On complete monotonicity of inverse powers of some stable polynomials

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Outline

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• Complete monotonicity (CM)
• Determinantal polynomials
• Quadratic forms
• Elementary symmetric polynomials
• Integral representation of CM functions
• Products of linear forms
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Motivation

In 1920's K. Friedrichs and H. Lewy were studying discretized wave equation in 2D:

$$(\Delta_k \Delta_\ell + \Delta_k \Delta_m + \Delta_\ell \Delta_m) a_k,\ell,m = 0,$$

where $\Delta_j a_j = a_j - a_{j-1}$.

They observed that Taylor coefficients $a_k,\ell,m$ of the function $$(1-x)(1-y) + (1-x)(1-z) + (1-y)(1-z) = \sum_{k,\ell,m \geq 0} a_k,\ell,m x^k y^\ell z^m$$
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$$\frac{1}{(1-x)(1-y) + (1-x)(1-z) + (1-y)(1-z)} = \sum_{k,\ell,m \geq 0} a_{k,\ell,m} x^k y^\ell z^m$$
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satisfy the difference equation \((\ast)\).
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They wanted to exploit positivity of \(a_{k,\ell,m}\) for proving convergence of this (discrete) solution to a (continuous) solution \(a\) of the wave equation (written in different coordinates)

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(\partial_x \partial_y + \partial_x \partial_z + \partial_y \partial_z)a = 0.
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In 1930 Lewy wrote to G. Szegö asking him to prove positivity of Taylor coefficients \((a_{k,\ell,m} > 0)\) in general.
Szegö’s solution

Shortly after (1932) Szegö proved

\[ \sum_{n}^{\infty} \prod_{j \neq i} \left(1 - x_i \right) = \sum_{k_1, \ldots, k_n \geq 0} a_{k_1, \ldots, k_n} x_1^{k_1} \ldots x_n^{k_n}. \]

For \( n = 3 \) this answers Lewy’s question.

Szegö also proved that for any \( \alpha \geq 1/2 \) the function

\[ \sum_{n}^{\infty} \prod_{j \neq i} \left(1 - x_i \right)^{\alpha} = \sum_{k_1, \ldots, k_n \geq 0} a_{k_1, \ldots, k_n}^{(\alpha)} x_1^{k_1} \ldots x_n^{k_n} \]

has nonnegative Taylor coefficients \( a_{k_1, \ldots, k_n}^{(\alpha)} \). He expresses coefficients as some integrals of products of Bessel functions which are shown to be positive (nonnegative).
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$$\frac{1}{\sum_{i=1}^{n} \prod_{j\neq i}(1 - x_{i})} = \sum_{k_{1},\ldots,k_{n} \geq 0} a_{k_{1},\ldots,k_{n}} x_{1}^{k_{1}} \cdots x_{n}^{k_{n}}.$$
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A new insight from Scott and Sokal

The spanning-tree polynomial of a connected graph $G = (E, V)$ is

$$T_G(x) = \sum T \prod_{e \in T} x^e,$$

where the sum is over all spanning trees of $G$.

**Theorem (Scott and Sokal, 2014)**

Let $G = (V, E)$ be a series-parallel graph. Then for any $\alpha \geq 1/2$ and any positive vector $x \in \mathbb{R}^E^+$, the function

$$y \mapsto T_G(x - y) - \alpha$$

has nonnegative Taylor coefficients.

If $G$ is the $n$-cycle, $T_G(x) = \sum_{i=1}^{n} \prod_{j \neq i} x^j$ and Szegő's result follows from Scott and Sokal's theorem with $x = (1, \ldots, 1)$. 
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Complete monotonicity

Nonnegativity of Taylor coefficients of $y \mapsto T_G(x - y)^{-\alpha}$, $x \in \mathbb{R}^n_>$.
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Nonnegativity of Taylor coefficients of $y \mapsto T_G(x - y)^{-\alpha}$, $x \in \mathbb{R}^n_>$, just means that $T_G(x)^{-\alpha}$, $x \in \mathbb{R}^n_>$, is completely monotone.
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A smooth function \( f : \mathbb{R}^n_{>0} \to \mathbb{R}_{>0} \) is completely monotone (CM).
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A smooth function $f : \mathbb{R}^n_> \to \mathbb{R}_>$ is completely monotone (CM) if for any positive vector $x \in \mathbb{R}^n_>$
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A smooth function $f : \mathbb{R}^n_0 \rightarrow \mathbb{R}_>$ is completely monotone (CM) if for any positive vector $x \in \mathbb{R}^n_>$ all coefficients in the Taylor expansion of $y \mapsto f(x - y)$ are nonnegative.
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A smooth function $f : \mathbb{R}^n_> \to \mathbb{R}_>0$ is completely monotone (CM) if for any positive vector $x \in \mathbb{R}^n_>$ all coefficients in the Taylor expansion of $y \mapsto f(x-y)$ are nonnegative or, equivalently, if

$\left( -1 \right)^k \frac{\partial^k f}{\partial x_{i_1} \ldots \partial x_{i_k}}(x) \geq 0, \quad x \in \mathbb{R}^n_>$,
Complete monotonicity

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for any \( k = 1, 2, \ldots \) and any \( i_1, \ldots, i_k = 1, \ldots, n \).

Simplest CM functions

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f(x) = \frac{1}{x}
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A smooth function $f : \mathbb{R}^n_0 \to \mathbb{R}_0^>$ is completely monotone (CM) if for any positive vector $x \in \mathbb{R}^n_>$ all coefficients in the Taylor expansion of $y \mapsto f(x - y)$ are nonnegative or, equivalently, if

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Simplest CM functions

- $f(x) = \frac{1}{x}$ : $(-1)^k f^{(k)}(x) = \frac{k!}{x^{k+1}} > 0$ for $x > 0$
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**Simplest CM functions**

- \( f(x) = \frac{1}{x} : \) \((-1)^k f^{(k)}(x) = \frac{k!}{x^{k+1}} > 0 \) for \( x > 0 \)

- \( f(x) = e^{-x} \)
Complete monotonicity

Nonnegativity of Taylor coefficients of $y \mapsto T_G(x - y)^{-\alpha}$, $x \in \mathbb{R}_n^+$, just means that $T_G(x)^{-\alpha}$, $x \in \mathbb{R}_n^+$, is completely monotone.

A smooth function $f : \mathbb{R}_n^+ \to \mathbb{R}_+^+$ is completely monotone (CM) if for any positive vector $x \in \mathbb{R}_n^+$ all coefficients in the Taylor expansion of $y \mapsto f(x - y)$ are nonnegative or, equivalently, if

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for any $k = 1, 2, \ldots$ and any $i_1, \ldots, i_k = 1, \ldots, n$.

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• $f(x) = \frac{1}{x}$ : $(-1)^k f^{(k)}(x) = \frac{k!}{x^{k+1}} > 0$ for $x > 0$

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Complete monotonicity and powers of homogeneous polynomials

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Necessary conditions for $p^{-\alpha}$ to be CM:

Conjecture (Michalek, Sturmfels, Uhler and Zwiernik, 2015)

Let $p$ be a real homogeneous polynomial that is stable and $p > 0$ on $\mathbb{R}^n > 0$. Then there is $\alpha = \alpha(p) > 0$ such that $p^{-\alpha}$ is CM.

Status: open in general!
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Necessary conditions for \( p^{-\alpha} \) to be CM:

- \( \alpha \geq 0 \) (enough to look at \( p(x) = x^d, \ x > 0 \))

Sufficient conditions for \( p^{-\alpha} \) to be CM:

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We are interested in functions of the form $p^{-\alpha}$, where $p$ is a real homogeneous polynomial. When is such function CM?

Necessary conditions for $p^{-\alpha}$ to be CM:

- $\alpha \geq 0$ (enough to look at $p(x) = x^d$, $x > 0$)
- $p$ must be a **stable polynomial**, that is, $p(x + iy) \neq 0$ for $x \in \mathbb{R}_{>0}^n$ and $y \in \mathbb{R}^n$ (Scott and Sokal, 2014).

What about sufficient conditions for $p^{-\alpha}$ to be CM?

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Let $p$ be a real homogeneous polynomial that is stable and $p > 0$ on $\mathbb{R}_{>0}^n$. Then there is $\alpha = \alpha(p) > 0$ such that $p^{-\alpha}$ is CM.

Status: open in general!
We are interested in functions of the form $p^{-\alpha}$, where $p$ is a real homogeneous polynomial. When is such function CM?

Necessary conditions for $p^{-\alpha}$ to be CM:

- $\alpha \geq 0$ (enough to look at $p(x) = x^d$, $x > 0$)
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Status: open in general!
Determinantal polynomials

Let $A_1, \ldots, A_n$ be $d \times d$ positive semi-definite real symmetric matrices such that their span contains a matrix of full rank. Then the determinantal polynomial $\det(x_1 A_1 + \cdots + x_n A_n)$ is stable.

Theorem (Scott and Sokal, 2014)

Let $\alpha = 0, 1, 2, 1, 3, 2, \ldots$ or $\alpha \geq d - 1, 2$. Then the function $x \in \mathbb{R}^n > 0 \mapsto \det(x_1 A_1 + \cdots + x_n A_n) - \alpha$ is CM. If $A_1, \ldots, A_n$ span the space of $d \times d$ real symmetric matrices, then CM for $p - \alpha$ implies that $\alpha$ is of the above form.

Corollary

If $p$ is a stable polynomial and some $r$th power $p(x)^r = \det(x_1 A_1 + \cdots + x_n A_n)$ of $p$ is determinantal with matrices $A_k$ of size $d \times d$, then $p - r \alpha$ is CM for $\alpha$ in the above range.
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Quadratic forms

Let $A$ be a non-singular real symmetric matrix of size $n \times n$ such that the quadratic form $p_A(x) = x^tAx$ is stable and $p_A(x) > 0$ on $\mathbb{R}^n > 0$ (the signature of $A$ is then necessarily $(+, -, \ldots, -)$).

Theorem (Scott and Sokal, 2014) $p_A(x) - \alpha$ is CM if and only if $\alpha = 0$ or $\alpha \geq \frac{n-2}{2}$.

Elementary symmetric polynomials $E^d_n(x) = \sum_{1 \leq i_1 < \cdots < i_d \leq n} x_{i_1} \cdots x_{i_d}, x \in \mathbb{R}^n$, are stable.

Corollary $E^{-\alpha}_2, n$ is CM if and only if $\alpha = 0$ or $\alpha \geq \left(\frac{n-2}{2}\right)$.

What about complete monotonicity of $E^{-\alpha}_d, n$ for other $d$?
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$p_A(x) - \alpha$ is CM if and only if $\alpha = 0$ or $\alpha \geq \frac{n - 2}{2}$.

Corollary

$E_{-\alpha, n}$ is CM if and only if $\alpha = 0$ or $\alpha \geq \left(\frac{n - 2}{2}\right)$.

What about complete monotonicity of $E_{-\alpha, n}$ for other $d$?
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Let $A$ be a non-singular real symmetric matrix of size $n \times n$ such that the quadratic form $p_A(x) = x^t A x$ is stable and $p_A > 0$ on $\mathbb{R}_n > 0$.
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**Theorem (Scott and Sokal, 2014)**

Elementary symmetric polynomials $E_d, n(x)$ are stable.

**Corollary** $E_{−α^2, n}$ is CM if and only if $α = 0$ or $α ≥ (n−2)/2$.

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Corollary

$E_{2,n}^{\alpha}$ is CM if and only if $\alpha = 0$
Quadratic forms

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What about complete monotonicity of $E_{d,n}^{-\alpha}$ for other $d$?
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- $d = 1$:

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Theorem (K., Michalek and Sturmfels, 2019)
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For any $n \geq 1$ and $1 \leq d \leq n$ there is $\alpha_{d,n} > 0$
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For any \( n \geq 1 \) and \( 1 \leq d \leq n \) there is \( \alpha_{d,n} > 0 \) such that for all \( \alpha \geq \alpha_{d,n} \) the function \( E_{d,n}^{-\alpha} = \left( \sum_{1 \leq i_1 < \ldots < i_d \leq n} x_{i_1} \ldots x_{i_d} \right)^{-\alpha} \) is CM.

Thus conjecture of Michalek et al. holds for all \( E_{d,n}! \)

However, \( \alpha_{d,n} \) from the theorem is very large (\( \alpha_{d,n} \geq \frac{(n-d)(n-1)!}{2(n-d+1)!} \))!

\[ E_{2,n}^{-\alpha} \text{ is CM iff } \alpha = 0 \text{ or } \alpha \geq \frac{n-2}{2} \]

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Based on these cases and some experiments Scott and Sokal conjectured the following.
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If $E_{d,n}^{−\alpha}$, $d \geq 2$, is completely monotone, then $\alpha = 0$ or $\alpha \geq \frac{n-d}{2}$.

Proof.
One can write

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Summary on complete monotonicity for inverse powers of stable polynomials

Conjecture (Michalek, Sturmfels, Uhler and Zwiernik, 2015)

Let $p$ be a real homogeneous polynomial that is stable and $p > 0$ on $\mathbb{R}_{>0}^n$. Then there is $\alpha = \alpha(p) > 0$ such that $p^{-\alpha}$ is CM.
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Integral representations of completely monotone functions

Recall that a smooth function $f: \mathbb{R}^n > 0 \rightarrow \mathbb{R} > 0$ is CM if
$$(−1)^k \frac{\partial^k f}{\partial x_{i_1} \ldots \partial x_{i_k}}(x) \geq 0,$$
$x \in \mathbb{R}^n > 0$, for any $k = 0, 1, 2, \ldots$ and any indices $i_1, \ldots, i_k = 1, \ldots, n$. So, it amounts to checking nonnegativity of infinitely many functions!

How to certify complete monotonicity?

Theorem (Bernstein-Hausdorff-Widder-Choquet, 1969)

Let $f: \mathbb{R}^n > 0 \rightarrow \mathbb{R} > 0$ be a smooth function. Then $f$ is CM if and only if there exists a (positive) Borel measure $\mu$ whose support is contained in the nonnegative orthant $\mathbb{R}^n \geq 0 = (\mathbb{R}^n > 0)^*$ and such that
$$f(x) = \hat{\mathbb{R}^n \geq 0} e^{-\langle y, x \rangle} d\mu(y), x \in \mathbb{R}^n > 0.$$
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Recall that a smooth function $f : \mathbb{R}^n_0 \to \mathbb{R}_{>0}$ is CM if

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for any $k = 0, 1, 2, \ldots$ and any indices $i_1, \ldots, i_k = 1, \ldots, n$. 

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Recall that a smooth function $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ is CM if

$(-1)^k \frac{\partial^k f}{\partial x_{i_1} \ldots \partial x_{i_k}}(x) \geq 0, \quad x \in \mathbb{R}^n_+$

for any $k = 0, 1, 2, \ldots$ and any indices $i_1, \ldots, i_k = 1, \ldots, n$. So, it amounts to checking nonnegativity of infinitely many functions!

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Theorem (Bernstein-Hausdorff-Widder-Choquet, 1969)

Let $f : \mathbb{R}^n_+ \to \mathbb{R}_+$ be a smooth function.
Integral representations of completely monotone functions

Recall that a smooth function $f : \mathbb{R}^n_{>0} \to \mathbb{R}_{>0}$ is CM if

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Let $f : \mathbb{R}^n_> \to \mathbb{R}_>$ be a smooth function. Then $f$ is CM if and only if there exists a (positive) Borel measure $\mu$ whose support is contained in the nonnegative orthant $\mathbb{R}^n_0 = (\mathbb{R}^n_>)^*$ and such that

$$f(x) = \int_{\mathbb{R}^n_0} e^{-\langle y, x \rangle} \, d\mu(y), \quad x \in \mathbb{R}^n_>.$$
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GOAL: understand the integral representation

\[ f(x) = \hat{R} \geq 0 e^{-\langle y, x \rangle} d\mu(y), \quad x \in R^n \geq 0. \]

for CM function \( f = p - \alpha \), where \( p \) is a product of linear forms.

Note that for any \( \alpha \geq 0 \) the function \( x - \alpha, x > 0 \) is CM

\[ x \alpha = \hat{R} \geq 0 e^{-y \cdot x} y \alpha - 1 \Gamma(\alpha) \ldots \Gamma(d) dy \]

(definition of the Gamma function)

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(Fubini's theorem)
Products of linear forms

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\[
\begin{pmatrix}
| & \cdots & | \\
y_1 & \cdots & y_d \\
| & \cdots & |
\end{pmatrix} : \mathbb{R}_{\geq 0}^d \to \mathbb{R}_{\geq 0}^n
\[
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| & \cdots & |
n1 \cdots y_d
| & \cdots & |
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\]

- The measure \( \mu \) is supported on the polyhedral cone

\[
C := \text{im}(L) = \mathbb{R}_{\geq 0} \cdot y_1 + \cdots + \mathbb{R}_{\geq 0} \cdot y_d \subset \mathbb{R}^n_{\geq 0}
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- If \(y_k\) span \(\mathbb{R}^n\),
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L = \begin{pmatrix}
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| & \cdots & |
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\[ C := \text{im}(L) = \mathbb{R}_{\geq 0} \cdot y_1 + \cdots + \mathbb{R}_{\geq 0} \cdot y_d \subset \mathbb{R}^n_{\geq 0} \]

- If \( y_k \) span \( \mathbb{R}^n \), then \( d\mu(z) = q(z) \, dz \),
\[ \ell_1^{-\alpha_1} \ldots \ell_d^{-\alpha_d} = \int_{\mathbb{R}_{\geq 0}^n} e^{-\langle z, x \rangle} \, d\mu(z), \quad \ell_k = \langle y_k, \cdot \rangle, \ y_k \in \mathbb{R}_\geq^n. \]

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- If \( y_k \) span \( \mathbb{R}^n \), then
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\begin{align*}
L &= \begin{pmatrix} y_1 & \cdots & y_d \\ \vdots & \ddots & \vdots \\ y_1 & \cdots & y_d \\ \vdots & \ddots & \vdots \\
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|L| &= \sqrt{\det(LL^t)}
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\[ |L| = \sqrt{\det(LL^t)} \] and \( dy \) is the Lebesgue measure on \( L^{-1}(z) \).
The chamber complex of a polyhedral cone $C = \mathbb{R}_{\geq 0} \cdot y_1 + \cdots + \mathbb{R}_{\geq 0} \cdot y_d \subset \mathbb{R}^n$, is the common refinement of all cones spanned by linearly independent subsets of \{y_1, \ldots, y_d\}.

Example: pentagonal cone and its chamber complex
Chamber complex

The chamber complex of a polyhedral cone
Chamber complex

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Example: pentagonal cone and its chamber complex
Volume function

\[ \ell^{\alpha_1} \cdots \ell^{\alpha_d} = \hat{C} e^{-\langle z, x \rangle} q(z) dz, \quad \ell_k = \langle y_k, \cdot \rangle, \quad y_k \in \mathbb{R}^n \geq 0. \]

\[ q(z) = |L| \hat{L} - 1(z) \cap \mathbb{R}^d \geq 0 y^{\alpha_1 - 1} \cdots y^{\alpha_d - 1} \Gamma(\alpha_1) \cdots \Gamma(\alpha_d) dy, \quad z \in C = \sum_{k=1}^{d} R \cdot y_k. \]

Theorem (K., Michalek and Sturmfels, 2019)

If \( \alpha_k > 0 \) are integers, then \( q(z), \quad z \in C, \) is a piecewise polynomial function that is differentiable of order \( \sum_{k=1}^{d} \alpha_k - n - 1. \) It is polynomial on each cone in the chamber complex of \( C. \)

If \( \alpha_1 = \cdots = \alpha_d = 1, \) the function \( q(z) \) measures the volume of the \( (d - n) \)-dimensional polytope \( L - 1(z) \cap \mathbb{R}^d \geq 0. \)
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\[ \ell_1^{-\alpha_1} \cdots \ell_d^{-\alpha_d} = \int_C e^{-\langle z, x \rangle} q(z) \, dz, \quad \ell_k = \langle y_k, \cdot \rangle, \quad y_k \in \mathbb{R}^n_{\geq 0}. \]
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If \( \alpha_k > 0 \) are integers, then \( q(z), \ z \in C, \) is a piecewise polynomial function.
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If \( \alpha_k > 0 \) are integers, then \( q(z), z \in C \), is a piecewise polynomial function that is differentiable of order \( \sum_{k=1}^d \alpha_k - n - 1 \).
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Questions

• Does some power of the elementary symmetric polynomial admit a determinantal representation?
• No power of the stable polynomial $p = x_1x_2 + 4(x_1 + x_2 + x_3 + x_4)(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)$ is determinantal (Branden, 2011).
• Is $p - \alpha$ CM for some $\alpha > 0$?
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• Does some power $E_{d,n}^r$ of the elementary symmetric polynomial admit a determinantal representation?

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Thank you!