How to choose a (good) random surface

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Plan of the talk

1. Path integral
   – based on [Polyakov ’81], [David-Kupiainen-Rhodes-Vargas ’14]

2. Legacy of Kahane
   – based on [Kahane ’85]

3. Case of the unit disk
   – based on [Huang-Rhodes-Vargas ’15]

Path integral

How to choose a random real number?

Theorem (Frivolous Theorem of Arithmetic)

Almost all real numbers are very, very, very far from 0*.*

"Proof".

$$\int_{\mathbb{R}} dx = \infty.$$

□

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Solution: penalize extreme values.

$$\forall z > 0, \quad \int_{\mathbb{R}} \exp(-(-zx + e^x))dx = \Gamma(z) < \infty.$$

Definition (Action)

$$S^z(x) = -zx + e^x.$$

How to choose a random path?

Space:

$$\Sigma = \{\sigma : [0,1] \to \mathbb{R}\}.$$

Probability law:

$$\mathbb{E}[F(B)] = \frac{1}{Z} \int_{\Sigma} F(\sigma) \exp(-S(\sigma)) d\sigma.$$

Action:

$$S(\sigma) = \frac{1}{2} \int_0^1 |\sigma'(t)|^2 dt.$$

Brownian motion

Every path σ in $\Sigma = \{\sigma : [0,1] \to \mathbb{R}\}$ is chosen with probability density

$$\exp\left(-\frac{1}{2}\int_0^1 |\sigma'(t)|^2 dt\right) d\sigma.$$

Most probable path:

$$\sigma(t) \equiv c.$$

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Fixing the constant: the convention is to restrict to

$$\Sigma_0 = \{\sigma : [0, 1] \to \mathbb{R}, \sigma(0) = 0\}.$$

Definition (Standard Brownian motion)

Gaussian process of covariance kernel $K(s, t) = s \wedge t$.

Random surface on the Riemann sphere: a first attempt

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Space:

$$\Sigma = \{X : \mathbb{S} = \mathbb{R}^2 \cup \{\infty\} \to \mathbb{R}\}.$$

Probability law:

$$\mathbb{E}[F(\phi)] = \frac{1}{Z}\int_{\Sigma} F(X)\exp(-S(X))DX.$$

Action:

$$S(X) = \frac{1}{4\pi}\int_{\mathbb{R}^2} |\partial X(z)|^2 dz.$$

We get the so-called Gaussian Free Field on $\mathbb{R}^2$.

– Starting point for many constructions in quantum field theory.

Definition (A formal definition)

The Gaussian Free Field on $\mathbb{R}^2$ is a Gaussian process with covariance kernel

$$G(x, y) = -\ln|x - y|$$

where $G(x, y) = \Delta^{-1}$ is the inverse of the Laplacian.

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Interesting properties:

- $G(x, x) = \infty$, so the Gaussian Free Field can not be defined point-wise;
  – Take average on compact sets.
- Scale-invariance properties: $G(x/r, y/r) = G(x, y) + \ln r$;
  – Multi-fractal nature.

Definition (A proper definition of the so-called whole plane GFF)

The Gaussian Free Field with 0-mean in the (normalized) metric g on $\mathbb{R}^2$ is a Gaussian process with covariance kernel

$$G_g(x,y) = -\ln|x-y| + \int_{\mathbb{R}^2} \ln|x-z| g(z)dz + \int_{\mathbb{R}^2} \ln|y-z| g(z)dz + Cst(g).$$

The Gaussian Free Field on $\mathbb{R}^2$ can be seen as

$$X = X_g + c$$

with c an independent additive constant distributed like $Leb(\mathbb{R})$.

Liouville action on the Riemann sphere [Polyakov '81, DKRV '14]

Problem: too many invariances (constant, conformal metric).

Definition (Liouville action)

$$S(X,g) = \frac{1}{4\pi}\int_{\mathbb{R}^2} |\partial^g X(z)|^2 dV_g + \frac{1}{4\pi}\int_{\mathbb{R}^2} QR_g(z)X(z)dV_g + \mu\int_{\mathbb{R}^2} e^{\gamma X}dV_g$$

with $\gamma \in [0,2)$, $Q = \frac{2}{\gamma} + \frac{\gamma}{2}$, R_g the Ricci curvature and V_g the volume form of g.
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We can seperate $S(X,g)$ into two parts: one S_{GFF} part and S_Γ part.

New problem:

- How do we define $e^{\gamma X}$?
  - Theory of Gaussian Multiplicative Chaos, [Kahane ’85].

Minisuperspace approximation

Replace $X = X_g + c$ by a constant c in the previous action:

$$S(c, g) = \frac{c}{4\pi} \int_{\mathbb{R}^2} Q R_g(z) dV_g + \mu \int_{\mathbb{R}^2} e^{\gamma c} dV_g.$$

Gauss-Bonnet formula:

$$\frac{c}{4\pi} \int_{\mathbb{R}^2} Q R_g(z) dV_g = 2Q \cdot c.$$

We get the action for the Gamma function! (almost...)

$$S(c, g) = 2Q \cdot c + \mu V_g(\mathbb{R}^2) e^{\gamma c}.$$

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New problem:

- Negative curvature on the Riemann sphere?
  - Vertex operator, i.e. insertion points verifying Seiberg bounds, [Seiberg '90].

Vertex operators and Seiberg bounds

Probability law: $\alpha_i > 0$, z_i fixed points on $\mathbb{R}^2$

$$\mathbb{E}[F(\phi)] = \frac{1}{Z} \int_\Sigma F(X) \prod_i e^{\alpha_i X(z_i)} \exp(-S(X)) DX.$$

Cameron-Martin theorem:

$$"\frac{1}{Z} \int_\Sigma F(X - \sum_i \alpha_i \ln|\cdot - z_i|) \exp(-S(X - \sum_i \alpha_i \ln|\cdot - z_i|)) DX".$$

Theorem (Seiberg bounds)

The partition function above is well-defined and non-trivial if and only if

$$\forall i, \quad \alpha_i < Q,$$
$$2Q - \sum_i \alpha_i < 0.$$

In particular, we need at least 3 insertions!

Legacy of Kahane

Let X be a Gaussian field on $\mathbb{R}^d$ with correlation

$$G(x,y) = -\ln|x-y| + O(1).$$

Goal: construct the exponential $e^{\gamma X}$ of X in $\mathbb{R}^d$ as a random measure.

Gaussian multiplicative chaos

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- First define the average X_ϵ of a log-correlated Gaussian field X on scale ϵ.
  - Example: circle-average $X_\epsilon(z) = \frac{1}{2\pi}\int_0^{2\pi} X(z + \epsilon e^{i\theta})d\theta$.

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- Look at $e^{\gamma X_\epsilon}$ with proper renormalization:
  - For all borelian $A \subset \mathbb{R}^d$, look at

$$M_\epsilon^\gamma(A) = \int_A \epsilon^{\gamma^2/2} e^{\gamma X_\epsilon(z)} dz.$$

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- Prove convergence result [Kahane '85]:

$$M_\epsilon^\gamma(A) \longrightarrow M^\gamma(A).$$

[Robert-Vargas; Duplantier-Sheffield; Shamov; Junnila-Saksman; Berestycki... '08–'17]

Theorem (Non-trivialness)

$M^\gamma(A)$ is non trivial iff $\gamma \in [0, \sqrt{2d})$.

Theorem (Multifractal nature)

Assume $\gamma < \sqrt{2d}$ and M^γ the Gaussian multiplicative chaos measure. For all $q \in [0, \frac{2d}{\gamma^2})$ we have

$$\mathbb{E}[M(B(x,r))^q] \asymp r^{\zeta(q)}$$

as $r \to 0$, with

$$\zeta(q) = (d + \frac{\gamma^2}{2})q - \frac{\gamma^2}{2}q^2.$$

Theorem (Support of the measure)

For $\gamma < \sqrt{2d}$, the measure $M^\gamma(A)$ almost surely lives on a set of Hausdorff dimension $d - \frac{\gamma^2}{2}$.

Moment bounds

Theorem (Moment bound for Gaussian multiplicative chaos measures in $\mathbb{R}^d$)
Let M^γ the Gaussian multiplicative chaos measure with $\gamma < \sqrt{2d}$:

$$\mathbb{E}[M^\gamma(B(0,1))^p] < \infty \quad \text{if and only if} \quad p < \frac{2d}{\gamma^2}.$$

Near a singularity (first Seiberg bound):

$$\int_{B(0,1)} \frac{1}{|x|^{\gamma\alpha}} M^\gamma(dx) < \infty \quad a.s.$$

if and only if $\alpha < \frac{2}{\gamma} + \frac{\gamma}{2}$. Furthermore for $p > 0$,

$$\mathbb{E}\left[\left(\int_{B(0,1)} \frac{1}{|x|^{\gamma\alpha}} M^\gamma(dx)\right)^p\right] < \infty$$

if and only if $p < \frac{2}{\gamma} + \frac{\gamma}{2} - \alpha$.

Application: unit volume Seiberg bounds, analytic continuations...

Case of the unit disk

Motivation: moment bounds for $M_\gamma(A) = \int_{A\in\mathbb{D}} e^{\gamma X}$ where X is a Gaussian field with the Green function on the unit disk.

Equivalently, we study locally on $\mathbb{H}$ a Gaussian field X with correlation

$$G_{\mathbb{H}}(x, y) = \ln \frac{1}{|x-y|} + \ln \frac{1}{|x-\overline{y}|}.$$

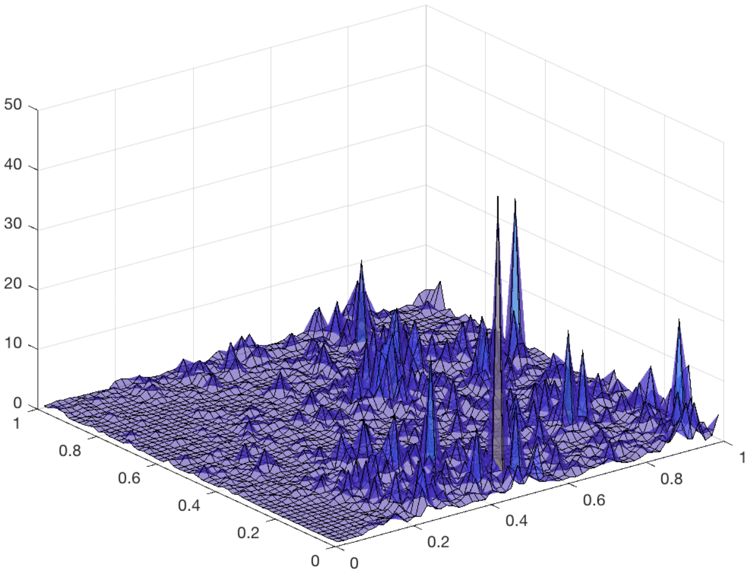
Gaussian Multiplicative Chaos measure:

$$M_\gamma(A) = \int_A e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X(z)^2]} |y|^{-\frac{\gamma^2}{2}} dz$$

with $z = x + iy$.

Problem: everything blows up approaching the boundary!

A local picture near the boundary (in $\mathbb{H}$)



Theorem (Moment bound for Gaussian Multiplicative Chaos on the unit disk)

Suppose $M_\gamma(A)$ is the Gaussian Multiplicative Chaos measure on the unit disk with parameter $\gamma \in (0, 2)$.

Recall that near the boundary $\mathbb{H}$, the random measure locally looks like

$$M_\gamma(A) = \int_A e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X(z)^2]} |y|^{-\frac{\gamma^2}{2}} \, dz.$$

Then

$$p < \frac{2}{\gamma^2} \quad \Longrightarrow \quad 0 < \mathbb{E}[M_\gamma(\mathbb{D})^p] < \infty.$$

Remark!

$$\frac{2}{(\sqrt{2}\gamma)^2} < \frac{2}{\gamma^2} < \frac{4}{\gamma^2}.$$

Thank you!

Polyakov introduced Liouville Conformal Field Theory in his theory of integration over 2d Riemann surfaces (1981). In this talk, we will gently explain a rigorous probabilistic approach by David-Kupiainen-Rhodes-Vargas (2014) based on Feynman's path integral formalism. In the construction of this path integral over surfaces with exponential interaction, a crucial ingredient is Kahane's Gaussian Multiplicative Chaos (1985), a natural multifractal random measure defined as the exponential of a log-correlated Gaussian field. We will also briefly explain some extensions of Kahane's theory to the case of the unit disk.