Point configurations in random fractals

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Motivation

General question: What conditions (measure, dimension, cardinality, structure, etc.) on $A \subset \mathbb{N}$ (or $A \subset [N] := \{1, \ldots, N\}$ or $A \subset \mathbb{R}^d$) guarantee that A contains certain geometric configurations?

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General question: What conditions (measure, dimension, cardinality, structure, etc.) on $A \subset \mathbb{N}$ (or $A \subset [N] := \{1, \ldots, N\}$ or $A \subset \mathbb{R}^d$) guarantee that A contains certain geometric configurations?

- When does $A \subset \mathbb{N}$ (or $A \subset [N]$ or $A \subset P$) contain arithmetic progressions? (Roth/Szemérdi, Green & Tao, etc.)
- When is the distance set $D(A) := \{|x - y| : x, y \in A\}$ of positive measure/non-empty interior for $A \subset \mathbb{R}^d$ (Falconer's distance set conjecture)?
- Erdős similarity problem: Recall that if $S \subset \mathbb{R}$ is finite, then each $A \subset \mathbb{R}$ with positive Lebesgue measure contains a homothetic copy of S. Does there exists an infinite set S with this property? (**Conjecture: NO**)

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- There are sets of full Hausdorff dimension which do not contain any arithmetic progressions (Keleti 1998).
- There are (even compact) sets of zero Hausdorff dimension containing homothetic copies of all finite subsets of $\mathbb{R}$ (Davies, Marstrand, Taylor, 1959). The same is true for polynomial patterns (Molter & Yavicoli 2016).
- For other patterns (e.g angles, distances), there are known dimension lower bounds implying the existence of these patterns. Even then, it is hopeless to characterize the existence of these patterns solely based on (Hausdorff or any other) dimension.

Random sets

Given $0 < p < 1$ and $N \in \mathbb{N}$, let $[N]_p$ be the 'canonical random set' $A \subset [N]$: Each $x \in [N]$ is selected to A with probability p with all choices independent of each other.

- The critical probability for the existence of k-arithmetic progression in $[N]_p$ is $p \approx N^{-2/k}$. Note that the expected number of k-AP's in $[N]_p$ is $\approx N^2 p^k$.

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- Conlon & Gowers (2016), Schacht (2016): For each $\delta > 0$, there is $C = C(\delta, k) > 0$ such that if $p \geq C(\delta, k) N^{-1/(k-1)}$, then the probability that each $A' \subset A$ with $|A'| \geq \delta |A|$ contains k-AP's tends to 1 as $N \to \infty$.

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- **Our target:** Analogues for random fractal sets $A \subset \mathbb{R}^d$, in terms of the (Hausdorff) dimension of A.

Fractal percolation

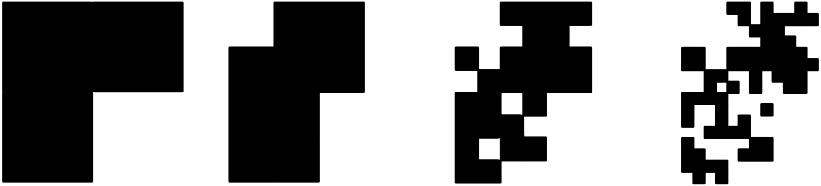

Figure: The first 4 steps in a fractal percolation process with $p = 0.7$.

- Fix $0 < p < 1$.
- Consider the dyadic sub-cubes of $[0,1]^d$ of side-length $\frac{1}{2}$ (there are 2^d of them).
- Retain each of them with probability p and discard otherwise, with all the choices independent.
- Continue inductively in the same fashion in the retained cubes.

- Denote by A_n the union of the retained sub-cubes of side-length 2^{-n} and define the fractal percolation (FP) limit set as

$$A = A^{\mathrm{perc(d,p)}} = \bigcap_{n\in\mathbb{N}} A_n \,.$$

- It is well known that $A \neq \varnothing$ is an event of positive probability if and only if $p > 2^{-d}$ and in this case

$$\dim_H A = d + \frac{\log p}{\log 2} =: s(d,p)$$

  almost surely on non-extinction.

- The process of defining A is *stochastically self-similar:* If Q is a dyadic cube of side-length 2^{-n}, then conditional on $Q \subset A_n$, the set $Q \cap A$ has the same law (up to scaling) as A.

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1. *If $m \geq 2$ and $s > d - (d+1)/m$, then a.s. A contains a homothetic copy of **all** m-point sets.*
2. *If $m \geq 2$, $s > d - d/m$ and $K \subset]0,1[^d$ is compact, then with positive probability, A contains a translation of $\{x_1, \ldots, x_m\}$ for each $(x_1, \ldots, x_m) \in K$.*
3. *If $s > 1/2$, then a.s. there is $\varepsilon > 0$ such that $(0, \varepsilon) \subset D(A)$. (**Rams and Simon, 2014**)*
4. *If $s > 1/(d+1)$, then a.s. there is $\varepsilon > 0$ such that A contains the vertices of a simplex of all volumes in $(0, \varepsilon)$.*

Theorem (continued)

5. *If* $d = 2$*,* $s > 1$ *and* $K \subset]0,1[^d$ *is compact, then with positive probability,* A *contains an isometric copy of* $\{x_1, x_2, x_3\}$ *for all* $(x_1, x_2, x_3) \in K$.

6. *If* $d \geq 2$ *and* $s > 1/3$*, then a.s. triples of points in* A *determine all angles in* $]0, \pi[$.

7. *If* $d \geq 2$ *and* $s > 2/3$*, then a.s.* A *contains the vertices of all non-degenerate triangles, up to similarities.*

8. *If* $m \geq 3$*,* $d = 2$ *and* $s > 2 - 4/m$*, then a.s. up to similarities* A *contains the vertices of all non-degenerate* m*-gons.*

Remarks

- Each range for the dimension $s = s(d,p)$ given by the above theorem is sharp: If s is smaller or equal to the given threshold, then a.s. A does not contain a given configuration in the corresponding class. For instance, if $s \leq \frac{1}{3}$, then a.s. $A \subset \mathbb{R}$ does not contain a 3-AP.

- For *packing dimension*, the range is sharp also for deterministic sets. E.g, if $A \subset \mathbb{R}^d$ contains homothetic copies of all triples, then $\dim_p A \geq \frac{2d}{3} - \frac{1}{3}$.

- Note that our theorem guarantees the existence of **ALL** patterns in the given configuration class simultaneously. This issue does not arise in the (random) discrete setting.

- Our results are analogous to the random discrete setting, but we use different methods.

Remarks

- An alternative definition of FP: Let (U_Q) be a sequence of independent random variables, uniformly distributed in $[0,1]$, where Q ranges over all dyadic cubes of all levels. Given $0 < p < 1$, we construct a set A_p by retaining cubes Q for which $U_Q \leq p$, and discarding those with $U_Q > p$. This leads to an increasing ensemble $(A_p)_{p\in[0,1]}$, where each A_p has the distribution of $A^{\mathrm{perc}(d,p)}$. Our theorem then shows that almost surely the sets A_p undergo a phase transition for the presence of geometric configurations at the corresponding critical value of p (or $s = s(d,p)$).

- In many cases, we can also determine (in terms of the dimension) how often each configuration arises in A. For instance, if $s > \frac{1}{3}$, then almost surely on $A \neq \varnothing$, for each triple $S = \{s_1, s_2, s_3\} \subset \mathbb{R}$,

$$\dim_H(\{(a,b) \in (0,\infty) \times \mathbb{R} : aS + b \subset A\}) = 3s - 1\,.$$

Rough idea of the proofs

- All the configurations arising in the Theorem can be realized as the zero set of a suitable polynomial, and the dimension thresholds are derived from a general statement about intersections (of the Cartesian powers of A) with algebraic varieties.

- Let us look at the existence of homothetic copies of $\{a_1, a_2, a_3\}$ in $A \subset [0,1]$ for $\mathbf{a} := (a_1, a_2, a_3) \in \mathbb{R}^3$. We observe that A contains a homothetic copy of $\{a_1, a_2, a_3\}$ if and only if $A \times A \times A \cap V_{\mathbf{a}} \setminus \Delta \neq \varnothing$ for the plane

$$V_{\mathbf{a}} = \{(x, x, x) + \lambda \mathbf{a} \,:\, x \in \mathbb{R}, \lambda \in \mathbb{R}\},$$

  where $\Delta = \{(x, x, x) \in \mathbb{R}^3 \,:\, x \in \mathbb{R}\}$.

- Instead of A^3, we reduce the problem to the study of the product $\widetilde{A} = A^1 \times A^2 \times A^3$ of three independent realizations of FP.

- Let us define $Y_n^{\mathbf{a}} = p^{-n} \int_{V_{\mathbf{a}}} \mathbf{1}[\widetilde{A}_n] \, d\mathcal{H}^2$. This s a martingale and a calculation shows it is L^2 bounded.
- Thus, $\lim Y_n^{\mathbf{a}} > 0$ and whence $\widetilde{A} \cap V_{\mathbf{a}} \neq \varnothing$ with positive probability.
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- …and verifies the claim *for a fixed pattern* $\mathbf{a} = (a_1, a_2, a_3)$.
- However, we need this *for all* patterns $\mathbf{a}$ similtaneously.
- A joint probabilistic induction (in n and the order of the cartesian power) is needed to bound these dependencies.
- Note that there are lots of dependencies due to the product structure (This is the main difference compared to FP in $\mathbb{R}^3$).

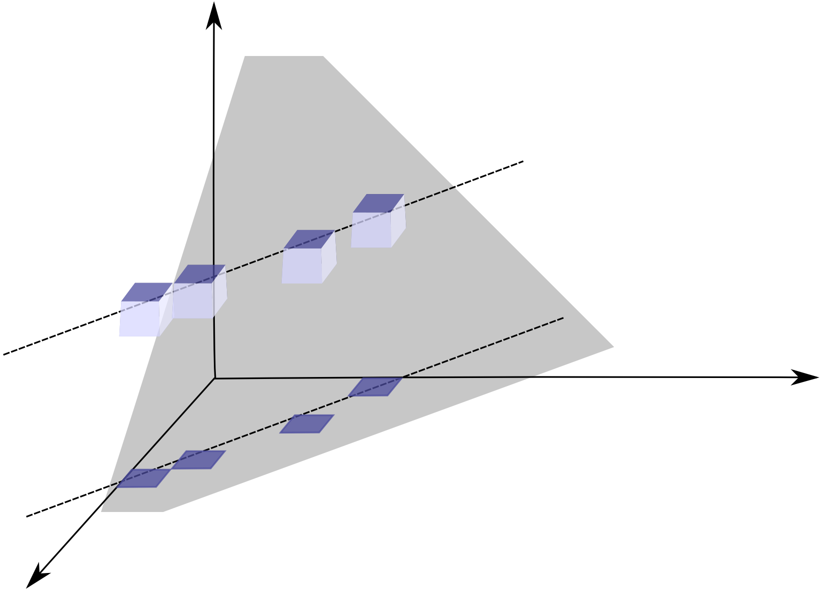

Figure: A plane $V_{\mathbf{a}}$ and 'dependent' cubes

Patterns in sets of positive measure

- Denote $\mu_n = p^{-n}\mathcal{L}|_{A_n}$.
- The random weak limit

$$\mu = \mu^{\mathrm{perc}(d,p)} = \lim_n \mu_n$$

exists almost surely and is called the *natural measure* (or the *branching measure* when it is attached to the underlying Galton-Watson process).

- $\mu^{\mathrm{perc}(d,p)}$ has dimension $s = s(d,p)$ almost surely on non-extinction:

$$\dim(\mu, x) = \lim_{r \downarrow 0} \frac{\log \mu(B(x,r))}{\log r} = s\,,$$

for μ-almost all x.

Theorem (Shmerkin and S. 2017)

Almost surely, the following holds for each Borel set A' *such that* $\mu(A') > 0$ *under the given conditions on* d *and* $s = s(d,p)$*:*

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Almost surely, the following holds for each Borel set A' such that $\mu(A') > 0$ under the given conditions on d and $s = s(d,p)$:

1. *If $m \geq 2$ and $s > d - \frac{1}{m-1}$, then A' contains a homothetic copy of all m point sets.*
2. *If $s > 1$, then the distance set of A' has non-empty interior.*
3. *If $s > \frac{1}{d}$, then the set of volumes of simplices with vertices in A' has non-empty interior.*
4. *If $d = 2$ and $s > \frac{3}{2}$, then there is an open set of triples $\{a_1, a_2, a_3\}$ such that A' contains an isometric image of $\{a_1, a_2, a_3\}$.*
5. *If $d \geq 2$ and $s > \frac{1}{2}$, then A' contains all angles in $]0, \pi[$.*
6. *If $d \geq 2$ and $s > 1$, then A' contains a similar copy of all non-degenerate triangles.*
7. *If $m \geq 3$, $d = 2$ and $s > 2 - \frac{2}{m-1}$, then A' contains a similar copy of all non-degenerate m-gons.*

Remark

Again, the range of s is sharp in each of the statements up to the endpoint. For example, if $s \leq 1$, then there is a full μ-measure set A' which does not contain any rational distances.

Thank you!