Hitting Probability and Hausdorff Dimension Results for Gaussian Random Fields

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(Based on joint works with H. Biermé, R. Dalang, C. Lacaux and C. Mueller)

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Outline

1. Hitting probabilities: existing results
2. Main results:
   - Hitting probability and capacity
   - Polarity of points
3. Applications to fBm and SPDEs

1. Hitting probabilities: some existing results

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. One can consider the following intersection problems:

(1) For Borel sets $E \subseteq \mathbb{R}^N$ and $F \subseteq \mathbb{R}^d$, when can one has

$$\mathbb{P}(X(E) \cap F \neq \emptyset) > 0? \tag{1}$$

(2) [k-multiple points] Given disjoint sets $E_1, \ldots, E_k \subseteq \mathbb{R}^N$, when does

$$\mathbb{P}(X(E_1) \cap \cdots \cap X(E_k) \cap F \neq \emptyset) > 0? \tag{2}$$

(3) Whenever an intersection set is non-empty, it is interesting to determine its Hausdorff dimension.

Some history about (1)

In the case when $E = [a, b]$, $(a, b \in \mathbb{R}^N)$, a necessary and sufficient condition for **(1)** in terms of certain kind of capacity of F has been established for X being

- Brownian motion (Kakutani, 1944; ...)
- Lévy processes (Port and Stone, 1971; ...)
- Some multiparameter Markov processes (Fitzsimmons and Salisbury, 1989)
- The Brownian sheet (Khoshnevisan and Shi, 1999)
- Additive Lévy processes (Khoshnevisan and Xiao, 2002, 2003, 2009)
- SPDEs (Dalang and Nualart, 2004; Dalang, Khoshnevisan and Nualart, 2007; ...)

Question **(1)** includes intersections of the graph set and level sets:

- Let $\mathrm{Gr}X(E) = \{(t, X(t)) : t \in E\}$ be the graph of X on E. Then **(1)** is equivalent to

$$\mathbb{P}(\mathrm{Gr}X(E) \cap (E \times F) \neq \emptyset) > 0.$$

- Take $F = \{0\}$, then (1) is equivalent to

$$\mathbb{P}(X^{-1}(0) \cap E \neq \emptyset) > 0.$$

- For general $E \subseteq \mathbb{R}$ and $F \subseteq \mathbb{R}^d$, a necessary and sufficient condition for (1) in terms of "thermal capacity" of $E \times F$ was established for Brownian motion $W = \{W(t), t \geq 0\}$ by Watson (1978). This is the only known complete characterization in this generality.

- The Hausdorff dimension $W(E) \cap F$ was determined by Khoshnevisan and Xiao (2015).

- In the special case of $F = \{0\}$, the hitting probability is characterized by Khoshnevisan and Xiao (2002) for a large class of additive Lévy processes, and by Khoshnevisan and Xiao (2007) for the Brownian sheet.

Another intersection problem: set of fast points

Recall that if $W = \{W(t), t \geq 0\}$ is Brownian motion, then its Uniform modulus of continuity and LIL state:

$$\lim_{\varepsilon\to 0} \frac{\max_{s,t,\in[0,1],|s-t|\leq\varepsilon} |W(s) - W(t)|}{\sqrt{2\varepsilon \log 1/\varepsilon}} = 1, \quad \text{a.s.}$$

and for any fixed $t \in [0, 1]$,

$$\limsup_{\varepsilon\to 0} \frac{\max_{|s-t|\leq\varepsilon} |W(s) - W(t)|}{\sqrt{2\varepsilon \log\log 1/\varepsilon}} = 1, \quad \text{a.s.}$$

The LIL and Fubini's theorem imply that almost surely

$$\limsup_{\varepsilon\to 0} \frac{\max_{|s-t|\leq\varepsilon} |W(s) - W(t)|}{\sqrt{2\varepsilon \log\log 1/\varepsilon}} = 1, \quad \text{a.e. } t \in [0, 1].$$

Set of fast points

Orey and Taylor (1974) showed that for $\lambda \in (0,1)$, the set of λ-fast points

$$F(\lambda) = \left\{ t \in [0,1] : \limsup_{\varepsilon \to 0} \frac{\max_{|s-t| \le \varepsilon} |W(s) - W(t)|}{\sqrt{2\varepsilon \log 1/\varepsilon}} \ge \lambda \right\}$$

is not empty and has Hausdorff dimension $1 - \lambda^2$.

(4) For a Borel set $E \subseteq [0,1]$, when does

$$\mathbb{P}\big(F(\lambda) \cap E \ne \emptyset\big) > 0?$$

This problem was considered by Kaufman (1975) and was solved by Peres, Khoshnevisan and Xiao (2000).

2. Our results on Question (1)

In the rest of this talk, we focus on Question (1) and present some results from the following three papers:

- Biermé, H., Lacaux, C. and Xiao, Y. (2009). Hitting probabilities and the Hausdorff dimension of the inverse images of anisotropic Gaussian random fields. *Bull. London Math. Soc.* **41**, 253–273.
- Dalang, R., Mueller, C. and Xiao Y. (2016). Polarity of points for Gaussian random fields. *Ann. Probab.*, to appear.

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field in $\mathbb{R}^d$ defined by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \qquad t \in \mathbb{R}^N, \tag{3}$$

where $X_1, \ldots, X_d$ are independent copies of a centered GF X_0.

Many sample path properties of X are determined by the behavior of the canonical metric $d_{X_0}(s, t)$ defined by

$$d_{X_0}(s, t) = \sqrt{\mathbb{E}\big(X_0(s) - X_0(t)\big)^2}.$$

Given constants $0 < H_1 \le \ldots \le H_N \le 1$, define a metric ρ on $\mathbb{R}^N$ by:

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \qquad \forall\, s, t \in \mathbb{R}^N. \tag{4}$$

We assume that X_0 satisfies the following conditions:

(C1). $\exists$ constants c_1 and c_2 such that $\forall\, s, t \in I (= [\varepsilon, 1]^N)$,

$$c_1\, \rho(s,t) \le d_{X_0}(s,t) \le c_2\, \rho(s,t).$$

(C2). $\exists\, c_3 > 0$ such that for all $s, t \in I$,

$$\mathrm{Var}\left(X_0(t) \middle| X_0(s)\right) \ge c_3\, \rho(s,t)^2.$$

- A class of Gaussian fields that satisfy (C1) and (C2) is fractional Brownian sheets.

- Solutions to SHE satisfy (C1) and (C2) with $H_1 = 1/4$ and $H_2 = 1/2$.

- Solutions to the stochastic wave equation satisfy (C1) and (C2) with $H_1 = 1/2$ and $H_2 = 1/2$.

- More examples can be found in X. (2009), Tudor and X. (2015), Allouba and X. (2016).

2.1 Hitting probabilities and Riesz capacity

The following result was motivated by Dalang, Khoshnevisan and Nualart (2007) for SPDEs and X. (1999) for fractional Brownian motion.

Theorem 2.1 [Biermé, Lacaux and X. (2009)]

If X is defined by (3) such that X_0 satisfies Conditions (C1) and (C2) on I. Then $\forall$ Borel set $F \subset \mathbb{R}^d$,

$$c_4\, \mathcal{C}^{d-Q}(F) \le \mathbb{P}\big\{X(I) \cap F \neq \emptyset\big\} \le c_5\, \mathcal{H}^{d-Q}(F), \qquad (5)$$

where $Q = \sum_{j=1}^{N} \frac{1}{H_j}$, $\mathcal{C}^{d-Q}$ is $(d-Q)$-dimensional Riesz capacity and $\mathcal{H}^{d-Q}$ is $(d-Q)$-dimensional Hausdorff measure.

Conjecture: $\mathcal{H}^{d-Q}(F)$ in (5) can be replaced by $\mathcal{C}^{d-Q}(F)$.

Recently, Dalang, Mueller and X. (2016) verified this conjecture for the case of $F = \{x\}$. Under certain conditions, they proved that, if $d = Q$, then for every $x \in \mathbb{R}^d$,

$$\mathbb{P}\Big\{X(I) \cap \{x\} \neq \emptyset\Big\} = \mathbb{P}\Big\{\exists t \in I : X(t) = x\Big\} = 0.$$

Proof of Theorem 2.1: the upper bound

For proving the upper bound in (5), we make use of a covering argument and the following lemma.

Lemma 2.1 [Biermé, Lacaux and X. (2009)]

Assume the conditions of Theorem 2.1 hold. For any constant $M > 0$, there exist positive constants c and δ_0 such that for all $r \in (0, \delta_0)$, $t \in I$ and all $x \in [-M, M]^d$,

$$\mathbb{P}\Big\{ \inf_{s \in B_\rho(t,r) \cap I} \|X(s) - x\| \le r \Big\} \le c\, r^d. \tag{6}$$

In the above $B_\rho(t, r) = \{s \in \mathbb{R}^N : \rho(s,t) \le r\}$ denotes the closed ball of radius r in the metric ρ in $\mathbb{R}^N$.

Proof of the upper bound

Only the case of $d > Q$ needs a proof. Choose and fix an arbitrary constant $\gamma > \mathcal{H}_{d-Q}(F)$. By the definition of $\mathcal{H}_{d-Q}(F)$, there is a sequence of balls $\{B(y_j, r_j), j \geq 1\}$ in $\mathbb{R}^d$ such that

$$F \subseteq \bigcup_{j=1}^{\infty} B(y_j, r_j) \quad \text{and} \quad \sum_{j=1}^{\infty} (2r_j)^{d-Q} \leq \gamma. \tag{7}$$

Notice that

$$\{F \cap X(I) \neq \varnothing\} \subseteq \bigcup_{j=1}^{\infty} \{B(y_j, r_j) \cap X(I) \neq \varnothing\}. \tag{8}$$

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Proof of the upper bound

For every $j \geq 1$, we divide the interval I into $c\, r_j^{-Q}$ intervals of side-lengths r_j^{-1/H_ℓ} $(\ell = 1, \ldots, N)$. Hence I can be covered by at most $c\, r_j^{-Q}$ many balls of radius r_j in the metric ρ.

It follows from Lemma 2.1 that

$$\mathbb{P}\big\{B(y_j, r_j) \cap X(I) \neq \varnothing\big\} \leq c\, r_j^{d-Q}. \tag{9}$$

Combining (8) and (9) we derive that $\mathbb{P}\big\{F \cap X(I) \neq \varnothing\big\} \leq c\gamma$. Since $\gamma > \mathcal{H}_{d-Q}(F)$ is arbitrary, the upper bound in (5) follows.

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Proof of Theorem 2.1: the lower bound

We make use of the following lemma.

Lemma 2.2 [Biermé, Lacaux and X. (2009)]

There exists a positive and finite constant c such that for all $\varepsilon \in (0,1)$, $s, t \in I$ and $x, y \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^{2d}} \exp\Big(-\frac{1}{2}(\xi,\,\eta)\,\big(\varepsilon I_{2d} + \mathrm{Cov}\big(X(s), X(t)\big)\big)\,(\xi,\,\eta)^T\Big) e^{-i(\langle \xi, x\rangle + \langle \eta, y\rangle)}\, d\xi d\eta \;\le \frac{c}{\big(\max\{\rho(s,t),\, \|x-y\|\}\big)^d}.$$

Here I_{2d} denotes the identity matrix of order $2d$, $\mathrm{Cov}\big(X(s), X(t)\big)$ denotes the covariance matrix of the random vector $(X(s), X(t))$.

Proof of the lower bound

Without loss of generality, we assume $\mathcal{C}_0(F) > 0$ and F is compact. Let $M > 0$ be a constant such that $F \subseteq [-M, M]^d$.

We only consider the critical case of $d = Q$. By definition of capacity, there is a Borel probability measure ν_0 on F such that

$$\begin{aligned} \mathcal{E}_0(\nu_0) &:= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \log\left(\frac{e}{\|x-y\| \wedge 1}\right) \nu_0(dx)\nu_0(dy) \\ &\le \frac{2}{\mathcal{C}_0(F)}. \end{aligned} \tag{10}$$

For all integers $n \geq 1$, we consider a family of random measures ν_n on I defined by

$$
\begin{aligned}
&\int_I g(t)\, \nu_n(dt) \\
&= \int_I \int_{\mathbb{R}^d} (2\pi n)^{d/2} \exp\left(- \frac{n \left\| X(t) - x \right\|^2}{2} \right) g(t)\, \nu_0(dx)\, dt \\
&= \int_I \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \exp\left(- \frac{\|\xi\|^2}{2n} + i\langle \xi, X(t) - x\rangle \right) g(t)\, d\xi \nu_0(dx) dt
\end{aligned}
$$

where g is an arbitrary measurable, nonnegative function on I.

Denote the total mass of ν_n by $\|\nu_n\| := \nu_n(I)$. We verify the following two inequalities hold:

$$\mathbb{E}\big(\|\nu_n\|\big) \geq c_4 \quad \text{and} \quad \mathbb{E}\big(\|\nu_n\|^2\big) \leq c_5 \mathcal{E}_0(\nu_0), \tag{11}$$

where the constants c_4 and c_5 are independent of ν_0 and n.

By (11) and the Paley-Zygmund inequality, one can show that there is an event Ω_0 with probability at least

$$c_4^2/(2c_5\,\mathcal{E}_0(\nu_0))$$

such that for every $\omega \in \Omega_0$, $\{\nu_n(\omega), n \geq 1\}$ has a subsequence that converges weakly to a finite positive measure ν which is supported on $X^{-1}(F) \cap I$.

Then, we have

$$\mathbb{P}\Big\{X(I) \cap F \neq \varnothing\Big\} \geq \mathbb{P}\{\|\nu\| > 0\} \geq \frac{c_4^2}{2c_5\,\mathcal{E}_0(\nu_0)}.$$

This proves the lower bound.

The Hausdorff dimension of $X^{-1}(F)$

For any Borel set $F \subseteq \mathbb{R}^d$, consider the inverse image

$$X^{-1}(F) = \{t \in \mathbb{R}^N : X(t) \in F\}.$$

Theorem 2.2 [Biermé, Lacaux and X. (2009)]

Let X be as in Theorem 2.1 and let $F \subseteq \mathbb{R}^d$ be a Borel set such that $\sum_{j=1}^N \frac{1}{H_j} > d - \dim_{\rm H} F$. Then with positive probability,

$$\dim_{\rm H}\big(X^{-1}(F) \cap I\big) = \min_{1 \le k \le N} \left\{ \sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(d - \dim_{\rm H} F) \right\}.$$

2.2 Polarity of points for Gaussian fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field in $\mathbb{R}^d$ defined by (3).

Assumption (D1) Let $I = \prod_{j=1}^N [c_j, d_j]$, where $c_j < d_j$. Let $I^{(\varepsilon)}$ denote an ε-enlargement of I, in Euclidean norm. There is an $\varepsilon_0 > 0$ and a Gaussian field $\{v(A,t), A \in \mathcal{B}(\mathbb{R}_+), t \in I^{(\varepsilon_0)}\}$ such that:

(a) for all $t \in I^{(\varepsilon_0)}$, $A \mapsto v(A,t)$ is an $\mathbb{R}^d$-valued Gaussian noise with i.i.d. components, $v(\mathbb{R}_+, t) = X(t)$, and when A and B are disjoint, $v(A, \cdot)$ and $v(B, \cdot)$ are independent.

(b) there are constants $c_0 \in \mathbb{R}_+$, $a_0 \in \mathbb{R}_+$ and $\gamma_j > 0$, $j = 1, \ldots, N$, such that for all $a_0 \le a \le b \le +\infty$, $s, t \in I^{(\varepsilon_0)}$,

$$\left\| v([a,b),\, s) - X(s) - v([a,b),\, t) + X(t) \right\|_{L^2(\mathbb{P})} \le c_0 \left[\sum_{j=1}^{N} a^{\gamma_j} \, |s_j - t_j| + b^{-1} \right]$$

and

$$\left\| v([0,a_0],\, s) - v([0,a_0],\, t) \right\|_{L^2(\mathbb{P})} \le c_0 \sum_{j=1}^{N} |s_j - t_j|.$$

Let $H_j = (\gamma_j + 1)^{-1}$ and, as before, $Q = \sum_{j=1}^{N} \frac{1}{H_j}$.

Assumption (D2)

(a) There is a constant $\tilde{c} > 0$ such that for all $t \in I^{(\varepsilon_0)}$, we have $\mathbb{E}\big(X_0(t)^2\big) \geq \tilde{c}$.

(b) There are constants $\eta > 0$, $\delta_j \in (H_j, 1]$, $j = 1, \ldots, N$, and $C > 0$ with the following property. For $t \in I$, there are $t' \in I^{(\varepsilon_0)}$, such that for all $s, \bar{s} \in I^{(\varepsilon_0)}$ with $\rho(s,\, t) \leq 2\eta$ and $\rho(\bar{s},\, t) \leq 2\eta$,

$$\Big|\mathbb{E}\big[(X_0(s) - X_0(\bar{s}))X_0(t')\big]\Big| \leq C \sum_{j=1}^{N} |s_j - \bar{s}_j|^{\delta_j}.$$

The following is our main theorem in Dalang, Mueller and X. (2016).

Theorem 2.3

Assume that (D1) and (D2) hold for a closed box I and $Q = d$. Then for all $x \in \mathbb{R}^d$,

$$\mathbb{P}\Big\{\exists\, t \in I : X(t) = x\Big\} = 0.$$

Proof of Theorem 2.3

The proof is based on a covering argument, but using balls of different sizes that are chosen based on local oscillations of the sample function $X(\cdot)$.

The main idea comes from Talagrand (1995, 1998), which was also used by X. (1996) to study the Hausdorff measure of the level set $X^{-1}(x)$.

Let $\eta > 0$ be the constant in (D2). Set

$$B_\rho(t, \eta) = \{s \in \mathbb{R}^N : \rho(s, t) \leq \eta\}.$$

Proposition 2.1

Assume that (D1) and (D2) hold, and $Q = d$. Then for every $t_0 \in I$ and $x \in \mathbb{R}^d$,

$$\mathbb{P}\Big\{\exists t \in B_\rho(t_0, \eta) : X(t) = x\Big\} = 0.$$

Equivalently, $X^{-1}(x) \cap B_\rho(t_0, \eta) = \varnothing$ almost surely.

Theorem 2.3 follows easily from Proposition 2.1.

Main ingredients for proving Proposition 2.1

(i) Extension of Talagrand's proposition

Proposition 2.2

Assume (D1) holds.Then there are constants $K < \infty$ and $\eta_0 \in (0, 1]$ such that for $0 < r_0 < \eta_0$ and $t \in I$, we have

$$\mathbb{P}\Big\{\exists r \in [r_0^2, r_0] : \sup_{s:\rho(s,t)<r} |X(s) - X(t)| \le \frac{K\, r}{(\log\log \frac{1}{r})^{1/Q}}\Big\}$$
$$\ge 1 - \exp\Big[-\Big(\log \frac{1}{r_0}\Big)^{1/2}\Big].$$

This proposition relies on small ball probability, and Gaussian arguments. It is also useful for studying fractal properties of X.

For $p \geq 1$, consider the random set

$$R_p = \Big\{ s \in B_\rho(t_0, 2\eta) : \exists r \in [2^{-2p}, 2^{-p}) \text{ such that } \sup_{\bar{s}:\, \rho(\bar{s}, s) < r} |X(\bar{s}) - X(s)| \leq \frac{K\, r}{(\log\log \frac{1}{r})^{1/Q}} \Big\},$$

and the event

$$\Omega_{p,1} = \Big\{ \lambda_N(R_p) \geq \lambda_N(B_\rho(x, 2\eta)) \Big(1 - \exp\big(-\frac{\sqrt{p}}{4}\big)\Big) \Big\}.$$

Proposition 2.2 implies that $\sum_{p=1}^{\infty} \mathbb{P}\big(\Omega_{p,1}^c\big) < \infty$. Hence the events $\Omega_{p,1}$ occur for all p large.

(ii) Conditioning argument

Let $t' \in \mathbb{R}^N$ be given by Part (b) of (D2). Define two $\mathbb{R}^d$-valued random fields

$$X^2(t) = \mathbb{E}(X(t) \mid X(t')), \quad X^1(t) = X(t) - X^2(t), \quad t \in \mathbb{R}^N.$$

The random fields X^1 and X^2 are independent. Further, X^1 is independent of the random vector $X(t')$.

The following lemma shows that X^2 is rather smooth.

Lemma 2.3

The random field $X^2 = \{X^2(t), t \in I\}$ has a continuous version, and there is a constant C such that for $s \in B_\rho(t_0, 2\eta)$ and $\bar{s} \in B_\rho(t_0, 2\eta)$,

$$|X^2(s) - X^2(\bar{s})| \le C|X(t')| \sum_{j=1}^{N} |s_j - \bar{s}_j|^{\delta_j}.$$

where $\delta_j \in (H_j, 1]$ are give by (D2).

Hence Proposition 2.2 still holds if X is replaced by X^1.

Using ingredients (i) and (ii), we are able to construct a covering of $X^{-1}(x) \cap B_\rho(t_0, \eta)$ and to prove Proposition 2.1.

3. Applications to fBm and SPDEs

Recall that a centered Gaussian field $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}^d$ is called a d-dimensional *fractional Brownian field with Hurst index* $H \in (0, 1)$ if it has continuous sample paths and covariance function given by

$$\mathbb{E}\big(B_\ell^H(s) B_j^H(t)\big) = \delta_{\ell,j}\,\frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right),$$

where $\delta_{\ell,j}$ is the Kronecker symbol. It follows that

$$\mathbb{E}(|B^H(s) - B^H(t)|^2) = d\,|s-t|^{2H}.$$

We make use of the following harmonizable representation (Talagrand, 1995):

$$B^H(t) = c\int_{\mathbb{R}^N} \frac{1-\cos(t\cdot\xi)}{|\xi|^{H+k/2}}\, W_1(d\xi) + c\int_{\mathbb{R}^N} \frac{\sin(t\cdot\xi)}{|\xi|^{H+k/2}}\, W_2(d\xi),$$

where W_1 and W_2 are independent $\mathbb{R}^d$-valued white noises on $\mathbb{R}^N$.

The random field $\{v(A,t), A\in\mathcal{B}(\mathbb{R}_+), t\in\mathbb{R}^N\}$ is defined by

$$v(A,t) = c\int_{|\xi|^H\in A} \frac{1-\cos(t\cdot\xi)}{|\xi|^{H+k/2}}\, W_1(d\xi) + c\int_{|\xi|^H\in A} \frac{\sin(t\cdot\xi)}{|\xi|^{H+k/2}}\, W_2(d\xi).$$

We can verify that B^H satisfies Assumptions (D1) and (D2) on every compact box $I \subset \mathbb{R}^N \backslash \{0\}$.

It follows from Theorem 2.3 that

Corollary 3.1

If $N = Hd$, then for every $x \in \mathbb{R}^d$,

$$\mathbb{P}\Big\{\exists\, t \in \mathbb{R}^N \backslash \{0\} \ \text{ such that } \ B^H(t) = x\Big\} = 0.$$

Linear SHE with colored noise

Fix $k \geq 1$ and suppose $\beta \in]0, k \wedge 2[$ or $k = 1 = \beta$. Let $\hat{W}$ be a spatially homogeneous Gaussian noise which has mean 0 and covariance informally given by

$$\mathbb{E}\Big(\hat{W}_\ell(t,x)\hat{W}_j(s,y)\Big) = \delta(t-s)\,|x-y|^{-\beta}\,\delta_{\ell,j},$$

where $\delta(\cdot)$ denotes the Dirac delta function and $\delta_{\ell,j}$ is the Kronecker symbol.

Let $\hat{v} = \{\hat{v}(t,x),\ t \in \mathbb{R}_+,\ x \in \mathbb{R}^k\}$ be the mild solution of a linear system of d uncoupled heat equations driven by this space-time white noise:

$$\begin{cases} \frac{\partial}{\partial t}\hat{v}_j(t,x) = \frac{1}{2}\Delta\hat{v}_j(t,x) + \dot{\hat{W}}_j(t,x), & j = 1,\ldots,d, \\ v(0,x) = 0, & x \in \mathbb{R}^k. \end{cases} \tag{12}$$

Here, $\hat{v}(t,x) = (\hat{v}_1(t,x),\ldots,\hat{v}_d(t,x))$ and Δ is the Laplacian in the spatial variables.

It is known (Dalang, 1999; Dalang, et al. 2013) that

$$\hat{v}(t,x) = \int_0^t \int_{\mathbb{R}^k} G_{t-r}(x-y)\, \hat{W}(drdy), \quad t > 0, x \in \mathbb{R}^k,$$

where $G_t(x)$ is the fundamental solution of the heat equation given by

$$G_t(x) = \frac{1}{(2\pi t)^{k/2}} \exp\Big(-\frac{|x|^2}{2t}\Big), \quad t > 0, x \in \mathbb{R}^k.$$

Dalang, Mueller and X. (2016) proved that $\{\hat{v}_j(t,x), t \geq 0, x \in \mathbb{R}^k\}$ has the following harmonizable representation:

$$\hat{v}(t,x) = \mathrm{Re} \int_{\mathbb{R}} \int_{\mathbb{R}^k} e^{-i\xi\cdot x} \frac{e^{-i\tau t} - e^{-t|\xi|^2}}{|\xi|^2 - i\tau} \frac{W(d\tau, d\xi)}{|\xi|^{(k-\beta-)/2}},$$

where $W(d\tau, d\xi)$ is a $\mathbb{C}^d$-valued space-time white noise.

It can be verified that $\{\hat{v}(t,x)\}$ satisfies Conditions (D1) and (D2) with

$$H_1 = \frac{2-\beta}{4} \text{ and } H_j = \frac{2-\beta}{2} \quad \text{for } j = 2, \ldots, k+1.$$

Consequently, Theorem 2.3 implies

Corollary 3.2

If $\frac{4+2k}{2-\beta} = d$, then for every $z \in \mathbb{R}^d$,

$$\mathbb{P}\{\exists\,(t,x) \in (0,\infty) \times \mathbb{R}^k \text{ such that } \hat{v}(t,x) = z\} = 0.$$

In Dalang, Mueller and X. (2016), similar results are also established for

- linear heat equations with non-constant coefficients
- linear wave equations with constant coefficients.

Thank you!