PDE session

The Brezis–Nirenberg phenomenon for fractional Laplacians

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Let m, s be two given real numbers, with $0 \le s < m < \frac{n}{2}$. Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Denote by $2^*_m = \frac{2n}{n-2m}$ the critical Sobolev exponent for the embedding $W_2^m \hookrightarrow L_q$.

We study equations

$$(-\Delta)^m_D u = \lambda(-\Delta)^s_D u + |u|^{2^*_m-2}u \qquad \text{in} \quad \Omega, \tag{1}$$

$$(-\Delta)^m_N u = \lambda(-\Delta)^s_N u + |u|^{2^*_m-2}u \qquad \text{in} \quad \Omega. \tag{2}$$

Here fractional Laplacians $(-\Delta)^m_D$ and $(-\Delta)^m_N$ (Dirichlet and Navier, respectively) are self-adjoint operators defined by their quadratic forms:

$$Q^D_m[u] \equiv \int\limits_\Omega (-\Delta)^m_D u \cdot u\, dx := \int\limits_{\mathbb{R}^n} |\xi|^{2m} |\mathcal{F}[u]|^2\, d\xi\,,$$

$$Q^N_m[u] \equiv \int\limits_\Omega (-\Delta)^m_N u \cdot u\, dx := \sum_{k\in\mathbb{N}} \lambda_k^m (u, \varphi_k)^2,$$

respectively. Here $\mathcal{F}$ stands for the Fourier transform while λ_k and φ_k are eigenvalues and (normalized) eigenfunctions of conventional Dirichlet–Laplacian in Ω. The domains of quadratic forms satisfy $Dom(Q^D_m) = \widetilde{H}^m(\Omega) \subset Dom(Q^N_m)$, where $\widetilde{H}^m(\Omega) = \{u \in W_2^m(\mathbb{R}^n)\ :\ \operatorname{supp} u \subset \overline{\Omega}\}$.

Theorem. *Let $s \ge 2m - \frac{n}{2}$. Then each of problems (1) and (2) has a nontrivial weak solution for arbitrarily small positive λ.*

The case $s = 0$ and m integer or $m \in (0, 1)$ was considered earlier in a number of papers beginning with the celebrated paper of H. Brézis and L. Nirenberg, 1983 (for $m = 1$).

An important auxiliary result of independent interest is coincidence of sharp Sobolev constants for Navier and Dirichlet fractional Laplacians.

This talk is based on a joint papers with Roberta Musina, see [1], [2], [3].

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References

[1] Musina R., Nazarov A. I., “Non-critical dimensions for critical problems involving fractional Laplacians”, to appear in Rev. Matem. Iberoamericana.

[2] Musina R., Nazarov A. I., “On fractional Laplacians–3”, to appear in ESAIM: COCV.

[3] Musina R., Nazarov A. I., “On the Sobolev and Hardy constants for the fractional Navier Laplacian”, Nonlin. Anal. – TMA, **121**. 2015. P.123-129.