Measures of maximal dimension on (uniformly) hyperbolic sets

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Repellers for Expanding Maps

M smooth Riemannian manifold;
$U \subset M$ an open set;
$f : U \to M$ a smooth map;
$\Lambda \subset U$ is a compact f-invariant set.
Λ is a repeller for f and f is expanding on Λ if:

1. $\Lambda = \{x \in U : f^n(x) \in U, \text{ for all } n \geq 0\}$;
2. there is $\lambda > 1$ such that for all $x \in \Lambda$ and $v \in T_xM$,

$$\|d_x f v\| \geq \lambda \|v\|.$$

Λ is conformal if $d_x f = a(x)\text{Isom}_x$ for all $x \in \Lambda$, where $a(x)$ is a Hölder continuous function, $|a(x)| > 1$, and Isom_x is an isometry on T_xM.

Hyperbolic sets

Λ is a hyperbolic set for f and f is hyperbolic on Λ if there is $\lambda > 1$ and for every $x \in \Lambda$ a splitting $T_xM = E^s(x) \oplus E^u(x)$ such that

- $dfE^s(x) = E^s(f(x))$ and $dfE^u(x) = E^u(f(x))$.
- For every $x \in \Lambda$ and $n \geq 0$,

$$\|d_x f v\| \leq \lambda^{-1}\|v\|, \quad v \in E^s(x);$$
$$\|d_x f^{-1} v\| \leq \lambda^{-1}\|v\|, \quad v \in E^u(x).$$

Note that $E^{s,u}(x)$ depend Hölder continuously on $x \in \Lambda$.

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Λ is locally maximal if $\Lambda = \{x \in U : f^n(x) \in U, \text{ for all } n \in \mathbb{Z}\}$, iff for every two points $x, y \in \Lambda$ that are sufficiently close, the intersection $[x, y] = V^s(x) \cap V^u(x)$ is a single point in Λ.

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Λ is u-conformal if $d_x f|E^u(x) = a^u(x)\text{Isom}_x$ for all $x \in \Lambda$, where $a^u(x)$ is a Hölder continuous function, $|a^u(x)| > 1$, and Isom_x is an isometry on T_xM. Similarly, one defines the notion of s-conformal.

Classical Thermodynamic Formalism (Sinai, Ruelle, Bowen)

X is a compact metric space.
$f : X \to X$ a continuous map of finite topological entropy.
φ a continuous function (potential) on X.
$\mathcal{M}(f, X)$ the space of all f-invariant Borel probability ergodic measures on X.
$\mu_\varphi \in \mathcal{M}(f, X)$ is an equilibrium measure if

$$P(\varphi) := \sup_{\mu \in \mathcal{M}(f,X)} \{h_\mu(f) + \int_X \varphi \, d\mu\} = h_{\mu_\varphi}(f) + \int_X \varphi \, d\mu_\varphi.$$

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Theorem

Let f be a diffeomorphism with an invariant set Λ that is either a repeller or a locally maximal hyperbolic set. If $f|\Lambda$ is topologically transitive, then for every Hölder continuous potential φ there is a unique equilibrium measure μ_φ for φ.

Hausdorff Dimension of Conformal Repellers

Λ is a conformal repeller for a $C^{1+\alpha}$ map.
$\varphi_t(x) := -t \log a(x)$ – the geometric t-potential – Hölder continuous for each t; in particular, it admits a unique equilibrium measure μ_t.

Theorem (Bowen-Ruelle)

1. *The pressure function* $P(t) := P(\varphi_t)$ *is convex, decreasing, and real analytic in* $t \in \mathbb{R}$ *and there is a (unique) number* $0 < \tau < \dim M$ *for which* $P(\tau) = 0$ *(Bowen's equation).*
2. $\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = \tau$.
3. $\dim_H \mu_\tau = \dim_H \Lambda$; *that is* μ_τ *is the unique invariant measure of maximal Hausdorff dimension; moreover,* $\mu_\tau(\cdot) \sim m_H(\cdot, \tau)$, *the Hausdorff measure at dimension* τ.

Hausdorff Dimension of Non-Conformal Repellers

Λ is a (not necessarily conformal) repeller for a $C^{1+\alpha}$ map.

1. $\dim_H \Lambda \le \underline{\dim}_B \Lambda \le \overline{\dim}_B \Lambda$ and the inequality may be strict.
2. There may not exist any invariant measure of maximal Hausdorff dimension.

Conjecture (Y. Peres) There exists a (not necessarily invariant) measure of maximal Hausdorff dimension.

Λ is a u-conformal locally maximal hyperbolic set for a $C^{1+\alpha}$ map and $f|\Lambda$ is topologically transitive.
$\varphi_t^u(x) := -t\log a^u(x)$ is a u-geometric t-potential and is Hölder continuous for each t; in particular, it admits a unique equilibrium measure μ_t^u.
Fix $x \in \Lambda$, a small $r > 0$, and consider the partition ξ of the set $X = B(x,r) \cap \Lambda$ by disjoint sets $C(y) := V^u(y) \cap X$ with $y \in X$.
This partition is measurable and the measure μ_t^u generates the system of conditional measures $\nu_t^u(y)$, $y \in X$ on elements $C(y)$ of the partition ξ such that for every measurable set $E \subset X$,

$$\mu_t^u(E) = \int_{X/\xi}\int_{C(y)} \chi_E(y,z)\, d\nu_t^u(y)(z)\, d\tilde{\nu}_t^u(y),$$

where $\tilde{\nu}_t^u$ is the factor-measure on the factor-space X/ξ.

Theorem (Bowen-Ruelle)

1. *The u-pressure function* $P^u(t) := P(\varphi_t^u)$ *is convex, decreasing, and real analytic in* $t \in \mathbb{R}$ *and there is a (unique) number* $0 < \tau^u < \dim V^u(x)$ *for which* $P^u(\tau^u) = 0$.
2. $\dim_H V^u(x) \cap \Lambda = \underline{\dim}_B V^u(x) \cap \Lambda = \overline{\dim}_B V^u(x) \cap \Lambda = \tau^u$ *independent of* $x \in \Lambda$.
3. $\dim_H \nu_{\tau^u}^u(x) = \dim_H V^u(x) \cap \Lambda$ *(*$\nu_{\tau^u}^u(x)$ *is the conditional measure generated by* $\mu_{\tau^u}^u$ *on* $V^u(x)$*); moreover,* $\nu_{\tau^u}^u(x)(\cdot) \sim m_H(\cdot, \tau^u)$*, the Hausdorff measure in* $V^u(x) \cap \Lambda$ *at dimension* τ^u.

Similar results hold for $\varphi_t^s(x) := -t \log a^s(x)$ and the equilibrium measure $\mu^s_{\tau^s}$. We have that

1. $\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda = \tau^u + \tau^s := \tau$.
2. The measure $\nu^u_{\tau^u}(x) \times \nu^s_{\tau^s}(x)$ is the measure of maximal Hausdorff dimension of $B(x, r) \cap \Lambda$ independent of x.
3. $m_H(E, \tau) = \nu^u_{\tau^u}(x) \times \nu^s_{\tau^s}(x)(E)$ for any measurable set $E \subset B(x, r) \cap \Lambda$.
4. If the potential $\varphi_t^u(x)$ is cohomologous to $\varphi_t^s(x)$, then $\mu := \mu^u_{\tau^u} = \mu^s_{\tau^s}$ is the unique invariant measure of maximal Hausdorff dimension of Λ and $\mu(\cdot) \sim m_H(\cdot, \tau)$, the Hausdorff measure of Λ at dimension τ.

Hausdorff Dimension of Non-Conformal Hyperbolic Sets

Λ is a (not necessarily conformal) locally maximal hyperbolic set for a $C^{1+\alpha}$ diffeomorphism.

1. $\dim_H \Lambda \le \underline{\dim}_B \Lambda \le \overline{\dim}_B \Lambda$ and the inequality may be strict.
2. There may be no invariant measure for which the conditional measures it generates on local unstable (respectively, stable) leaves $V^u(x)$ (respectively, $V^s(x)$) are measures of maximal Hausdorff dimension on $V^u(x) \cap \Lambda$ (respectively, of $V^s(x) \cap \Lambda$, Das and Simmons).

Conjecture (Y. Peres) For every $x \in \Lambda$ there exists a measure of maximal Hausdorff dimension in $V^u(x) \cap \Lambda$.

Lower and Upper Bounds on Dimension of Repellers

To obtain lower and upper bounds on the dimension, one can consider other types of geometric t-potentials, e.g., $\psi_s(x) = -s \log \|(d_x f)^{-1}\|^{-1}$ and $\phi_t(x) = -t \log \|d_x f\|$. If s^* and t^* are the unique roots of Bowen's equations

$$P(\psi_s) = 0 \text{ and } P(\phi_t) = 0,$$

then

$$s^* \le \dim_H \Lambda \le \underline{\dim}_B \Lambda \le \overline{\dim}_B \Lambda \le t^*.$$

Singular Valued Potentials.

For $x \in \Lambda$ and $n \geq 1$, consider the singular values of the differentiable operator $D_x f^n : T_x M \to T_{f^n(x)} M$

$$\alpha_1(x, f^n) \geq \alpha_2(x, f^n) \geq \cdots \geq \alpha_{m_0}(x, f^n).$$

For $s \in [0, m_0]$, set

$$\psi^s(x, f^n) := \sum_{i=1}^{[s]} \log \alpha_i(x, f^n) + (s - [s]) \log \alpha_{[s]+1}(x, f^n)$$

and

$$\varphi^t(x, f^n) := \sum_{i=m_0-[t]+1}^{m_0} \log \alpha_i(x, f^n) + (t - [t]) \log \alpha_{m_0-[t]}(x, f^n)$$

The sequences of functions

$$\Psi(s) := \{-\psi^s(\cdot, f^n)\}_{n\geq 1} \text{ and } \Phi(t) := \{-\varphi^t(\cdot, f^n)\}_{n\geq 1}$$

are super- and sub-additive. They are called supper- and sub-additive singular valued potentials.

We consider the super- and sub-additive pressure functions

$$P_{\sup}(s) := P_{\mathrm{var}}(f, \Psi(s)) \text{ and } P_{\mathrm{sub}}(t) := P(f, \Phi(t)),$$

where $P_{\mathrm{var}}(f, \Psi(s))$ is the supper-additive topological pressure and $P(f, \Phi(t))$ is the sub-additive topological pressure. $P_{\sup}(s)$ and $P_{\mathrm{sub}}(t)$ are continuous and strictly decreasing in s, and in t. Let s^* and t^* be the (unique) roots of Bowen's equations $P_{\sup}(s) = 0$ and $P_{\mathrm{sub}}(t) = 0$ respectively.

Theorem (Falconer, Zhang, Ban-Cao-Hu, Cao-P.-Zhao)

- *If f is C^1, then $\dim_H \Lambda \le t^*$.*
- *If f is $C^{1+\gamma}$, then $\dim_H \Lambda \ge s^*$.*
- *If f is $C^{1+\gamma}$, then $\overline{\dim}_B \Lambda \le t^*$, provided $\frac{1}{C} \le \frac{\alpha_i(x, f^n)}{\alpha_i(y, f^n)} \le C$ for some $C > 0$ and every $1 \le i \le m_0$, $n > 0$, and $x, y \in P_{i_0 i_1 \ldots i_{n-1}}$ (a cylinder associated with a Markov partition of Λ).*

The Geometric Approach to Equilibrium Measures

The idea of the geometric approach is to follow the classical Bogolyubov-Krylov procedure for constructing invariant measures by pushing forward a given reference measure. To this end fix $x \in \Lambda$ and let $\kappa^u(x)$ be a Borel probability measure on $V^u(x) \cap \Lambda$. We can extend this measure to a measure on the whole Λ (still denoted by $\kappa^u(x)$). Consider the sequence of probability measures on Λ

$$\kappa_n^u(x) = \frac{1}{n}\sum_{k=0}^{n-1} f_*^k \kappa^u(x). \tag{1}$$

Any weak* limit κ of this sequence is an invariant measure on Λ. The sequence of measures $\kappa_n^u(x)$ describe the evolution of the reference measure $\kappa^u(x)$ under the dynamics.

To see that this procedure can be used to obtain equilibrium measures assume that μ_φ is an equilibrium measure for a potential φ and consider the system of conditional measures $\nu^u(x)$ on $V^u(x) \cap \Lambda$ that is generated by μ_φ.

Theorem

For almost every $x \in \Lambda$, setting the reference measure $\kappa^u(x)$ to be the conditional measure $\nu^u(x)$, the sequence of measures $\kappa_n^u(x)$ converges to the equilibrium measure μ_φ.

Therefore, to construct an equilibrium measure using the geometric approach, one needs to "guess" the conditional measure generated by the desired equilibrium measure on unstable leaves, choose it as a reference measure and then apply the above theorem.

In the particular case $\Lambda = M$ (i.e., f is an Anosov diffeomorphism) we can choose the leaf-volume m^u_x on $V^u(x)$ as a reference measure (i.e., $\kappa^u(x) = m^u_x$) and applying the push forward procedure (1), we obtain a limit measure κ for which conditional measures it generates on unstable leaves are equivalent to leaf-volume. Thus κ is the Sinai-Ruelle-Bowen (SRB) measure for f and it is an equilibrium measure for the potential $\varphi_1(x) = -\log|\mathrm{Jac} df|E^u(x)|$, so that (by the entropy formula):

$$h_\kappa(f) + \int_M \varphi_1 d\kappa = \sup_{\mu\in\mathcal{M}(f,M)} \left\{ h_\mu(f) + \int_M \varphi_1 d\mu \right\} = P(\varphi_1) = 0.$$

Carathéodory Dimension Structure (Ya. P.)

X a set.

$\mathcal{F}$ a collection of subsets of X called admissible.

$\eta, \psi : \mathcal{F} \to [0, \infty)$ set functions satisfying

(A1) $\emptyset \in \mathcal{F}$; $\eta(\emptyset) = \psi(\emptyset) = 0$ and $\eta(U), \psi(U) > 0$ for any $U \in \mathcal{F}, U \neq \emptyset$;

(A2) for any $\delta > 0$ one can find $\varepsilon > 0$ such that $\eta(U) \leq \delta$ for any $U \in \mathcal{F}$ with $\psi(U) \leq \varepsilon$;

(A3) there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$, one can find a finite or countable subcollection $\mathcal{G} \subset \mathcal{F}$ covering X such that $\psi(U) \leq \varepsilon$ for any $U \in \mathcal{G}$.

Let $\xi : \mathcal{F} \to [0, \infty)$ be a set function. The collection of subsets $\mathcal{F}$ and the functions ξ, η, ψ, satisfying (A1), (A2), (A3) introduce a Carathéodory dimension structure or C-structure $\tau = (\mathcal{F}, \xi, \eta, \psi)$ on X.

η is a potential set function, ξ measures the weight, and ψ the size of $U \subset \mathcal{F}$.

For any subcollection $\mathcal{G} \subset \mathcal{F}$ let $\psi(\mathcal{G}) := \sup\{\psi(U) : U \in \mathcal{G}\}$. Given $Z \subset X$ and numbers $\alpha \in \mathbb{R}$ and $\varepsilon > 0$, define

$$M_C(Z, \alpha, \varepsilon) := \inf_{\mathcal{G}, \psi(\mathcal{G}) \leq \varepsilon} \left\{ \sum_{U \in \mathcal{G}} \xi(U) \eta(U)^\alpha \right\},$$

where the infimum is taken over all finite or countable subcollections $\mathcal{G} \subset \mathcal{F}$ covering Z. Set

$$m_C(Z, \alpha) := \lim_{\varepsilon \to 0} M_C(Z, \alpha, \varepsilon).$$

If $m_C(\emptyset, \alpha) = 0$, the set function $m_C(\cdot, \alpha)$ becomes an outer measure on X, which induces a measure called the α-Carathéodory measure. In general, this measure may not be σ-finite or it may be a zero measure.

Furthermore, there exists $\alpha_C \in \mathbb{R}$ s.t. $m_C(Z, \alpha) = \infty$ for $\alpha < \alpha_C$ and $m_C(Z, \alpha) = 0$ for $\alpha > \alpha_C$ (while $m_C(Z, \alpha_C)$ may be 0, ∞, or a finite positive number). The quantity $\dim_C Z = \alpha_C$ is the Carathéodory dimension of Z.

Examples of C-structures

C-structures can be generated by other structures on the set X.

1. X is a metric space, the C-structure is given by

$$\mathcal{F} := \{\text{open sets}\}, \quad \xi(U) = 1, \quad \eta(U) = \psi(U) = \text{diam } U.$$

   $\dim_C Z = \dim_H Z$ is the Hausdorff dimension of Z.

2. X is a metric space, $f : X \to X$ continuous, $\varphi : X \to \mathbb{R}$ continuous. Fix $r > 0$ and set

$$B_n(x, r) := \{y \in X : d(f^k(x), f^k(y)) \le r, 0 \le k \le n\}$$
$$\mathcal{F} := \{\emptyset\} \cup \{B_n(x, r) : x \in X, n \ge 0\},$$
$$\xi(B_n(x, r)) := e^{S_n\varphi(x)},$$
$$\eta(B_n(x, r)) := e^n, \ \psi(B_n(x, r)) := \frac{1}{n}.$$

$\dim_C Z = \limsup_{r\to 0} \dim_{C,r} Z = P_Z(\varphi)$, the topological pressure of φ on Z (Pitskel', P.).

Other examples of Carathéodory dimensions are dimension spectra for pointwise dimensions and dimension of Poincaré recurrences.

Carathéodory Structure on Local Unstable Manifolds (Climenhaga, P., Zelerowicz)

Fix $x_0 \in \Lambda$ and set $X := V^u(x_0) \cap \Lambda$. Fix a small number $r > 0$ and define the Bowen's u-ball by

$$B_n^u(x, r) := \{y \in V^u(x) \cap \Lambda : d(f^k(y), f^k(x)) < r \text{ for } k = 0, \dots, n\}.$$

Then set define the collection $\mathcal{F}$ of admissible sets by

$$\mathcal{F} := \{\emptyset\} \cup \{B_n^u(x, r) : x \in V^u(x_0) \cap \Lambda, n \in \mathbb{N}\},$$
$$\xi(B_n^u(x, r)) := \exp(S_n\varphi(x)),$$
$$\eta(B_n^u(x, r)) := e^{-n}, \ \psi(B_n^u(x, r)) := \frac{1}{n}.$$

It is easy to see that the collection of subsets $\mathcal{F}$ and set functions ξ, η, ψ satisfy (A1), (A2), (A3), and hence, introduce a C-structure in X.

Thus we obtain the Carathéodory measure $m^u_{C,x_0}(\cdot) := m_{C,x_0}(\cdot, P)$ on X at the Carathéodory dimension $P = \dim_C X$. For every $Z \subset X$ we have

$$m^u_{C,x_0}(Z) = \lim_{N\to\infty} \inf_{\{B^u_{n_i}(x_i,r)\}} \left\{ \sum_i \exp\left(-Pn_i + \sum_{k=0}^{n_i-1} \varphi(f^k(x_i)) \right) \right\},$$

where the infimum is taken over all collections $\{B^u_{n_i}(x_i, r)\}$ of Bowen's u-balls with $x_i \in X$, $n_i \geq N$, which cover Z that is $Z \subset \bigcup_i B^u_{n_i}(x_i, r)$.
The Carathéodory structure and hence, the Carathéodory measure depend on the potential φ but do not depend on r due to expansivity of the map $f|\Lambda$: there is $\delta > 0$ such that no two trajectories can stay within the distance δ from each other.

Theorem (Climenhaga, P., Zelerowicz))

The P-Carathéodory measure m^u_{C,x_0} on $X = V^u(x_0) \cap \Lambda$ is finite and positive independently of the choice of the point x_0 and the number r provided it is sufficiently small. Moreover, $P = P(\varphi)$.

As an immediate corollary we obtain that for any set $Z \subset X$ of positive m^u_{C,x_0}-measure we have that $\dim_C Z = P$.

Obtaining Equilibrium Measures by Pushing Forward Carathéodory Measures on Unstable Leaves

Theorem (Climenhaga, P., Zelerowicz))

Let Λ be a topologically transitive locally maximal hyperbolic set and φ a Hölder continuous potential function. Then for any $x \in \Lambda$:

1. *The sequence of measures (1) with the reference measure $\kappa^u(x) = m^u_{C,x}$ converges as $n \to \infty$ to a limiting measure μ_φ, which is the unique equilibrium measure for φ.*
2. *The measure μ_φ has the Gibbs property: there is $Q > 0$ such that for all $x' \in \Lambda$,*

$$Q^{-1} \le \frac{\mu_\varphi(B_n(x', \varepsilon))}{e^{-nP(\varphi)+S_n\varphi(x)}} \le Q. \qquad (2)$$

3. *The conditional measure ν^u generated by μ_φ on $V^u(x')$ and the measure $m^u_{C,x}$ are equivalent for μ_φ-almost every $x' \in \Lambda$.*

Measures of Maximal Carathéodory Dimension

Let Λ be a topologically transitive locally maximal hyperbolic set. Consider the u-geometric t-potential $\varphi_t^u = -\log \mathrm{Jac} df|E^u(x)$. The pressure function $P^u(t) = P(\varphi_t^u)$ is monotonically decreasing, convex and real analytic in t. Moreover, $P^u(t) \to +\infty$ as $t \to -\infty$ and $P^u(t) \to -\infty$ as $t \to +\infty$ with $P^u(1) \leq 0$. Therefore, there is a number $0 < t^u \leq 1$ which is the unique solution of *Bowen's equation* $P^u(t^u) = 0$. This implies that the Carathéodory measure $m_{C,x}^u$ becomes

$$m_{C,x}^u(Z) = \lim_{N\to\infty} \inf \left\{ \sum_{\{B_{n_i}^u(x_i,r)\}} \left(\prod_{k=0}^{n_i-1} \mathrm{Jac}\left(df|E^u(f^k(x_i)\right)\right)^{-t^u} \right\}. \tag{3}$$

where the infimum is taken over all collections $\{B_{n_i}^u(x_i, r)\}$ of Bowen's u-balls with $x_i \in X$, $n_i \geq N$, which cover Z.

Relation (3) shows that the measure $m^u_{C,x}$ can be viewed as the Carathéodory measure generated by yet another C-structure: $\tau' = (\mathcal{F}, \xi', \eta', \psi)$, where $\mathcal{F}$ is the collection of Bowen's u-balls, $\xi'(B^u_n(x,r)) := 1$, and

$$\eta'(B^u_n(x,r)) := \prod_{k=0}^{n_i-1} \operatorname{Jac}\left(df|E^u(f^k(x_i)\right)^{-1}.$$

It is easy to see that with respect to the $C-$structure τ' we have that $\dim_{C,\tau'} X = t^u$ and the measure $m^u_{C,x} = m_{C,\tau'}(\cdot, t^u)$ is the measure of maximal Carathéodory dimension. In particular, the Carathéodory dimension of $X = V^u(x) \cap \Lambda$ does not depend of the choice of the point $x \in \Lambda$.

In the particular case when the map f is u-conformal the direct calculation involving (3) shows that the Carathéodory measure $m^u_{C,x}$ is the measure of full Hausdorff dimension and that $\tau^u = t^u \dim E^u = \dim_H X$.