Generalized Permutohedra: Ehrhart Positivity and Minkowski Linear Functionals

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IML, Unimodality, Log Concavity and Beyond

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Valuations on Convex Sets

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A map \( \phi: \mathcal{K}_d \to \mathbb{R} \) is called a valuation if for every \( A, B \in \mathcal{K}_d \),

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\phi(A \cup B) = \phi(A) + \phi(B) - \phi(A \cap B),
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Some examples of valuations:

1. Volume.
2. Surface area.
3. Volume of slice with a fixed hyperplane.
4. Expected Volume of intersection with random \( k \) dimensional subspace (uniformly distributed with respect to the unique \( O(d) \) invariant measure on \( G(d, k) \)).
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Let us call the valuations in the last item $V_k$ for $0 \leq k \leq d$.

Note that $V_0 = 1$ and $V_{d-1}$ is the surface area and $V_d$ is the usual volume.
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One reason for studying valuations is Hilbert’s third problem.
Let $D_2$ be the group of isometries of $\mathbb{R}^2$.

**Definition**

A dissection of a polytope $A \subset \mathbb{R}^2$ is a collection of polytopes $A_1, \ldots, A_k$ such that $A = \bigcup_{i=1}^{k} A_i$ and such that the interiors of the $A_i$ are disjoint.
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Two polytopes $A, B \subset \mathbb{R}^2$ are called $\mathcal{D}_2$ equidissectable if there are dissections $A = \bigcup_{i=1}^{k} A_i$ and $B = \bigcup_{i=1}^{n} B_i$ and such that $A_i \sim_{\mathcal{D}_2} B_i$ for each $i$.

Caveat: What we are calling equidissectability is usually called *Scissors Congruence*.
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The most elementary way of computing volume is to break down a set into elementary sets and sum up their elementary volumes. To this end, we have

### Theorem (Bolyai-Gerwein)

*Two plane polygons are of equal area iff they are $D_2$ equidissectable.*
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This was the first of Hilbert’s 23 problems to be solved, actually in the same year!
**Dehn Invariants**

**Definition**

Let $f : \mathbb{R} \to \mathbb{R}$ be an additive map that is zero at $\pi$ but is not identically zero. Such a $f$ must be non-measurable. The associated Dehn invariant for a polytope $P \subset \mathbb{R}^3$ is

$$f^*(P) = \sum_{i=1}^{k} \sigma_i f(\alpha_i),$$

where the $\sigma_i$ are the edge lengths and the $\alpha_i$ are the dihedral angles.
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Dehn showed the necessity of the following (settling Hilbert’s third problem). The sufficiency was settled by Sydler in 1965.

Theorem (Dehn, Sydler)

Polytopes $P, Q \subset \mathbb{R}^3$ are equidissectable under $D_3$ iff all their Dehn invariants are the same. It is easy to see that every Dehn invariant of the standard cube is zero while every Dehn invariant of the regular tetrahedron is non-zero.
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Some related results: Two sets are equidecomposable (as opposed to equidissectable) if they can be partitioned into $D_2$ congruent sets.

**Theorem (Laczkovich)**

*Two plane polygons of equal area are equidecomposable: Indeed, one only needs to take translations in place of $D_2$.*
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*A circular disk and a square of the same area are equidecomposable, even under the group of translations.*
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And of course, we have the Banach-Tarski paradox.

**Theorem**

*Any two bounded sets $X, Y \subset \mathbb{R}^n$ with non-empty interior, for $n \geq 3$ are equidecomposable.*
Valuations

Hadwiger’s theorem

Given $A \in \mathbb{K}^d$, the Steiner polynomial is the polynomial given by

$$|A + xB|_d^2 = d \sum_{k=0}^{d} S_k x^k.$$ 

The coefficients are called Quermassintegrals and are equal to the $V_k$ up to a scaling.

Hadwiger’s theorem says

Theorem

The collection of all continuous rigid motion invariant valuations on $\mathbb{K}^d$ is a $d+1$ dimensional vector space, spanned by the coefficients of the Steiner Polynomial.

In an earlier paper that motivated Hadwiger’s work, Blaschke characterized continuous valuations on $\mathbb{K}^d$ that are $SL(d)$ and translation invariant.

Theorem (Blaschke)

Any such valuation is a linear combination of $1$ and $S_0$. (Note that $S_0$ is the volume).

The study of valuations that take values in other semigroups is a beautiful and well developed area.
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The natural question here is: What are the translation invariant and \( SL(d) \) invariant valuations?
Valuations on Lattice Polytopes

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This has a beautiful and clean answer. To do this, let us introduce the Ehrhart polynomial of a lattice polytope. Let $\mathcal{L}_d$ be the set of all lattice polytopes in $\mathbb{Z}^d$. Theorem (Ehrhart's theorem)

Let $P$ be a lattice polytope: Then there is a polynomial $Ehr(P)$ of degree equal to the affine dimension of $P$ such that $Ehr(P)(n) = |nP \cap \mathbb{Z}^d|$, where $|A|$ is the number of lattice points in $A$. This polynomial has rational coefficients.

If we write $Ehr(P)(x) = \sum_{k=0}^{d} E_k(P) x^k$, we see that $E_d$ is the volume and $E_0 = 1$. 

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$$R_t := \text{conv}\{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, r)\}.$$
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This was introduced by John Reeve in 1902 to show that there is no analogue of Pick’s theorem in higher dimensions. There are no lattice points in the Reeve tetrahedron apart from the vertices but the volume is $r/6$ which can be arbitrarily large.

$$Ehr_{R_t} = \frac{rt^3}{6} + t^2 + \left(2 - \frac{r}{6}\right)t + 1.$$
Betke and Kneser proved the following beautiful analogue of Hadwiger’s theorem.

**Theorem (Betke-Kneser)**

Any \( SL(d) \) and translation invariant valuation on \( L_d \) is a linear combination of the coefficients of the Ehrhart polynomial.
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Betke and Kneser proved the following beautiful analogue of Hadwiger’s theorem.

**Theorem (Betke-Kneser)**

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Stanley in 1974 showed that the usual monomial basis is **not the best basis** to write the Ehrhart polynomial in.

**Theorem (Stanley)**

*Let $P$ be a lattice polynomial of dimension $d$. Then, there are natural numbers $h_0^*, \ldots, h_d^*$ such that*

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\text{Ehr}(n) = h_0^* \binom{n + d}{d} + h_1^* \binom{n + d - 1}{d} + \ldots + h_d^* \binom{n}{d}.
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$$Ehr(n) = h_0^* \binom{n + d}{d} + h_1^* \binom{n + d - 1}{d} + \ldots + h_d^* \binom{n}{d}.$$ 

Note that the elements of the $h^*$ vector are not valuations: The $h^*$ vector depends upon the dimension of the polytope.
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**Theorem (McMullen)**

Let \( \phi : \mathcal{L}_d \to \mathbb{R} \) be a translation invariant valuation. Then there is a polynomial \( \varphi_P \) of degree at most the affine dimension of \( P \) such that

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\varphi_P(n) = \phi(nP), \quad \forall n \in \mathbb{N} \cup \{0\}.
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The Euler Characteristic

Instead of considering valuations on convex sets, may we consider valuations on the larger class of \textbf{PolyConvex} sets.

\textbf{Definition}

A relatively open polyhedron is a polyhedral convex set that is open in the affine space that it lies in. A polyconvex set is a finite union of relatively open polyhedra. In practical terms, a polyconvex set is a disjoint union of finitely many sets that are polyhedra with the interiors of certain facets removed.

The most important valuation by far is the following.

\textbf{Theorem}

There is a unique integer valued valuation on polyconvex sets in \( \mathbb{R}^d \) which assigns the value 1 to every (closed) polytope. This valuation is called the Euler Characteristic.
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Let $P \in \mathcal{L}_d$. Then

$$\text{Ehr}_P(-n) = (-1)^{\dim(P)} \text{Ehr}_{P^\circ}(n),$$

where $P^\circ$ is the relative interior of $P$.

This was conjectured by Ehrhart and proved by him in several cases. The full proof was given by Ian Macdonald in 1971.
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This theorem was generalized by McMullen to all translation invariant valuations.

**Theorem (McMullen)**

Let $\phi$ be a valuation on $\mathcal{L}_d$. Then

$$\varphi_P(-n) = \sum_{F \subseteq P} (-1)^{\dim(F)} \varphi_F(n).$$
Recall the definition of the Ehrhart polynomial,

\[ E_{nP}(n) = |nP \cap \mathbb{Z}^d|, \]

**Question**

*Which lattice polytopes have Ehrhart polynomials with non-negative coefficients?*
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**Theorem (Liu, 2009)**

*Any lattice polytope is combinatorially equivalent to a lattice polytope that is Ehrhart positive.*
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This shows that Ehrhart Positivity is not a combinatorial property, but a geometric one.
Given a vector $a = (a_1, \ldots, a_d) \in \mathbb{N}^d$, the associated Stanley-Pitman polytope is

$$PS_d(a) = \{x \in \mathbb{R}_{\geq 0}^d \mid \sum_{j=1}^i x_i \leq \sum_{j=1}^i a_i, \ i \in [d]\}.$$ 

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Crosspolytopes and several derived polytopes are Ehrhart positive: This uses an interesting fact, namely that their Ehrhart polynomials have roots on $S^1$. That this implies positivity is easy to see.
Zonotopes are Minkowski sums of line segments,

\[ Z = \sum_{i=1}^{m} [0, v_i], \]

where the \( v_i \in \mathbb{Z}^d \). In this case, the coefficients of the Ehrhart polynomial have meaning.
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**Theorem (Stanley)**

The coefficient of $t^k$ in $\text{Ehr}_Z$ is

$$\sum_{X} h(X),$$

where $X$ is the collection of all linearly independent size $k$ subsets and $h(X)$ is the gcd of all $k \times k$ minors of the matrix whose column vectors are the elements in $X$. 

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Generalized Permutohedra: Ehrhart Positivity and Minkowski Linear Functionals
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\[ \Pi_d = \text{conv}\{(\sigma(1), \ldots, \sigma(d + 1)) : \sigma \in S_{d+1}\} = \sum_{i<j} [0, e_j - e_i]. \]
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Stanley’s theorem specializes to: \( E_Z(k) \) is the \# forests on \([d + 1]\) with exactly \( d + 1 - k \) trees.
A matroid $M$ is a finite set $X$ and a collection of subsets $T$ (called independent sets) which are
1. Downward closed.
2. For every $e \in T$ and $i \in X \setminus e$, there is a $j \in e$ such that $e \cup i \setminus \{j\} \in T$. 

The matroid (base) polytope is $P_M = \text{conv}\{e \in B\}$. 

For instance, the following are Linear Programs over Matroid Polytopes.
1. The Travelling Salesman problem,
2. Finding maximum weight matchings in graphs,
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Optimization problems where the associated matroid polytopes have compact facet description are tractable. If a problem is intractable, then the corresponding matroid polytope has complex facet structure.
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Yet another definition is the following: A polytope $P$ is a generalized permutohedron if there is another polytopes $Q$ such that $P + Q = \lambda \Pi_d$. In other words, $P$ is a weak Minkowski summand of $\Pi_d$. 
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**Definition**

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**Theorem (Ardila, Benedetti, Doker)**

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be the **standard simplices** where $e_1, \ldots, e_d$ are the standard basis vectors in $\mathbb{R}^d$. 

Theorem (Jochemko, Ravichandran, 2019) Let $\{y_I\}_{I \in [d]}$ be a vector of real numbers. Then the following are equivalent.

(i) The signed Minkowski sum $\sum_{I \subseteq [d]} y_I \Delta_I$ defines a generalized permutahedron.

(ii) For all $2$-element subsets $E \in \binom{[d]}{2}$ and all $T \subseteq [d]$ such that $E \subseteq T$

$$\sum_{E \subseteq I \subseteq T} y_I \geq 0.$$ 

Further, every generalized permutohedron is of the above form. In particular, the collection of all coefficients $\{y_I\}_{I \in [d]}$ such that $\sum_{I \subseteq [d]} y_I \Delta_I$ defines a generalized permutahedron is a polyhedral cone. The inequalities are facet-defining.
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Characterization of Generalized Permutohedra via signed Minkowski sums

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This theorem uses the supermodular characterization of Permutohedra due to Postnikov together with a theorem of Schneider on facets of Minkowski sums of polytopes.
Our second contribution exploits a fundamental property of the linear term

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The linear term of the Ehrhart polynomial is Minkowski linear.
The proof of this uses a deep theorem of McMullen that is a generalization of Ehrhart’s theorem

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Given lattice polytopes $P_1, \ldots, P_m \in \mathbb{Z}^d$, the function $(\mathbb{Z}^d \geq 0)^m \ni (k_1, \ldots, k_m) \rightarrow |k_1P_1 + \cdots + k_mP_m|$ agrees with a polynomial.

With this in hand, we can calculate the linear term of the Ehrhart polynomial of any generalized Permutohedron.

The linear term of the $n$-simplex is easily calculated to be $E(\Delta_{i+1}) = 1 + \frac{1}{2} + \cdots + \frac{1}{i} =: h_i$. 

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Given lattice polytopes $P_1, \ldots, P_m \in \mathbb{Z}^d$, the function

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With this in hand, we can calculate the linear term of the Ehrhart polynomial of any generalized Permutohedron.

The linear term of the $n$ simplex is easily calculated to be

\[
\mathcal{E}(\Delta_{i+1}) = 1 + \frac{1}{2} + \cdots + \frac{1}{i} =: h_i
\]
For any 2-element subset $E \in \binom{[d]}{2}$ and any $T \subseteq [d]$ such that $E \subseteq T$ let $v^T_E$ be the Minkowski linear functional defined by

$$v^T_E(\Delta_I) = \begin{cases} 1 & \text{if } E \subseteq I \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

We characterize all positive, translation-invariant Minkowski linear functionals on $\mathcal{P}_d$.

**Proposition**

Let $\varphi: \mathcal{P}_d \to \mathbb{R}$ be a Minkowski linear functional. Then $\varphi$ is positive and translation-invariant if and only if there are nonnegative real numbers $c^T_E$ such that

$$\varphi = \sum_{E \in \binom{[d]}{2}} \sum_{T \supseteq E} c^T_E v^T_E.$$ 

In particular, the family of positive, translation-invariant Minkowski linear functionals is a polyhedral cone with rays $v^T_E$. 
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The proof is not difficult: It essentially uses Conic Duality.
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For all $1 \leq k \leq d - 1$ let $f_k : \mathcal{P}_d \to \mathbb{R}$ be the symmetric, translation-invariant Minkowski linear functional defined by

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(f_k)(\Delta_{i+1}) = \binom{i+1}{2} \binom{d-i-1}{k-i} \tag{3.2}
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**Theorem**

Let \(\varphi : \mathcal{P}_d \rightarrow \mathbb{R}\) be a Minkowski linear functional. Then \(\varphi\) is positive, translation-invariant and symmetric if and only if there are real numbers \(c_1, \ldots, c_{d-1} \geq 0\) such that

\[
\varphi = \sum_{k=1}^{d-1} c_k f_k.
\]

In particular, the family of all positive, Minkowski linear, translation- and symmetric functionals form a simplicial cone of dimension \(d - 1\).
The final step is showing that the functional

$$\sum_{I \subset [d]} y_I \Delta_I \rightarrow \sum_{I \subset [d]} y_I h_{|I|},$$

satisfies the above condition. This uses some basic combinatorics with univariate polynomials.
Let $q \in \mathbb{R}^d$ be a point, $P \subseteq \mathbb{R}^d$ be a polytope and let $B_\epsilon(q)$ denote the ball with radius $\epsilon$ centered at $q$. The solid angle of $q$ with respect to $P$ is defined by

$$\omega_q(P) = \lim_{\epsilon \to 0} \frac{(P \cap B_\epsilon(q))}{B_\epsilon}.$$ 

We note that the function $q \mapsto \omega_q(P)$ is constant on relative interiors of the faces of $P$. In particular, if $q \not\in P$ then $\omega_q(P) = 0$, if $q$ is in the interior of $P$ then $\omega_q(P) = 1$ and if $q$ lies inside the relative interior of a facet then $\omega_q(P) = \frac{1}{2}$. The solid angle sum of $P$ is defined by

$$A(P) = \sum_{q \in \mathbb{Z}^d} \omega_q(P) \quad \text{Fact: McMullen’s theorem shows this is a polynomial}$$

**Proposition**

There is a 3-dimensional generalized permutahedron in $\mathbb{R}^4$ such that the linear term of its solid angle polynomial is negative.
Definition (Birkhoff Polytope)

The Birkhoff polytope is the set of all $n \times n$ matrices with non-negative entries so that each row and column sum is 1. This is a polytope of dimension $(n - 1)^2$. 

Perhaps the most basic problem in the theory of Ehrhart positivity is

Conjecture (Folklore)

The Birkhoff Polytope is Ehrhart positive.

This has been experimentally verified for $n \leq 11$.

The Birkhoff polytope is the Bipartite Matching polytope of the complete graph $K_n$, $n$. With very little evidence, we rashly conjecture

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- **Deletion**: Given $e \in E$, the matroid $M \setminus e$ is the matroid on the ground set $E \setminus e$ with independent sets being independent sets in $M$ not containing $e$.

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Thanks for Listening!