Multifractal properties of typical continuous functions

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D.: Let $f \in L^\infty([0,1]^d)$. For $h \geq 0$ and $\mathbf{x} \in [0,1]^d$, the function f belongs to $C^h(\mathbf{x})$ if there are a polynomial P of degree strictly less than $[h]$ and a constant $C > 0$ such that, for all $\mathbf{x}'$ close to $\mathbf{x}$, $|f(\mathbf{x}') - P(\mathbf{x}' - \mathbf{x})| \leq C|\mathbf{x}' - \mathbf{x}|^h$.

The pointwise Hölder exponent of f at $\mathbf{x}$ is $h_f(\mathbf{x}) = \sup\{h \geq 0 : \ f \in C^h(\mathbf{x})\}$.

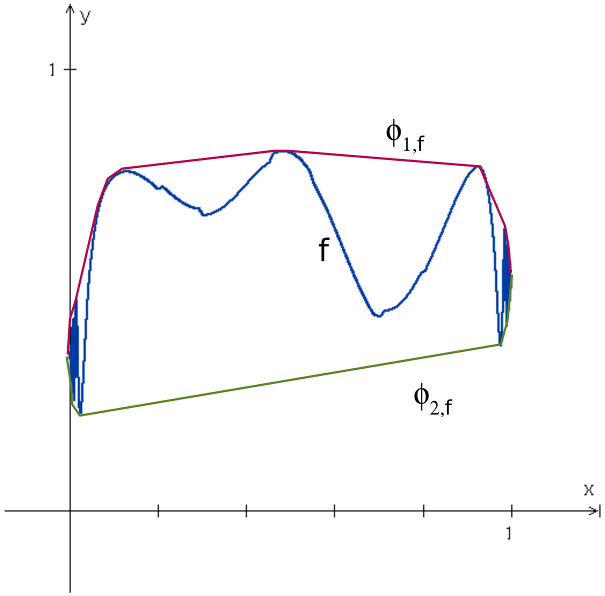
In the following, $\dim = \dim_H$ denotes the Hausdorff dimension.

D.: The singularity spectrum of f is the mapping

$d_f(h) = \dim E_f(h)$, where $E_f(h) = \{\mathbf{x} : h_f(\mathbf{x}) = h\}$.

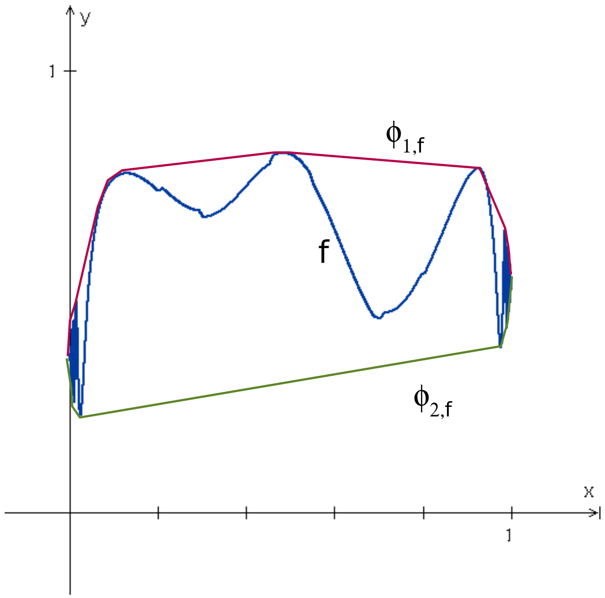
By convention $\dim \emptyset = -\infty$.

We will also use the sets $E_f^{\leq}(h) = \{\mathbf{x} : h_f(\mathbf{x}) \leq h\} \supset E_f(h)$.



The convex hull, H_f of typical continuous functions $f \in C[0,1]$ was considered by A. M. Bruckner and J. Haussermann. Recall that a property is typical, or generic in a complete metric space E, when it holds on a residual set, i.e. a set with a complement of first Baire category.

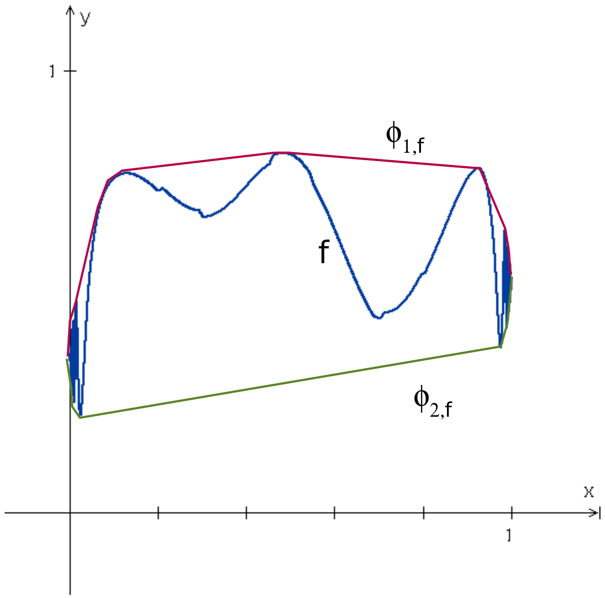
We are interested in two functions:
$\varphi_{1,f}(\mathbf{x}) = \max\{y : (\mathbf{x}, y) \in H_f\}$ is the function on the top of H_f,
and $\varphi_{2,f}(\mathbf{x}) = \min\{y : (\mathbf{x}, y) \in H_f\}$ is the function on the bottom of H_f.
The upper one, $\varphi_{1,f}$ is concave the lower one $\varphi_{2,f}$ is convex.
It was shown that for the typical f these functions are continuously differentiable on $(0, 1)$ and at the endpoints they have infinite derivatives.
We want to describe the multifractal spectrum of these functions in the multidimensional setting.



Generic/typical continuous functions f in $C[0,1]^d$ are considered in the sense of Baire category.
The topology on $C[0,1]^d$ is the supremum metric.
We also proved the multidimensional version of the above mentioned results of A. M. Bruckner and J. Haussermann.

The points where f and $\varphi_{i,f}$ coincide are denoted by
$\mathbf{E}_{i,f} = \{\mathbf{x} : \varphi_{i,f}(\mathbf{x}) = f(\mathbf{x})\}$, $i = 1, 2$.

The faces of $[0,1]^d$ are $F_{0,j} = \{(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_d) \in [0,1]^d\}$ and
$F_{1,j} = \{(x_1, ..., x_{j-1}, 1, x_{j+1}, ..., x_d) \in [0,1]^d\}$.
Then $\partial([0,1]^d) = \cup_{i=1}^{2} \cup_{j=1}^{d} F_{i,j}$.



Our main result:

T.: *There exists a dense G_δ set $\mathcal{G} \subset C[0,1]^d$ such that for every $f \in \mathcal{G}$ for $i = 1, 2$*

- $\varphi_{i,f}$ *is continuously differentiable on* $(0,1)^d$;
- *if* $\mathbf{x} \in \partial([0,1]^d)$ *then* $h_{\varphi_{i,f}}(\mathbf{x}) = 0$ *hence* $d_{\varphi_{i,f}}(0) = d-1$ *and* $E^0_{\varphi_{i,f}} \cap (0,1)^d = \emptyset$;
- $d_{\varphi_{i,f}}(1) = d-1$;
- $d_{\varphi_{i,f}}(+\infty) = d$;
- $d_{\varphi_{i,f}}(h) = -\infty$, *that is,* $E^h_f = \emptyset$ *for* $h \in (0, +\infty) \setminus \{1\}$;
- *for* $j = 1, ..., d$ *if* $\mathbf{x} \in F_{0,j}$ *then* $\partial_{j,+}\varphi_{i,f}(\mathbf{x}) = (-1)^{i+1}(+\infty)$ *if* $\mathbf{x} \in F_{1,j}$ *then* $\partial_{j,-}\varphi_{i,f}(\mathbf{x}) = (-1)^{i}(+\infty)$;
- $\dim_H \mathbf{E}_{i,f} = 0$.

In

Z. Buczolich and J. Nagy. Hölder spectrum of typical monotone continuous functions. *Real Anal. Exchange*, 26(1):133–156, 2000.

it was proved that for typical monotone continuous $f : [0, 1] \to \mathbb{R}$ we have

$$d_f(h) = \dim E_f^{\leq}(h) = \begin{cases} h & \text{if } h \in [0,1] \\ -\infty & \text{otherwise.} \end{cases}$$

The same holds true for typical monotone functions (not necessarily continuous).

We remark that it also follows, for example from results in the above paper, that for arbitrary monotone functions there is an upper estimate

$\bigstar$ $\dim E_f^{\leq}(h) \leq h$ for $h \in [0, 1]$.

It is interesting to extend the previous results to higher dimensions.
The first natural way is to consider Borel measures on the cube.
The local regularity of a positive measure μ at a given $\mathbf{x} \in [0,1]^d$ is given by a local dimension (or a local Hölder exponent) $h_\mu(\mathbf{x})$, defined as

$h_\mu(\mathbf{x}) = \liminf_{r \to 0^+} \frac{\log \mu(B(\mathbf{x},r))}{\log r}$, where $B(\mathbf{x},r)$ denotes the ball with center $\mathbf{x}$ and radius r.

The singularity spectrum of μ is the map $d_\mu : h \geq 0 \mapsto \dim E_\mu(h)$, where $E_\mu(h) := \{\mathbf{x} \in [0,1]^d : h_\mu(\mathbf{x}) = h\}$.

In Z. Buczolich and S. Seuret. Typical Borel measures on $[0,1]^d$ satisfy a multifractal formalism. *Nonlinearity.*, 23(11):7–13, 2010.
it is proved that typical measures μ supported on $[0,1]^d$ satisfy a multifractal formalism, and

$$d_\mu(h) = \begin{cases} h & \text{if } h \in [0,d] \\ -\infty & \text{otherwise.} \end{cases}$$

See also F. Bayart. Multifractal spectra of typical and prevalent measures. *Nonlinearity*, 26:353–367, 2013.
for a nice generalization to all compact sets in $\mathbb{R}^d$ (instead of $[0,1]^d$).

Another interesting class is constituted of the continuous monotone increasing in several variables (in short: MISV) functions.

These functions extend to higher dimensions in a different direction the one-dimensional monotone functions.

A function $f : [0, 1]^d \to \mathbb{R}$ is MISV when for all $i \in \{1, ..., d\}$, the functions $f^{(i)}(t) = f(x_1, ..., x_{i-1}, t, x_{i+1}, ..., x_d)$ are continuous monotone increasing.

We use the notation $\mathcal{M}^d = \{f \in C([0, 1]^d) : f \text{ MISV}\}$.

The space $\mathcal{M}^d$ is a separable complete metric space when equipped with the supremum L^∞ norm for functions.

In Z. Buczolich and S. Seuret. Multifractal spectrum and generic properties of functions monotone in several variables. *J. Math. Anal. Appl.*, 382 (10):110–126, 2011.

we showed that typical MISV functions satisfy

$$d_f(h) = \begin{cases} d - 1 + h & \text{if } h \in [0, 1] \\ -\infty & \text{otherwise.} \end{cases}$$

Typical Convex functions

We denote by $\mathcal{CC}^d$ the set of continuous convex functions $f : [0,1]^d \to \mathbb{R}$.
Equipped with the supremum norm $\|\cdot\|$, $\mathcal{CC}^d$ is a separable complete metric space.

An open ball in $\mathcal{CC}^d$ of center $f \in \mathcal{CC}^d$ of radius $r \geq 0$ is written as $B_{\|\cdot\|}(f,r)$, and a closed ball is $\overline{B}_{\|\cdot\|}(f,r)$

An upper bound for the multifractal spectrum of all functions in $\mathcal{CC}^d$:

T.: *For any function $f \in \mathcal{CC}^d$, one has*

$$d_f(h) \leq \begin{cases} d-1 & \text{if } h \in [0,1) \\ d+h-2 & \text{if } h \in [1,2] \\ d & \text{if } h > 2. \end{cases}$$

The multifractal spectrum of typical functions in $\mathcal{CC}^d$:

T.: *For typical functions $f \in \mathcal{CC}^d$, one has*

$$d_f(h) = \begin{cases} d-1 & \text{if } h = 0 \\ d+h-2 & \text{if } h \in [1,2] \\ -\infty & \text{otherwise.} \end{cases}$$

More precisely, one has $E_f(0) = \partial([0,1]^d)$.

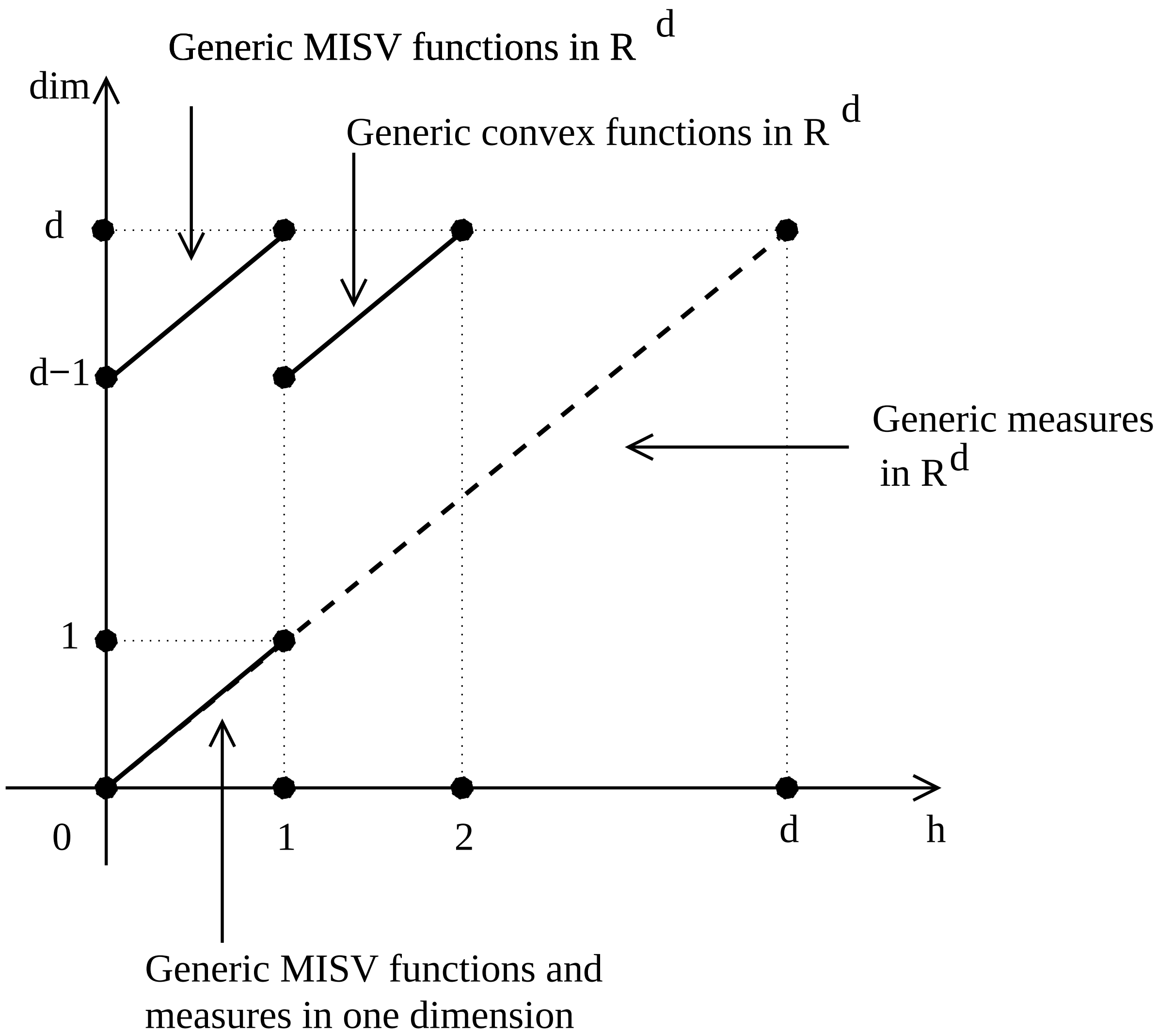
Rem.:

One cannot directly infer the results about typical convex functions by integrating MISV functions or measures on $[0,1]^d$.

For instance, letting $f(x_1,x_2) = 10(x_1^2 + x_2^2) + x_1^2 x_2^2$,

its second differential
$d^2 f(x_1,x_2,h_1,h_2) = (20 + 2x_2^2)h_1^2 + (20 + 2x_1^2)h_2^2 + 4x_1x_2h_1h_2$
is positive definite for any $(x_1,x_2) \in [-1,1]^2$.
Hence this function f is strictly convex on $[-1,1]^2$,
but $\partial_1 f = 20x_1 + 2x_1x_2^2$ is monotone only in x_1
and $\partial_2 f = 20x_2 + 2x_2x_1^2$ is monotone only in x_2.

The previous results are illustrated by this figure:



Usually upper estimates, valid for arbitrary functions, are easier to obtain.
In this case it is not true.

The next proposition asserts that a convex function cannot have exceptional isolated directional pointwise regularity.

Consider the d-dimensional unit sphere $S_d = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x}\| = 1\}$.

We select a finite set of pairwise distinct points
$(\mathbf{z}_1, \mathbf{z}_2, ...\mathbf{z}_N) \in (S_d)^N$ for some integer $N \geq 1$ such that
the convex hull of $\{\mathbf{z}_1, \mathbf{z}_2, ...\mathbf{z}_N\}$ contains the d-dimensional ball $B(\mathbf{0}, 1/2)$.

Let us choose $\varepsilon_c > 0$ so small that :

- $0 < \varepsilon_c \leq \frac{1}{1000} \min(\|\mathbf{z}_i - \mathbf{z}_j\|, \ i \neq j, \ i,j \in \{1, ..., N\})$.
- Setting for every $i \in \{1, ..., N\}$ $C_i = S_d \cap B(\mathbf{z}_i, \varepsilon_c)$,

then for any choice of $\mathbf{z}'_i \in C_i$, the convex hull of $\{\mathbf{z}'_1, \mathbf{z}'_2, ...\mathbf{z}'_N\}$ contains the ball $B(\mathbf{0}, 1/4)$.

Prop.: *If $h_f(\mathbf{x}) = h$, then there exists $i \in \{1, ..., N\}$ such that for every $\mathbf{z}'_i \in C_i$, the restriction of f to the straight line passing through $\mathbf{x}$ parallel to the vector $\mathbf{z}'_i$ has a pointwise Hölder exponent equal to h.*

The upper estimate

Prop.: *If* $1 < h \leq 2$ *and* $f \in \mathcal{C}\mathcal{C}^d$, *then* $\dim E_f^{\leq}(h) \leq d + h - 2$.

Proof.: It is sufficient to treat the case $h \in (1, 2)$.
Assume that $\dim E_f^{\leq}(h) > d + h - 2$.
For every $\mathbf{x} \in E_f^{\leq}(h)$, by the previous lemma,
there exists a cone of direction C_{i_x}, where $i_x \in \{1, ..., d\}$
such that for every $\mathbf{z} \in C_{i_x}$, the restriction of f to the straight line passing through $\mathbf{x}$ parallel to $\mathbf{z}$ has exponent less than h.
Let us call E_i the set of elements of $E_f^{\leq}(h)$ satisfying this property with $i_x = i \in \{1, ..., N\}$.
Obviously, $E_f^{\leq}(h) = \bigcup_{i=1}^N E_i$, so there exists at least one $i \in \{1, ..., N\}$ such that $\dim E_i > d + h - 2$.

Let us recall the following special case of Marstrand's slicing theorem:
Recall that S_d is the unit sphere in $\mathbb{R}^d$.
T.: *Let $E \subset [0,1]^d$ be a Borel set with Hausdorff dimension $\alpha \in (d-1, d)$.*
Then for almost every $\mathbf{z} \in S_d$
(in the sense of $(d-1)$-dimensional "surface" measure),
there exists a set $E_{\mathbf{z}}$ of positive $(d-1)$-dimensional Hausdorff measure
in the hyperplane orthogonal to $\mathbf{z}$
such that for every $\mathbf{x} \in E_{\mathbf{z}}$, $\dim E \cap (\mathbf{x} + \mathbb{R}\mathbf{z}) = \alpha - (d-1)$.
Each C_i has non-empty interior in the subspace topology of S_d, hence it is of positive $d-1$-dimensional measure.
Applying Marstrand's slicing theorem to E_i, one can find $\mathbf{z} \in C_i$ and $\overline{\mathbf{x}} \in [0,1]^d$ such that
if $\mathcal{D} = (\overline{\mathbf{x}} + \mathbb{R}\mathbf{z})$, then $\dim E_i \cap \mathcal{D} \geq d + h - 2 - (d-1) = h - 1$.
Let us call g the restriction of f to $\mathcal{D}$.
Then g is still a convex function of one variable.
By definition of E_i, every $\mathbf{x} \in \mathcal{D} \cap E_i$ satisfies $h_g(\mathbf{x}) \leq h$.

Next, by applying

L.: *If g is convex on $(a,b) \subset \mathbb{R}$ and $h_g(x) \in [1,2)$ for some $x \in (a,b)$, then $\min(h_{g'_+}(x), h_{g'_-}(x)) \leq h_g(x) - 1$.*

we deduce that $\min(h_{g'_+}(\mathbf{x}), h_{g'_-}(\mathbf{x})) \leq h - 1$, for every $\mathbf{x} \in \mathcal{D} \cap E_i$.

Hence, at least one of the two sets $E^{\leq}_{g'_+}(h-1)$ and $E^{\leq}_{g'_-}(h-1)$ has Hausdorff dimension strictly greater than $h-1$.

But this is impossible, since both functions g'_+ and g'_- are monotone, and for such functions, by $\bigstar$,

the Hausdorff dimension of $E^{\leq}_{g'_+}(h-1)$ and $E^{\leq}_{g'_-}(h-1)$ is necessarily less than $h-1 \in [0,1]$.

Hence a contradiction, and the conclusion that $\dim E^{\leq}_f(h) \leq d + h - 2$. $\square$

Prop.: *If* $0 \leq h \leq 1$, $f \in \mathcal{CC}^d$, *then* $\dim E_f^{\leq}(h) \leq d-1$.

Proof.: The proof is immediate: if $f \in \mathcal{CC}^d$, the pointwise exponent of f at any $\mathbf{x} \in (0,1)^d$ is necessarily larger or equal than 1.
The remaining points are located on the boundary, whose dimension is $d-1$.
And for every $h \in [0,1]$, it is easy to build examples of convex functions such that $h_f(\mathbf{x}) = h$ for every $\mathbf{x}$ satisfying $x_1 = 0$,
so the upper bound $d-1$ for the Hausdorff dimension of $E_f^{\leq}(h)$ is optimal.
$\square$

Typical properties of continuous convex functions

It is very easy to see that the typical function in $\mathcal{CC}^d$ is continuously differentiable on $(0,1)^d$. This is well-known. We gave another proof for completeness.

Prop.: *There is a dense G_δ set $\mathcal{G}$ in $\mathcal{CC}^d$ such that every $f \in \mathcal{G}$ is continuously differentiable on $(0,1)^d$.*

However, the next lemma shows that typical functions f in $\mathcal{CC}^d$ are not differentiable on $[0,1]^d$. The problem comes from the boundary of the domain.

Prop.: *There is a dense G_δ set $\mathcal{G}_\infty$ in $\mathcal{G}$ such that every $f \in \mathcal{G}_\infty$ satisfies the following:*
for every $j \in \{1,...,d\}$, for every $x_i \in [0,1]$ with $i \in \{1,...,d\} \setminus \{j\}$, one has
$\partial_{j,+} f(x_1,...,x_{j-1},0,x_{j+1},...,x_d) = -\infty$
and $\partial_{j,-} f(x_1,...,x_{j-1},1,x_{j+1},...,x_d) = +\infty$.
Moreover, $h_f(x_1,...,x_{j-1},0,x_{j+1},...,x_d) = 0 = h_f(x_1,...,x_{j-1},1,x_{j+1},...,x_d)$.

Typical convex functions have only pointwise exponents less than 2.

Prop.: *There exists a G_δ-set $\mathcal{G}^2$ such that for every $f \in \mathcal{G}^2$, $E_f(h) = \emptyset$ for every $h > 2$.*

Some simple tools

A first lemma allows one to control the left and right partial derivatives of all functions in a neighborhood of a convex differentiable function in $\mathcal{CC}^d$. The notations $\partial_{j,+}f$ and $\partial_{j,-}f$ are used for the right and left partial j-th derivatives of a convex function f.

L.: *Suppose $f \in \mathcal{CC}^d \cap C^1([0,1]^d)$ and $\varepsilon > 0$.*
There exists $\varrho_{f,\varepsilon} > 0$, such that
for all $j \in \{1, ..., d\}$, if $g \in B_{\|\cdot\|}(f, \varrho_{f,\varepsilon})$,
then for every $x_j \in [\varepsilon, 1-\varepsilon]$, $x_i \in [0,1]$, $i \in \{1, ..., d\} \setminus \{j\}$
we have $|\partial_{j,\pm} g(x_1, ..., x_d) - \partial_j f(x_1, ..., x_d)| < \varepsilon.$

The one-dimensional version of the above Lemma is stated as follows:
L.: *Suppose $f \in \mathcal{CC}^1 \cap C^1([0,1])$.*
For $\varepsilon > 0$ there exist $\varepsilon > 0$ and $\varrho_{\varepsilon,f} > 0$ such that
for any $g \in B_{\|\cdot\|}(f, \varrho_{\varepsilon,f})$ and $x \in [\varepsilon, 1-\varepsilon]$ we have $|g'_{\pm}(x) - f'(x)| < \varepsilon.$

Next, one compares the pointwise exponents of a differentiable convex function f and its derivative f'.

It is a general property that $h_{f'}(x) \leq h_f(x) - 1$, for every differentiable f.

A surprising property is that equality necessarily holds when f is convex and $h_f(x) \in [1, 2)$:

L.: *If f is convex and differentiable on (a, b) and $h_f(x) \in [1, 2)$ for some $x \in (a, b)$, then $h_f(x) = h_{f'}(x) + 1$.*

For non-differentiable convex functions we obtain a Lemma used earlier:

L.: *If f is convex on $(a, b) \subset \mathbb{R}$ and $h_f(x) \in [1, 2)$ for some $x \in (a, b)$, then* $\min(h_{f'_+}(x), h_{f'_-}(x)) \leq h_f(x) - 1$.