Multifractal analysis of the Birkhoff sums of Saint-Petersburg potential

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Outline

Introduction

I. Saint Petersburg game (paradox)

- Proposed by Nicolas Bernoulli on Sep. 9th 1713 in a letter to Pierre Raymond de Montmort.
- A solution by Daniel Bernoulli, who lived at Saint Petersburg, was published in the Commentaries of the Imperial Academy of Science of Saint Petersburg in 1738.

The game : tossing a fair coin at each stage.

- if head appears at the first time, you gain 1 dollar, and stop,
- if tails appears at the first time, go on the tossing, if at the second time head appears, you gain 2 dollars (double) and stop,
- if tails appears for both the first two times, go on the tossing, if at the third time head appears, you gain 2^2 dollars (again double) and stop,
- ...

Question : How much do you want to pay for entering the game ?

II. Infinite expectation of the gain function

Let φ be the gain function. Then

- φ takes the value 2^{n-1} with the probability 2^{-n},
- the expectation of φ is

$$\mathbb{E}(\varphi) = \sum_{n=1}^{\infty} 2^{n-1} \times 2^{-n} = \sum_{n=1}^{\infty} \frac{1}{2} = +\infty.$$

Conclusion : for a fair play, we should pay infinite money for entering the game !

III. A model on the unit interval

We could construct the probability model on the unit interval.

- T : the doubling map on $(0, 1]$ defined by $Tx = 2x - \lceil 2x \rceil + 1$.
- $\varepsilon_1(x) = \lceil 2x \rceil - 1$ and $\varepsilon_n(x) := \varepsilon_1(T^{n-1}x)$ for $n \geq 2$.
- each $x \in (0, 1]$ can be written as

$$x = \frac{\varepsilon_1(x)}{2} + \cdots + \frac{\varepsilon_n(x)}{2^n} + \cdots .$$

The tossing results can be identified with sequences $\varepsilon_1(x)\varepsilon_2(x)\ldots$.

The gain function which will be called the Saint-Petersburg potential is a function $\varphi : (0, 1] \to \mathbb{R}$ defined as

$$\varphi(x) = 2^n \text{ if } x \in (2^{-n-1}, 2^{-n}], \ \forall n \geq 0.$$

IV. Birkhoff sums and law of convergence

Let

$$S_n(x) = S_n\varphi(x) := \varphi(x) + \varphi(T(x)) + \cdots + \varphi(T^{n-1}(x)), \quad x \in (0,1].$$

We can see that for Lebesgue a.e. $x \in (0,1]$,

$$\lim_{n\to\infty} \frac{S_n\varphi(x)}{n} = +\infty.$$

By using a result in **Feller**'s 1968 book, page 253, we have for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \text{Leb}\left\{x \in (0,1] \ : \ \left|\frac{S_n\varphi(x)}{n\log n} - \frac{1}{\log 2}\right| \geq \varepsilon\right\} = 0.$$

By applying **Feller 1946** : we obtain that if $\{\Psi_n\}_{n\geq 1}$ is an increasing sequence such that $\Psi_n \to \infty$ as $n \to \infty$, then almost surely either

$$\lim_{n\to\infty} \frac{S_n\varphi(x)}{\Psi_n} = 0 \quad \text{or} \quad \limsup_{n\to\infty} \frac{S_n\varphi(x)}{\Psi_n} = \infty,$$

according as

$$\sum_{n\geq 1} \text{Leb}\{x \in (0,1] : \varphi(x) \geq \Psi_n\} < \infty \quad \text{or} \quad = \infty.$$

V. Question of multifractal analysis

From multifractal analysis point of view, we are interested in the following level sets

$$E(\alpha) = \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\} \quad \alpha \geq 1.$$

Question :

- What is the Hausdorff dimension of $E(\alpha)$?
- What is the nature of the dimension spectrum function $\alpha \mapsto \dim_H E(\alpha)$?

Classical results on multifractal analysis of Birkhoff averages

I. General setting

- (X, d) a metric space,
- $T : X \to X$ a transformation.
- $\varphi : X \to \mathbb{R}$ a real-valued function, called potential.
- Birkhoff average of φ at x :

$$A_\varphi(x) := \lim_{n\to\infty} \frac{S_n\varphi(x)}{n}, \quad \text{with } S_n\varphi(x) := \sum_{j=0}^{n-1} \varphi(T^j x).$$

- Level sets :

$$E_\varphi(\alpha) := \Big\{x : A_\varphi(x) = \alpha\Big\}, \quad \alpha \in \mathbb{R} \cup \{-\infty, \infty\}.$$

Question : **Dimension spectrum :** $f(\alpha) := \dim_H(E_\varphi(\alpha)) =?$

II. A typical classical example

Borel normal numbers (typical Birkhoff average)

- For each number $x = \sum_{n=1}^{\infty} \frac{x_n}{2^n} \in [0,1]$ $(x_n = 0,1)$, define

$$A(x) = \lim_{n\to\infty} \frac{x_1 + \cdots + x_n}{n} \quad \left(= \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} 1_{[1/2,1]}(T^j x)\right),$$

  (with $Tx = 2x \mod 1$).

- **E. Borel 1909 :**

$$A(x) = \frac{1}{2} \quad \mathcal{L} - a.e.$$

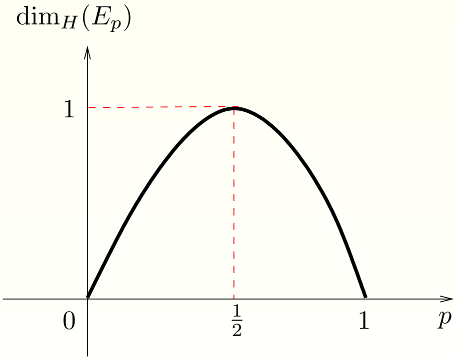
- **A. Besicovitch, 1934 ; H. G. Eggleston, 1949 :**

$$\dim_H(E_p) = \frac{-p\log p - (1-p)\log(1-p)}{\log 2}$$

  where

$$E_p := \Big\{x \in [0,1] : A(x) = p\Big\}.$$

Spectrum of Besicovitch-Eggleston sets



III. Contributions to multifractal analysis of Birkhoff averages

(Hölder) continuous potential, in mixing subshift of finite type, $C^{1+\varepsilon}$-mixing conformal repeller : **Barreira–Schmeling 2000, Barreira–Saussol 2001, Barreira–Saussol–Schmeling 2002, Fan–Feng 2000, Fan–Feng–Wu 2001, Feng–Lau–Wu 2002, Olivier 1999, Olsen 2003, Pesin–Weiss 2001, Tempelman 2001, ...**

Continuous potential, in systems with (weak, almost) specification property : **Taken-Verbytzky 2003, Chen–Küpper–Shu 2005, Pfister–Sullivan 2007, Fan–Liao–Peyrière 2008, Thompson 2009, Climenhaga–Thompson 2012, ...**

Piecewise continuous (constant) potential in continued fractions : **Kinney–Pitcher 1966, Billingsley–Henningsen 1975, Pollicott–Weiss 1999, Kesseböhmer–Stratmann 2007, Liao–Ma–Wang 2008, Fan–Liao–Wang–Wu 2009, Fan–Liao–Ma 2010, Fan–Jordan–Liao–Rams 2015, Iommi–Jordan 2015, ...**

IV. Typical results

Conditional variational principle :

$$\dim_H E_\varphi(\alpha) = \sup\left\{ \frac{h_\mu}{\int_X \log|T'|d\mu} : \ \mu \text{ is invariant and } \int_X \varphi d\mu = \alpha \right\}.$$

For continued fractions :

$$\dim_H E_\varphi(\alpha) = \max\left\{ \frac{1}{2}, \sup_{\mu \in \mathcal{M}(T)} \left\{ \frac{h_\mu}{-2\int \log x d\mu} : \int \varphi \mathrm{d}\mu = \alpha, \ h_\mu < \infty \right\}\right\}.$$

Legendre transform version :

$$\dim_H \left\{ x : \lim_{n\to\infty} \frac{S_n\varphi(x)}{S_n \log|T'|(x)} = \alpha \right\} = \inf_q \{T(q) + q\alpha\},$$

where $T(q)$ is the number such that $P(T(q), q) = 0$, and $P(t, q)$ is the pressure function associated to the potential $-t\log|T'| + q\varphi$.

Multifractal analysis of the Birkhoff averages of St. Petersburg potentiial

I. Result of the Birkhoff averages of St. Petersburg potential

For $t \in \mathbb{R}$ and $q > 0$, define

$$P(t,q) := \log \sum_{j=1}^{\infty} 2^{-tj-q(2^j-1)}.$$

- P is a real-analytic function,
- for each $q > 0$, there is a unique $t(q) > 0$ such that $P(t(q), q) = 0$,
- $q \mapsto t(q)$ is real-analytic, strictly decreasing and convex.

Theorem (Kim–L–Rams–Wang, arXiv :1707.06059)

For any $\alpha \geq 1$ we have

$$\dim_H E(\alpha) = \inf_{q>0} \{t(q) + q\alpha\}.$$

Consequently, $\dim_H E(1) = 0$ and the function $\alpha \mapsto \dim_H E(\alpha)$ is real-analytic, strictly increasing, concave, and has limit 1 as $\alpha \to \infty$.

II. Transference to an interval map of infinitely many branches

Define for $x \in (0,1]$, $n(x) := \inf\{n \geq 0 : T^n x \in (1/2, 1]\}$. Then

$$n(x) = n \quad \text{if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \text{ for all } n \geq 0.$$

Define $\widehat{T} : (0,1] \to (0,1]$ (called the acceleration of T) by

$$\widehat{T}(x) = T^{n(x)+1}(x) = 2^{n+1}\left(x - \frac{1}{2^{n+1}}\right) \text{ if } x \in \left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right], \text{ for all } n \geq 0.$$

Define a new potential function

$$\phi(x) := 2^{n(x)+1} - 1, \; x \in (0,1].$$

In fact, ϕ is nothing but the function satisfying $\phi(x) = \sum_{j=0}^{n(x)} \varphi(T^j x)$.
Let $n_1 = n(x) + 1 \geq 1$, and $n_k = n(\widehat{T}^{k-1} x) + 1$ for $k \geq 2$, we have

$$\sum_{j=0}^{n_1 + \cdots + n_\ell - 1} \varphi(T^j x) = \sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x) = 2^{n_1} + \cdots + 2^{n_\ell} - \ell.$$

III. Transference lemma

Recall the set in question :

$$E(\alpha) = \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(T^j x) = \alpha \right\} \qquad (\alpha \geq 1).$$

Define

$$\widetilde{E}(\alpha) := \left\{ x \in (0,1] : \lim_{\ell\to\infty} \frac{\sum_{k=0}^{\ell-1} \phi(\widehat{T}^k x)}{\sum_{k=0}^{\ell-1} \log_2 |\widehat{T}'|(\widehat{T}^k x)} = \alpha \right\} \qquad (\alpha \geq 1).$$

Transference Lemma (KLRW, arXiv :1707.06059)

For all $\alpha \geq 1$, we have $E(\alpha) = \widetilde{E}(\alpha)$.

IV. Ruelle theory for interval maps of infinite many branches

We consider the potential function with two parameters $t > 0, q > 0$

$$\psi_{t,q} := -t \log |\widehat{T}'| - (\log 2) \cdot q\phi.$$

Define Ruelle transfer operator

$$\mathcal{L}_{t,q} f(x) := \sum_{y \in \widehat{T}^{-1} x} e^{\psi_{t,q}(y)} f(y).$$

By Ruelle theory (see ex. Hanus–Mauldin–Urbański 2002)

- $\exists$ eigenvalue $\lambda_{t,q}$ and eigenfunction $h_{t,q}$ for $\mathcal{L}_{t,q}$,
- $\exists$ eigenfunction $\nu_{t,q}$ for the conjugate operator $\mathcal{L}^*_{t,q}$.
- pressure function $P(t,q) = \log \lambda_{t,q}$,
- Gibbs measure $\mu_{t,q} = h_{t,q} \cdot \nu_{t,q}$.

In the present case, the pressure function can be computed by

$$P(t,q) = \lim_{\ell \to \infty} \frac{1}{\ell} \log \sum_{\widehat{T}^\ell x = x} \exp(S_\ell \psi_{t,q}(x)) = \log \sum_{j=1}^{\infty} 2^{-tj - q(2^j - 1)}.$$

V. Upper bound of $\dim_H \widetilde{E}(\alpha)$

For $(n_1, \cdots, n_\ell) \in \mathbb{N}^\ell$, a $\widehat{T}$-*dyadic* cylinder of order ℓ is defined as

$$D_\ell(n_1, \cdots, n_\ell) = \{x \in (0,1] : n_k(x) = n_k, 1 \le k \le \ell\} = [0^{n_1} 1 0^{n_2} \dots 0^{n_\ell} 1].$$

Let $D_\ell(x)$ be the $\widehat{T}$-dyadic cylinder containing x of order ℓ. By the Gibbs property of $\mu_{t,q}$,

$$\begin{aligned} \frac{\log \mu_{t,q}(D_\ell(x))}{\log |D_\ell(x)|} &= \frac{S_\ell \psi_{t,q}(x) - \ell P(t,q)}{-S_\ell \log |\widehat{T}'|(x)} \\ &= \frac{-tS_\ell \log |\widehat{T}'|(x) - (\log 2) \cdot q S_\ell \phi(x) - \ell P(t,q)}{-S_\ell \log |\widehat{T}'|(x)} \\ &= t + q \frac{S_\ell \phi(x)}{S_\ell \log_2 |\widehat{T}'|(x)} + \frac{\ell P(t,q)}{S_\ell \log |\widehat{T}'|(x)}. \end{aligned}$$

For each $q > 0$, let $t(q)$ be the the number such that $P(t(q), q) = 0$. Then for all $x \in \widetilde{E}(\alpha)$,

$$\begin{aligned} \liminf_{r \to 0} \frac{\log \mu_{t,q}(B(x,r))}{\log r} &\le \liminf_{\ell \to \infty} \frac{\log \mu_{t,q}(B(x, |D_\ell(x)|))}{\log |D_\ell(x)|} \\ &\le \liminf_{\ell \to \infty} \frac{\log \mu_{t,q}(D_\ell(x))}{\log |D_\ell(x)|} = t(q) + q\alpha, \end{aligned}$$

VI. Question and remark

Question : More general potentials ? What about $\varphi(x) = 1/x$?

Remark : Fan–Schmeling, personal communications : multifractal analysis of the Birkhoff averages of the potential $\varphi(x) = \log|\sin(x)|$ on $(0, 1)$. It is unbounded but integrable.

Fast increasing Birkhoff sums of St. Petersburg potentiial

I. Questions and results

Let $\Psi : \mathbb{N} \to \mathbb{R}^+$ be an increasing function such that $\Psi(n)/n \to \infty$ as $n \to \infty$.

Question : For $\beta > 0$, what is the Hausdorff dimension of the level set

$$E(\Psi, \beta) := \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{S_n\varphi(x)}{\Psi(n)} = \beta \right\}?$$

Theorem (Kim–L–Rams–Wang, arXiv :1707.06059)

If $\Psi(n)$ is one of the following

$$\Psi(n) = n \log n, \;\; \Psi(n) = n^a \;\, (a > 1), \;\; \Psi(n) = 2^{n^\gamma} \;\, (0 < \gamma < 1/2),$$

then for any $\beta > 0$, $\dim_H E(\Psi, \beta) = 1$.

If $\Psi(n) = 2^{n^\gamma}$ with $\gamma \geq 1/2$, then for any $\beta > 0$, $E(\Psi, \beta) = \emptyset$.

II. General result for the case of full dimension

Lemma (Kim–L–Rams–Wang, arXiv :1707.06059)

Let $\Psi : \mathbb{N} \to \mathbb{N}$ be an increasing function such that $\Psi(n)/n \to \infty$ as $n \to \infty$. Assume that there exists a subsequence N_k satisfying the following conditions

$$N_k - N_{k-1} \to \infty, \ \Psi(N_k) - \Psi(N_{k-1}) \to \infty, \tag{1}$$

and

$$\frac{\Psi(N_{k-1})}{\Psi(N_k)} \to 1, \quad \frac{\log\big(\Psi(N_k) - \Psi(N_{k-1})\big)}{N_k - N_{k-1}} \to 0, \tag{2}$$

as $k \to \infty$. Then the set

$$E(\Psi, 1) = \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{\Psi(n)} S_n\varphi(x) = 1 \right\}$$

has Hausdorff dimension 1.

III. Key observations

Lemma 1

Let W be an integer such that $2^t \leq W < 2^{t+1}$ for some positive integer t. For any $0 \leq n \leq t$, among the integers between W and $W(1+2^{-n})$, there is one $V = V(W,n)$ whose binary expansion of V has at most $n+2$ digits 1 and ends with at least $t-n$ zeros.

Lemma 2

For each integer W, and any integer $n \leq \log W$, we can find a word w with length

$$|w| \leq (n+2)(2+\log W)$$

and for any $x \in I_{|w|}(w)$

$$W \leq \sum_{j=0}^{|w|-1} \varphi(T^j x) \leq W(1+2^{-n}).$$

IV. Cantor type set

For each $k \geq 1$, we write $W_k := \Psi(N_k) - \Psi(N_{k-1})$ and let $\{n_k\}$ be a sequence of integers tending to ∞ such that

$$n_k \leq \log W_k, \quad n_k \cdot \frac{\log\left(\Psi(N_k) - \Psi(N_{k-1})\right)}{N_k - N_{k-1}} \to 0.$$

Now for W_k and n_k, let w_k be the word given in Lemma 2. Then the length a_k of w_k satisfies

$$\begin{aligned} a_k \leq & (n_k + 2)(2 + \log W_k) \\ = & (n_k + 2)\left(2 + \log(\Psi(N_k) - \Psi(N_{k-1}))\right) = o(N_k - N_{k-1}) \end{aligned}$$

and for any $x \in I_{a_k}(w_k)$,

$$W_k \leq \sum_{j=0}^{a_k - 1} \varphi(T^j x) \leq W_k(1 + 2^{-n_k}).$$

Define t_k, ℓ_k to be the integers satisfying

$$N_k - N_{k-1} - a_k = t_k m + \ell_k, \text{ for some } 0 \leq \ell_k < m.$$

V. Cantor type set - continued

Fix a large integer m and write
$\mathcal{U} = \Big\{ u = (\varepsilon_1, \cdots, \varepsilon_m) : \varepsilon_m = 1, \varepsilon_i \in \{0,1\}, i \neq m \Big\}$.
Level 1 of the Cantor subset. Define

$$E_1 = \Big\{ I_{N_1}(u_1, \cdots, u_{t_1}, 1^{\ell_1}, w_1) : u_i \in \mathcal{U}, 1 \leq i \leq t_1 \Big\}.$$

Denote by $I_{N_1}(U_1)$ a general cylinder in E_1.

Level 2 of the Cantor subset. Fix an element $I_{N_1} = I_{N_1}(U_1) \in E_1$. Define

$$E_2(I_{N_1}(U_1)) = \Big\{ I_{N_2}(U_1, u_1, \cdots, u_{t_2}, 1^{\ell_2}, w_2) : u_i \in \mathcal{U}, 1 \leq i \leq t_2 \Big\}.$$

Then $E_2 = \bigcup_{I_{N_1} \in E_1} E_2(I_{N_1})$. Let $I_{N_2}(U_2)$ be a general cylinder in E_2.

From Level k to $k+1$. Fix $I_{N_k}(U_k) \in E_k$. Define

$$E_{k+1}(I_{N_k}(U_k)) = \Big\{ I_{N_{k+1}}(U_k, u_1, \cdots, u_{t_{k+1}}, 1^{\ell_{k+1}}, w_{k+1}) : u_i \in \mathcal{U}, 1 \leq i \leq t_{k+1} \Big\}.$$

Then

$$E_{k+1} = \bigcup_{I_{N_k} \in E_k} E_{k+1}(I_{N_k}).$$

VI. Empty set case

Now suppose that $\Psi(n) = 2^{n^\gamma}$ with $1/2 \le \gamma < 1$. Let $\beta > 0$ be given. Then, for $x \in E(\Psi, \beta)$, if x has binary expansion

$$x = [0^{n_1-1}10^{n_2-1}1 \cdots 0^{n_\ell-1}1 \cdots]$$

then

$$\frac{S_{n_1+n_2+\cdots+n_\ell}(x)}{\Psi(n_1+n_2+\cdots+n_\ell)} = \frac{2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell}{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}} \to \beta,$$
$$\frac{S_{n_1+n_2+\cdots+n_\ell+1}(x)}{\Psi(n_1+n_2+\cdots+n_\ell+1)} = \frac{2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell+2^{n_{\ell+1}-1}}{2^{(n_1+n_2+\cdots+n_\ell+1)^\gamma}} \to \beta,$$

which implies that

$$\frac{S_{n_1+n_2+\cdots+n_{\ell+1}}(x)}{S_{n_1+n_2+\cdots+n_\ell}(x)} = 1 + \frac{2^{n_{\ell+1}}-1}{2^{n_1}+2^{n_2}+\cdots+2^{n_\ell}-\ell} \to 1.$$

This further implies

$$\frac{2^{(n_1+n_2+\cdots+n_\ell+n_{\ell+1})^\gamma}}{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}} = \frac{\Psi(n_1+\cdots+n_{\ell+1})}{\Psi(n_1+\cdots+n_\ell)} \to 1.$$

VII. Empty set case -continued

Thus

$$\frac{\gamma n_{\ell+1}}{(n_1+\cdots+n_\ell)^{1-\gamma}} \approx (n_1+\cdots+n_\ell+n_{\ell+1})^\gamma - (n_1+\cdots+n_\ell)^\gamma \to 0.$$

Therefore, for any $\varepsilon > 0$, there exists $k_0 \geq 1$ such that for all $j > k_0$,

$$n_j < \varepsilon(n_1+n_2+\cdots+n_{j-1})^{1-\gamma}.$$

Then for any $k_0 < j \leq \ell$

$$n_j < \varepsilon(n_1+n_2+\cdots+n_\ell)^{1-\gamma}.$$

This implies

$$S_{n_1+n_2+\cdots+n_\ell}(x) = 2^{n_1}+2^{n_2}+\cdots+2^{n_\ell} - \ell \leq M + \ell 2^{\varepsilon(n_1+n_2+\cdots+n_\ell)^{1-\gamma}} - \ell,$$

with $M := 2^{n_1}+\cdots+2^{n_{k_0}}$. Thus

$$\frac{S_{n_1+n_2+\cdots+n_\ell}(x)}{\Psi(n_1+n_2+\cdots+n_\ell)} < \frac{M+\ell 2^{\varepsilon(n_1+n_2+\cdots+n_\ell)^{1-\gamma}} - \ell}{2^{(n_1+n_2+\cdots+n_\ell)^\gamma}}. \quad (3)$$

Observing $n_j \geq 1$, we deduce that the righthand of (3) converges to 0 for $1/2 \leq \gamma < 1$, a contradiction ! Hence $E(\Psi, \beta) = \emptyset$.