integer such that $(\alpha, \beta, \gamma) \in T_j$.

By the definition of $T_j$ in part (c) of Step $(j-1)$, $(\alpha, \beta, \gamma)$ must be in one of the sets

\[\{(\alpha', \beta', b_{j-1}) | (\alpha', \beta') \in S'_{j-1}\}\]

and $T_{j-1}$. Moreover, by the definition of these sets in part (a) and (b) of Step $(j-1)$, $\gamma$ is not greater than $\alpha$ or $\beta$.

Suppose $(\alpha, \beta, \gamma) \in \{(\alpha', \beta', b_{j-1}) | (\alpha', \beta') \in S'_{j-1}\}$, that is, $(\alpha, \beta) \in S'_{j-1}$, and there exists an $\alpha\beta$-edge on a cycle $C$ of $G_{2b_{j-1}}$. The inflated cycle in $G_{2b_{j-1}}$ corresponding to $C$ can be contracted to a complete graph on $\alpha + \beta + b_{j-1}$ vertices, and so $\eta(G') \geq \alpha + \beta + \gamma$.

Now suppose $(\alpha, \beta, \gamma) \in T'_{j-1}$. Then, by definition of $T'_{j-1}$, we have $\beta = \gamma = b_{j-1}$ and $\alpha \in A'_{j-1}$, that is, there is an $\alpha$-vertex on a cycle $C$ in $G_{2b_{j-1}}$. The inflated cycle in $G'_{2b_{j-1}}$ corresponding to $C$ can be contracted to a complete graph on $\alpha + 2b_{j-1}$ vertices, and so $\eta(G') \geq \alpha + \beta + \gamma$.

By Lemma 4 and 5, $\chi(G') \leq \max\{\alpha + \beta + \gamma | (\alpha, \beta, \gamma) \in T_{n+1}\}$, and so, by Lemma 6, $\eta(G') \geq \chi(G')$. Thus, $G'$ is not a counterexample to Hadwiger's Conjecture, and we have obtained a contradiction from which the theorem follows.

Algorithm 1 together with the proof of Lemma 4 can be used to produce a proper coloring $\varphi$ of any inflation of any 3-connected 3-chromatic graph such that the number of colors used in $\varphi$ is at most $\max\{\alpha + \beta + \gamma | (\alpha, \beta, \gamma) \in T_{n+1}\}$. (The case when the graph is not 2-connected can be handled by Lemma 2.) Since a triple $(\alpha, \beta, b_j)$ is in $T_{i+1}$ at Step $i$ of the algorithm, where $j \leq i$, if and only there is an $\alpha$-vertex and a $\beta$-vertex in $G$ that are adjacent and lie on a cycle $C$ of $G$, which satisfies that every vertex in $C$ is replaced by a clique of size at least $b_j$ in $G'$, we in fact have that the number of colors used in $\varphi$ is at most

\[\max\{\alpha + \beta + \gamma | \text{there is an } \alpha\beta\text{-edge in } G \text{ that lies on a cycle where every vertex is replaced by a clique of size at least } \gamma\}.\]
References


