Approximate Counting via Polynomial Capacity

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Unimodality, Log-concavity, and Beyond
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Outline

1. **Gurvits’ Capacity Method**
   - Motivating example: perfect matchings of regular bipartite graphs
   - Real stable polynomials
   - Polynomial capacity
   - Gurvits’ Theorem

2. **Generalizing Gurvits’ Method**
   - Generalization of Gurvits’ capacity theorem
   - Generalized method
   - Application: imperfect matchings of biregular bipartite graphs

3. **Approximation More Generally**
   - Two-sided Bounds
   - Application: perfect matchings of irregular bipartite graphs

4. **Further Questions**
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Motivating Example: Bipartite Perfect Matchings

Let $G$ be a $d$-regular bipartite graph on $2n$ vertices and consider:

- Bipartite adjacency matrix, $A$:
  \[
  \begin{bmatrix}
  1 & 1 & 0 & 1 \\
  1 & 0 & 1 & 1 \\
  1 & 1 & 1 & 0 \\
  0 & 1 & 1 & 1 \\
  \end{bmatrix}
  \]
  # perfect matchings = permanent
  $d$-regular $\iff$ doubly $d$-stochastic

**Goal:** bound/approximate number of perfect matchings of $G$
(exactly counting is #P-hard)

**How?** Properties of $p_G(x) = \prod_{i=1}^{n} \sum_{j=1}^{n} A_{ij}x_j$
Real Stable Polynomials

**Definition**

A polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is *real stable* if $p(x_1, \ldots, x_n) \neq 0$ whenever $\text{Im}(x_i) > 0$ for all $i$.

Some nice properties:

- **Log-concavity**: Generalizes real-rootedness and hence Newton’s inequalities (“ultra” log-concavity of coefficients)
- **“Minor-closed”**: preserved by partial derivatives and real evaluations
- **Rich inductive structure**: linear operators which preserve this property completely characterized by Borcea-Brändén, ’09

Back to our example...
Motivating Example: Bipartite Perfect Matchings

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$$

# perfect matchings $=$ permanent $\iff$ $d$-regular $\iff$ doubly $d$-stochastic

Connection to real stability:

- **Real stable polynomial:** $p_G(x) = \prod_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_j$
- **Real stability preservers:** $\#\text{pm}(G) = \left. \partial x_1 \right|_{x_1=0} \cdots \left. \partial x_n \right|_{x_n=0} p_G$
For $p \in \mathbb{R}_+[x_1, \ldots, x_n]$ and $\alpha \in \mathbb{R}_+^n$ define:

$$\text{Cap}_\alpha(p) := \inf_{x_1, \ldots, x_n > 0} \frac{p(x_1, \ldots, x_n)}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}} = \inf_{x > 0} \frac{p(x)}{x^\alpha}$$

Some nice properties:

- **Combinatorial:** $\text{Cap}_\alpha(p) > 0$ iff $\alpha$ is in the Newton polytope of $p$
- **Probabilistic:** $\text{Cap}_\alpha(p) \leq p(1)$ with equality iff $\alpha = \nabla \log(p)|_{x=1}$ (the vector of marginals of the distribution associated to $p$)
- **Convexity:** $-\log(\text{Cap}_\alpha(p))$ is the convex conjugate of $\log(p(e^x))$
- **Entropic:** $\log(\text{Cap}_\alpha(p)) = \sup_{\tilde{\mu}} \left[ -D_{KL}(\tilde{\mu}||\mu) \right]$ where $\mu \sim p$ and $\tilde{\mu}$ ranges over distributions with marginals $\alpha$
From Capacity to Perfect Matchings

Theorem (Gurvits, ’07)

Let \( p \in \mathbb{R}_+ [x_1, ..., x_n] \) be homogeneous of degree \( n \) and real stable. If \( p \) is of degree at most \( d \) in \( x_k \), then:

\[
\text{Cap}_1\left( \partial_{x_k} \big|_{x_k=0} p \right) \geq \left( \frac{d-1}{d} \right)^{d-1} \text{Cap}_1(p)
\]

Inductively apply the theorem in the \( d \)-regular case:

\[
\#\text{pm}(G) = \left( \prod_{k=1}^{n} \partial_{x_k} \big|_{x_k=0} \right) p_G \geq \cdots \geq \left( \frac{d-1}{d} \right)^{n(d-1)} \cdot \text{Cap}_1(p_G)
\]

Finally, \( d \)-stochasticity implies marginals = \( \nabla \log(p_G)|_{x=1} = 1 \). So:

\[
\text{Cap}_1(p_G) = p_G(1) = d^n
\]
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Generalizing Gurvits’ Theorem

Theorem (Gurvits-L, ’18)

For all (nondegenerate) real stability preservers $T$ and all non-negative vectors $\alpha, \beta$, there exists $K = K(T, \alpha, \beta) \geq 0$ such that

$$\text{Cap}_\beta(T(p)) \geq K \cdot \text{Cap}_\alpha(p)$$

for all real stable $p \in \mathbb{R}_+[x]$ (and the optimal $K$ has an explicit formula).

“Capacity-preserving linear operators”: can be interpreted as an analytic extension of the Borcea-Brändén characterization.

Generalization of inner product lower bound for real stable polynomials.

- Original result due to Anari-Gharan, ’17.
- Follows from log-concavity: reverse Cauchy-Schwarz type inequality.
- Equivalent form of the Bethe approximation (Straszak-Vishnoi, ’17).
How do we apply this result to counting and approximation?

**General plan:**

1. Real stable polynomial which **encodes our object**.
2. **Apply operator(s)**; keep track of capacity. (Use previous theorem.)
3. At the end, we’re left with a **constant we care about**.

**Note:** To actually approximate, you would also want an upper bound. We will discuss this later.

Let’s consider an example.
Let $G$ be a $(a, b)$-biregular $(m, n)$-bipartite graph $(am = bn)$ and consider:

Bipartite adjacency matrix, $A$:

$$
\begin{bmatrix}
1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1
\end{bmatrix}
$$

$\# k$-matchings $= k$-subpermanents $(a, b)$-regular $\iff$ $(a, b)$-stochastic

We can associate a polynomial to $G$ in the same way as before:

$$
p_G(x) := \prod_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_j
$$
Question: Can we emulate what Gurvits did for perfect matchings?

For \((a, b)\)-regular \((m, n)\)-bipartite \(G\):

\[
p_G(x) := \prod_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_j
\]

\# perfect matchings = permanent = \(\partial_x^{[n]}\big|_{x=0} p_G\)

\# \(k\)-matchings = sum of \(k\)-subpermanents \(\approx \sum_{S \in [n\choose k]} \partial_x^S\big|_{x=1} p_G\)

Evaluation at 1 instead of 0: requires regularity of the graph.

\[
\sum_{S \in [n\choose k]} \partial_x^S\big|_{x=1} p_G = \mu_k(G) \cdot a^{m-k} \quad (a \text{ is the row sum})
\]
Applying the Capacity Method

Recall the general plan:

1. **Real stable polynomial which encodes our object:** $p_G$

2. **Stability preserving operator:** $\sum_{S \in \binom{[n]}{k}} \partial_x^S \big|_{x=1}$

3. **Constant we care about:** $k$-matchings of $G$ (up to scalar)

**Theorem (Gurvits-L, ’18; see Csikvári, ’14)**

*This achieves the best known bound on biregular bipartite $k$-matchings.*

Some info about the actual proof:

- **Con:** Explicit formulas for $\text{Cap}(p_G)$ and $K(T, \alpha, \beta)$ rely on regularity and symmetry.

- **Pro:** For approximation, both can be computed by a convex program.

- **???:** No intuition required; “just” algebra, calculus, AM-GM.
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Two-sided Bounds

Natural “stability-preserving” functional on polynomials:

\[ T_\mu : p \mapsto p_\mu \text{ (the coefficient of } x^\mu) \]

Recall the main theorem: \( \text{Cap}_\beta(T_\mu(p)) \geq K(T_\mu, \alpha, \beta) \cdot \text{Cap}_\alpha(p) \)

Straightforward: \( K > 0 \) iff \( \alpha = \mu \) and \( \beta = 0 \) \( \implies p_\mu \geq K_\mu \text{Cap}_\mu(p) \)

Also easy: positive coefficients \( \implies p_\mu \leq \inf_{x > 0} \frac{p(x)}{x^\mu} = \text{Cap}_\mu(p) \)

Corollary (Anari-Gharan, ’17; Gurvits-L, ’18)

For all multi-indices \( \mu \), there exists \( K_\mu > 0 \) such that

\[ \text{Cap}_\mu(p) \geq p_\mu \geq K_\mu \text{Cap}_\mu(p) \]

for all real stable \( p \in \mathbb{R}_+[x] \) (and the optimal \( K \) has an explicit formula).
Irregular Bipartite Perfect Matchings

Constant can be explicitly computed:

\[ K_\mu = \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} e^{-\mu_i} \sim \sqrt{\prod_{i=1}^{n} \frac{1}{2\pi \mu_i}} \]

Recall: \#pm(G) = coefficient of \( x_1 x_2 \cdots x_n \) in \( p_G \) for any \( G \) \( \Rightarrow \)

\[ \text{Cap}_1(p_G) \geq \#\text{pm}(G) \geq e^{-n} \text{Cap}_1(p_G) \]

Implies: \( e^{-n} \) multiplicative approximation to \#pm(G)

- Works for any bipartite multigraph
- Works for the permanent of any matrix with nonnegative entries
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What about non-bipartite graphs?

The multivariate matching polynomial is real stable: is there some way to use this to count/approximate matchings of general graphs?

Other applications of the capacity-preservation theorem?

Currently the only applications of the theorem and related results are for differential operators. What else can we bound like this?