Approximate Counting via Polynomial Capacity

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Unimodality, Log-concavity, and Beyond
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Outline

1. Gurvits' Capacity Method
   - Motivating example: perfect matchings of regular bipartite graphs
   - Real stable polynomials
   - Polynomial capacity
   - Gurvits' Theorem
2. Generalizing Gurvits' Method
   - Generalization of Gurvits' capacity theorem
   - Generalized method
   - Application: imperfect matchings of biregular bipartite graphs
3. Approximation More Generally
   - Two-sided Bounds
   - Application: perfect matchings of irregular bipartite graphs
4. Further Questions

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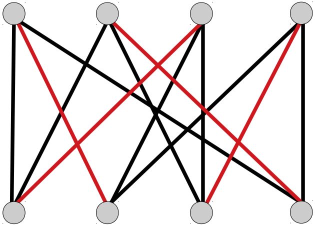
Motivating Example: Bipartite Perfect Matchings

Let G be a d-regular bipartite graph on $2n$ vertices and consider:



Bipartite adjacency matrix, A:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

perfect matchings = permanent

d-regular $\iff$ doubly d-stochastic

Goal: bound/approximate number of perfect matchings of G
(exactly counting is #P-hard)

How? Properties of $p_G(\mathbf{x}) = \prod_{i=1}^{n} \sum_{j=1}^{n} A_{ij} x_j$

Real Stable Polynomials

Definition

A polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ is *real stable* if $p(x_1, \ldots, x_n) \neq 0$ whenever $\text{Im}(x_i) > 0$ for all i.

Some nice properties:

- **Log-concavity:** Generalizes real-rootedness and hence Newton's inequalities ("ultra" log-concavity of coefficients)
- **"Minor-closed":** preserved by partial derivatives and real evaluations
- **Rich inductive structure:** linear operators which preserve this property completely characterized by Borcea-Brändén, '09

Back to our example...

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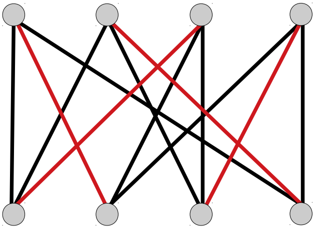

Bipartite adjacency matrix, A:

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$\#$ perfect matchings $=$ permanent

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Connection to real stability:

- **Real stable polynomial:** $p_G(\mathbf{x}) = \prod_{i=1}^n \sum_{j=1}^n A_{ij} x_j$
- **Real stability preservers:** $\#\text{pm}(G) = \partial_{x_1}|_{x_1=0} \cdots \partial_{x_n}|_{x_n=0}\, p_G$

Polynomial Capacity

For $p \in \mathbb{R}_+[x_1, ..., x_n]$ and $\alpha \in \mathbb{R}_+^n$ define:

$$\text{Cap}_\alpha(p) := \inf_{x_1,...,x_n>0} \frac{p(x_1, ..., x_n)}{x_1^{\alpha_1} \cdots x_n^{\alpha_n}} = \inf_{\mathbf{x}>0} \frac{p(\mathbf{x})}{\mathbf{x}^\alpha}$$

Some nice properties:

- **Combinatorial:** $\text{Cap}_\alpha(p) > 0$ iff α is in the Newton polytope of p
- **Probabilistic:** $\text{Cap}_\alpha(p) \leq p(\mathbf{1})$ with equality iff $\alpha = \nabla \log(p)|_{\mathbf{x}=\mathbf{1}}$ (the vector of marginals of the distribution associated to p)
- **Convexity:** $-\log(\text{Cap}_\alpha(p))$ is the convex conjugate of $\log(p(e^\mathbf{x}))$
- **Entropic:** $\log(\text{Cap}_\alpha(p)) = \sup_{\tilde{\mu}} [-D_{KL}(\tilde{\mu}||\mu)]$ where $\mu \sim p$ and $\tilde{\mu}$ ranges over distributions with marginals α

From Capacity to Perfect Matchings

Theorem (Gurvits, '07)

Let $p \in \mathbb{R}_+[x_1, ..., x_n]$ be homogeneous of degree n and real stable. If p is of degree at most d in x_k, then:

$$\mathrm{Cap}_{\mathbf{1}}(\partial_{x_k}|_{x_k=0}\, p) \geq \left(\frac{d-1}{d}\right)^{d-1} \mathrm{Cap}_{\mathbf{1}}(p)$$

Inductively apply the theorem in the d-regular case:

$$\#\mathrm{pm}(G) = \left(\prod_{k=1}^{n} \partial_{x_k}|_{x_k=0}\right) p_G \geq \cdots \geq \left(\frac{d-1}{d}\right)^{n(d-1)} \cdot \mathrm{Cap}_{\mathbf{1}}(p_G)$$

Finally, d-stochasticity implies marginals $= \nabla \log(p_G)|_{\mathbf{x}=\mathbf{1}} = \mathbf{1}$. So:

$$\mathrm{Cap}_{\mathbf{1}}(p_G) = p_G(\mathbf{1}) = d^n$$

Outline

Generalizing Gurvits' Theorem

Theorem (Gurvits-L, '18)

For all (nondegenerate) real stability preservers T and all non-negative vectors α, β, there exists $K = K(T, \alpha, \beta) \geq 0$ such that

$$\mathrm{Cap}_\beta(T(p)) \geq K \cdot \mathrm{Cap}_\alpha(p)$$

for all real stable $p \in \mathbb{R}_+[\mathbf{x}]$ (and the optimal K has an explicit formula).

"Capacity-preserving linear operators": can be interpreted as an analytic extension of the Borcea-Brändén characterization.

Generalization of inner product *lower* bound for real stable polynomials.

- Original result due to Anari-Gharan, '17.
- Follows from log-concavity: reverse Cauchy-Schwarz type inequality.
- Equivalent form of the Bethe approximation (Straszak-Vishnoi, '17).

Generalizing the Method

How do we apply this result to counting and approximation?

General plan:

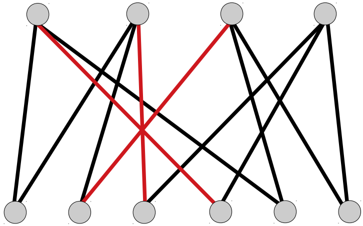
1. Real stable polynomial which **encodes our object**.
2. **Apply operator(s)**; keep track of capacity. (Use previous theorem.)
3. At the end, we're left with a **constant we care about**.

Note: To actually approximate, you would also want an upper bound. We will discuss this later.

Let's consider an example.

Imperfect Matchings, Biregular Bipartite Graphs

Let G be a (a, b)-biregular (m, n)-bipartite graph $(am = bn)$ and consider:



Bipartite adjacency matrix, A:

$$\begin{bmatrix} 1 & 0 & 0 & \color{red}{1} & 1 & 0 \\ 1 & 1 & \color{red}{1} & 0 & 0 & 0 \\ 0 & \color{red}{1} & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

$\#$ k-matchings $=$ k-subpermanents

(a, b)-regular $\iff$ (a, b)-stochastic

We can associate a polynomial to G in the same way as before:

$$p_G(\mathbf{x}) := \prod_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_j$$

Applying the Capacity Method

Question: Can we emulate what Gurvits did for perfect matchings?

For (a, b)-regular (m, n)-bipartite G:

$$p_G(\mathbf{x}) := \prod_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_j$$

$\#$ perfect matchings $=$ permanent $= \partial_x^{[n]}\Big|_{x=0} p_G$

$\#$ k-matchings $=$ sum of k-subpermanents $\approx \sum_{S \in \binom{[n]}{k}} \partial_x^S\big|_{x=1} p_G$

Evaluation at 1 instead of 0: requires regularity of the graph.

$$\sum_{S \in \binom{[n]}{k}} \partial_x^S\Big|_{x=1} p_G = \mu_k(G) \cdot a^{m-k} \qquad (a \text{ is the row sum})$$

Applying the Capacity Method

Recall the general plan:

1. **Real stable polynomial which encodes our object:** p_G
2. **Stability preserving operator:** $\sum_{S \in \binom{[n]}{k}} \partial_x^S \big|_{x=1}$
3. **Constant we care about:** k-matchings of G (up to scalar)

Theorem (Gurvits-L, '18; see Csikvári, '14)

This achieves the best known bound on biregular bipartite k-matchings.

Some info about the actual proof:

- **Con:** Explicit formulas for $\text{Cap}(p_G)$ and $K(T, \alpha, \beta)$ rely on regularity and symmetry.
- **Pro:** For approximation, both can be computed by a convex program.
- **???:** No intuition required; "just" algebra, calculus, AM-GM.

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Two-sided Bounds

Natural "stability-preserving" functional on polynomials:

$$T_\mu : p \mapsto p_\mu \text{ (the coefficient of } x^\mu\text{)}$$

Recall the main theorem: $\mathrm{Cap}_\beta(T_\mu(p)) \geq K(T_\mu, \alpha, \beta) \cdot \mathrm{Cap}_\alpha(p)$

Straightforward: $K > 0$ iff $\alpha = \mu$ and $\beta = 0 \implies p_\mu \geq K_\mu \,\mathrm{Cap}_\mu(p)$

Also easy: positive coefficients $\implies p_\mu \leq \inf_{\mathbf{x}>0} \frac{p(\mathbf{x})}{\mathbf{x}^\mu} = \mathrm{Cap}_\mu(p)$

Corollary (Anari-Gharan, '17; Gurvits-L, '18)

For all multi-indices μ, there exists $K_\mu > 0$ such that

$$\mathrm{Cap}_\mu(p) \geq p_\mu \geq K_\mu \,\mathrm{Cap}_\mu(p)$$

for all real stable $p \in \mathbb{R}_+[\mathbf{x}]$ (and the optimal K has an explicit formula).

Irregular Bipartite Perfect Matchings

Constant can be explicitly computed:

$$K_\mu = \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} e^{-\mu_i} \sim \sqrt{\prod_{i=1}^{n} \frac{1}{2\pi\mu_i}}$$

Recall: $\#\mathrm{pm}(G) =$ coefficient of $x_1 x_2 \cdots x_n$ in p_G for *any* $G \implies$

$$\mathrm{Cap}_{\mathbf{1}}(p_G) \geq \#\mathrm{pm}(G) \geq e^{-n}\,\mathrm{Cap}_{\mathbf{1}}(p_G)$$

Implies: e^{-n} multiplicative approximation to $\#\mathrm{pm}(G)$

- Works for any bipartite multigraph
- Works for the permanent of any matrix with nonnegative entries

Outline

Further Questions

What about non-bipartite graphs?

The multivariate matching polynomial is real stable: is there some way to use this to count/approximate matchings of general graphs?

Other applications of the capacity-preservation theorem?

Currently the only applications of the theorem and related results are for differential operators. What else can we bound like this?