ON THE VISIBILITY OF PLANAR SETS

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ABSTRACT. Assume that $E, K \subset \mathbb{R}^2$ are Borel sets with $\dim_H K > 0$. Is a positive dimensional part of $K$ visible from some point in $E$? Not necessarily, since $E$ can be zero-dimensional, or $E$ and $K$ can lie on a common line. I prove that these are the only obstructions: if $\dim_H E > 0$, and $E$ does not lie on a line, then there exists a point in $E$ from which a $(\dim_H K)/2$ dimensional part of $K$ is visible. Applying the result with $E = K$ gives the following corollary: if $K \subset \mathbb{R}^2$ is Borel set, which does not lie on a line, then the set of directions spanned by $K$ has Hausdorff dimension at least $(\dim_H K)/2$.

1. INTRODUCTION

This paper studies visibility and radial projections in the plane. Given $p \in \mathbb{R}^2$, define the radial projection $\pi_p : \mathbb{R}^2 \setminus \{p\} \to S^1$ by

$$\pi_p(q) = \frac{p - q}{|p - q|}.$$ 

A Borel set $K \subset \mathbb{R}^2$ will be called

- invisible from $p$, if $\mathcal{H}^1(\pi_p(K \setminus \{p\})) = 0$, and
- totally invisible from $p$, if $\dim_H \pi_p(K \setminus \{p\}) = 0$.

Above, $\dim_H$ and $\mathcal{H}^s$ stand for Hausdorff dimension and $s$-dimensional Hausdorff measure, respectively. I will only consider Hausdorff dimension in this paper, as many of the results below would be much easier for box dimension. The study of (in-)visibility has a long tradition in geometric measure theory. For many more results and questions than I can introduce here, see Section 6 of Mattila’s survey [8]. The basic question is the following: given a Borel set $K \subset \mathbb{R}^2$, how large can the sets

$$\text{Inv}(K) = \{p \in \mathbb{R}^2 : K \text{ is invisible from } p\}$$

and

$$\text{Inv}_T(K) := \{p \in \mathbb{R}^2 : K \text{ is strongly invisible from } p\}$$

be? Clearly $\text{Inv}_S(K) \subset \text{Inv}(K)$, and one generally expects $\text{Inv}_S(K)$ to be significantly smaller than $\text{Inv}(K)$. The existing results fall roughly into the following three categories:

1. What happens if $\dim_H K > 1$?
2. What happens if $\dim_H K \leq 1$?
3. What happens if $0 < \mathcal{H}^1(K) < \infty$?

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Cases (1) and (3) are the most classical, having already been studied in the 1954 paper [6] of Marstrand. Given $s > 1$, Marstrand proved that any Borel set $K \subset \mathbb{R}^2$ with $0 < \mathcal{H}^s(K) < 1$ is visible (that is, not invisible) from Lebesgue almost every point $p \in \mathbb{R}^2$, and also from $\mathcal{H}^s$ almost every point $p \in K$. Unifying Marstrand’s results, the following sharp bound was recently established by Mattila and the author in [9] and [10]:

$$\dim_H \text{Inv}(K) \leq 2 - s, \quad \dim_H K = s > 1. \quad (1.1)$$

The visibility of sets $K$ in Case (3) depends on their rectifiability. It is easy to show that 1-rectifiable sets, which are not $\mathcal{H}^1$ almost surely covered by a single line, are visible from all points in $\mathbb{R}^2$, with possibly one exception, see [11]. On the other hand, if $K \subset \mathbb{R}^2$ is purely 1-unrectifiable, then the sharp bound

$$\dim_H [\mathbb{R}^2 \setminus \text{Inv}(K)] = \dim_H \{p \in \mathbb{R}^2 : K \text{ is visible from } p\} \leq 1.$$  

was obtained by Marstrand, building on Besicovitch’s projection theorem. For generalisations, improvements and constructions related to the bound above, see [7, Theorem 5.1], and [3, 4]. Marstrand raised the question – which remains open to the best of my knowledge – whether it is possible that $\mathcal{H}^1(\mathbb{R}^2 \setminus \text{Inv}(K)) > 0$; in particular, can a purely 1-unrectifiable set be visible from a positive fraction of its own points? For purely 1-unrectifiable self-similar sets $K \subset \mathbb{R}^2$ one has $\text{Inv}(K) = \mathbb{R}^2$, as shown by Simon and Solomyak [13].

Case (3) has received less attention. To simplify the discussion, assume that $\dim_H K = 1$ and $\mathcal{H}^1(K) = 0$, so that the considerations of Case (3) no longer apply, and $\text{Inv}(K) = \mathbb{R}^2$. Then, the relevant question becomes the size of $\text{Inv}_T(K)$. The radial projections $\pi_p$ fit the influential generalised projections framework of Peres and Schlag [12], so one should start by checking, what bounds follow from [12, Theorem 7.3]. If $K \subset \mathbb{R}^2$ is a Borel set with arbitrary dimension $s \in [0, 2]$, then it follows from [12, Theorem 7.3] that

$$\dim_H \text{Inv}_T(K) = \dim_H \{p \in \mathbb{R}^2 : \dim_H \pi_p(K \setminus \{p\}) < \dim_H K/2\} \leq 2 - s. \quad (1.2)$$

When $s > 1$, the bound (1.2) is a weaker version of (1.1), but the benefit of (1.2) is that it holds without any restrictions on $s$. In particular, if $s = 1$, one obtains

$$\dim_H \text{Inv}_T(K) \leq 1. \quad (1.3)$$

This bound is sharp, and quite trivially so: consider the case, where $K$ lies on a single line $\ell \subset \mathbb{R}^2$. Then, $\text{Inv}_T(K) = \ell$. The starting point for this paper was the question: are there essentially different examples manifesting the sharpness of (1.3)? The answer turns out to be negative in a very strong sense. Here are the main results of the paper:

**Main Theorem 1.4 (Weak version).** Assume that $K \subset \mathbb{R}^2$ is a Borel set with $\dim_H K > 0$. Then, at least one of the following holds:

- $\dim_H \text{Inv}_T(K) = 0$.
- $\text{Inv}_T(K)$ is contained on a line.

In fact, more is true. For $K \subset \mathbb{R}^2$, define

$$\text{Inv}_{1/2}(K) := \left\{p \in \mathbb{R}^2 : \dim_H \pi_p(K \setminus \{p\}) < \frac{\dim_H K}{2}\right\}.$$ 

Then, if $\dim_H K > 0$, one evidently has $\text{Inv}_T(K) \subset \text{Inv}_{1/2}(K) \subset \text{Inv}(K)$. 
Main Theorem 1.5 (Strong version). Theorem 1.4 holds with $\text{Inv}_T(K)$ replaced by $\text{Inv}_{1/2}(K)$. That is, if $E \subset \mathbb{R}^2$ is a Borel set with $\text{dim}_H E > 0$, not contained on a line, then there exists $p \in E$ such that $\text{dim}_H \pi_p(K \setminus \{p\}) \geq (\text{dim}_H K)/2$.

Remark 1.6. A closely related result is Theorem 1.6 in the paper [1] of Bond, Łaba and Zahl; with some imagination, Theorem 1.6(a) in [1] can be viewed as a “single scale” variant of Theorem 1.5. As far as I can tell, proving the Hausdorff dimension statement in this context presents a substantial extra challenge, so Theorem 1.5 is not easily implied by the results in [1].

Example 1.7. Figure 1 depicts the main challenge in the proofs of Theorems 1.4 and 1.5. The set $E$ has $\text{dim}_H E > 0$, and consists of something inside a narrow tube $T$, plus a point $p \not\in T$. Then, Theorem 1.4 states that $E \not\subset \text{Inv}_T(K)$ for any compact set $K \subset \mathbb{R}^2$ with $\text{dim}_H K > 0$. So, in order to find a counterexample to Theorem 1.5, all one needs to do is find $K$ by a standard “Venetian blind” construction, in such a way that $\text{dim}_H K > 0$ and $\text{dim}_H \pi_q(K) = 0$ for all $q \in E$. The first steps are obvious: to begin with, require that $K \subset T^*$ for another narrow tube parallel to $T$, see Figure 1. Then $\pi_q(K)$ is small for all $q \in T$. To handle the remaining point $p \in E$, split the contents of $T^*$ into a finite collection of new narrow tubes in such a way that $\pi_p(K)$ is small. In this manner, $\pi_q(K)$ can be made arbitrarily small for all $q \in E$ (in the sense of $\epsilon$-dimensional Hausdorff content, for instance, for any prescribed $\epsilon > 0$). It is quite instructive to think, why the construction cannot be completed: why cannot the “Venetian blinds” be iterated further (for both $E$ and $K$) so that, at the limit, $\text{dim}_H \pi_q(K) = 0$ for all $q \in E$?

Theorem 1.5 has the following immediate consequence:

Corollary 1.8 (Corollary to Theorem 1.5). Assume that $K \subset \mathbb{R}^2$ is a Borel set, not contained on a line. Then the set of unit vectors spanned by $K$, namely

$$S(K) := \left\{ \frac{p-q}{|p-q|} \in S^1 : p, q \in K \text{ and } p \neq q \right\},$$

satisfies $\text{dim}_H S(K) \geq \frac{\text{dim}_H K}{2}$.

Proof. If $\text{dim}_H K = 0$, there is nothing to prove. Otherwise, Theorem 1.5 implies that $K \not\subset \text{Inv}_{1/2}(K)$, whence $\text{dim}_H S(K) \geq \text{dim}_H \pi_p(K \setminus \{p\}) \geq (\text{dim}_H K)/2$ for some $p \in K$. \qed

Corollary 1.8 is probably not sharp, and the following conjecture seems plausible:

Conjecture 1.9. Assume that $K \subset \mathbb{R}^2$ is a Borel set, not contained on a line. Then $\text{dim}_H S(K) = \min\{\text{dim}_H K, 1\}$.
This follows from Marstrand’s result, discussed in Case (1) above, when \( \dim H K > 1 \). For \( \dim H K \leq 1 \), Conjecture 1.9 is closely connected with continuous sum-product problems, which means that significant improvements over Corollary 1.8 will, most likely, require new technology. An \( \epsilon \)-improvement may be possible, combining the proof below with ideas from the paper [5] of Katz and Tao, and using the discretised sum-product theorem of Bourgain [2].

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2. Proofs

If \( \ell \subset \mathbb{R}^2 \) is a line, I denote by \( T(\ell, \delta) \) the open (infinite) tube of width \( 2\delta \), with \( \ell \) "running through the middle", that is, \( \text{dist}(\ell, \mathbb{R}^2 \setminus T(\ell, \delta)) = \delta \). The notation \( B(x, r) \) stands for a closed ball with centre \( x \in \mathbb{R}^2 \) and radius \( r > 0 \). The notation \( A \lesssim B \) means that there is an absolute constant \( C \geq 1 \) such that \( A \leq CB \).

Lemma 2.1. Assume that \( \mu \) is a Borel probability measure on \( B(0, 1) \subset \mathbb{R}^2 \), and \( \mu(\ell) = 0 \) for all lines \( \ell \subset \mathbb{R}^2 \). Then, for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( \mu(T(\ell, \delta)) \leq \epsilon \) for all lines \( \ell \subset \mathbb{R}^2 \).

Proof. Assume not, so there exists \( \epsilon > 0 \), a sequence of positive numbers \( \delta_1 > \delta_2 > \ldots > 0 \), and a sequence of lines \( \{\ell_i\}_{i \in \mathbb{N}} \subset \mathbb{R}^2 \) with \( \mu(T(\ell_i, \delta_i)) \geq \epsilon \). Since \( \text{spt} \mu \subset B(0, 1) \), one has \( \ell_i \cap B(0, 1) \neq \emptyset \) for all \( i \in \mathbb{N} \). Consequently, there exists a subsequence \( (i_j)_{j \in \mathbb{N}} \), and a line \( \ell \subset \mathbb{R}^2 \) such that \( \ell_j \to \ell \) in the Hausdorff metric. Then, for any given \( \delta > 0 \), there exists \( j \in \mathbb{N} \) such that

\[
B(0, 1) \cap T(\ell_{i_j}, \delta_{i_j}) \subset T(\ell, \delta),
\]

so that \( \mu(T(\ell, \delta)) \geq \epsilon. \) It follows that \( \mu(\ell) \geq \epsilon \), a contradiction. \( \square \)

The following lemma contains most of the proof of Theorem 1.5:

Lemma 2.2. Assume that \( \mu, \nu \) are Borel probability measures with compact supports \( K, E \subset B(0, 1) \), respectively. Assume that both measures \( \mu \) and \( \nu \) satisfy a Frostman condition with exponents \( \kappa_\mu, \kappa_\nu \in (0, 2] \), respectively:

\[
\mu(B(x, r)) \leq C_\mu r^{\kappa_\mu} \quad \text{and} \quad \nu(B(x, r)) \leq C_\nu r^{\kappa_\nu}
\]

for all balls \( B(x, r) \subset \mathbb{R}^2 \), and for some constants \( C_\mu, C_\nu \geq 1 \). Assume further that \( \mu(\ell) = 0 \) for all lines \( \ell \subset \mathbb{R}^2 \). Fix also

\[
0 < \tau < \frac{\kappa_\mu}{2} \quad \text{and} \quad \epsilon > 0,
\]

and write \( \delta_k := 2^{-(1+\epsilon)k} \).

Then, there exist numbers \( \beta = \beta(\kappa_\mu, \kappa_\nu, \tau) > 0 \), \( \eta = \eta(\epsilon, \kappa_\mu, \kappa_\nu, \tau) > 0 \), and an index \( k_0 = k_0(\epsilon, \mu, \kappa_\nu, \tau) \in \mathbb{N} \) with the following properties. For all \( k \geq k_0 \), there exist

(a) compact sets \( K \supset K_{k_0} \supset K_{k_0+1} \ldots \) with

\[
\mu(K_k) \geq 1 - \sum_{k_0 \leq j < k} \left(\frac{1}{4}\right)^{j-k_0+1} \geq \frac{1}{2},
\]

(2.4)
(b) compact sets $E \supset E_{k_0} \supset E_{k_0+1} \ldots$ with \( \nu(E_k) \geq k^3 \)
with the following property: if $k > k_0$, $p \in E_k$, and $T(\ell_1, \delta_k), \ldots, T(\ell_N, \delta_k)$ is a family of tubes of cardinality $N \leq \delta_k^{-1}$, each containing $p$, then
\[
\mu \left( K_k \cap \bigcup_{j=1}^N T(\ell_j, \delta_k) \right) \leq \delta_k^\eta.
\] (2.5)

Remark 2.6. The index $k_0$ can be chosen as large as desired; this will be clear from the proof below. It will also be used on many occasions, without separate remark, that $\delta_k$ can be assumed very small for all $k \geq k_0$.

Proof. The proof is by induction, starting at the largest scale $k_0$, which will be presently defined. Fix $\eta = \eta(\epsilon, \kappa_\mu, \kappa_\nu, \tau) > 0$ and
\[
\Gamma = \Gamma(\epsilon, \kappa_\mu, \kappa_\nu, \tau) \in \mathbb{N}
\] (2.7)
The number $\Gamma$ will be specified at the very end of the proof, right before (2.32), and there will be several requirements for the number $\eta$, see (2.22), (2.28), and (2.31). Applying Lemma 2.1, first pick an index $k_1 = k_1(\epsilon, \mu, \kappa_\nu, \tau) \in \mathbb{N}$ such that $\mu(T(\ell, \delta_{k_1})) \leq (\frac{1}{4})^{\Gamma+1}$ for all tubes $T(\ell, \delta_{k_1}) \subset \mathbb{R}^2$, and
\[
\delta_{k_1-\Gamma}^\eta \leq (\frac{1}{4})^{k_1-\Gamma+1}, \quad k \geq k_1.
\] (2.8)
Set $k_0 := k_1 + \Gamma$. Then, the following holds for all $k \in \{k_0, \ldots, k_0 + \Gamma\}$. For any subset $K' \subset K$, and any tube $T(\ell, \delta_{k-\Gamma}) \subset \mathbb{R}^2$, one has
\[
\mu(K' \cap T(\ell, \delta_{k-\Gamma})) \leq \mu(T(\ell, \delta_{k_1})) \leq (\frac{1}{4})^{\Gamma+1} \leq (\frac{1}{4})^{k-k_0+1}.
\] (2.9)
Define
\[
K_k := K \quad \text{and} \quad E_k := E, \quad k_1 \leq k \leq k_0.
\]
(The definitions of $E_k, K_k$ for $k_1 \leq k < k_0$ are only given for notational convenience.)

I start by giving an outline of how the induction will proceed. Assume that, for a certain $k \geq k_0$, the sets $K_k$ and $E_k$ have been constructed such that
(i) the condition (2.9) is satisfied with $K' = K_k$, and for all tubes $T(\ell, \delta_{k-\Gamma})$ with $T(\ell, \delta_{k-\Gamma}) \cap E_{k-\Gamma} \neq \emptyset$.
(ii) $K_k$ and $E_k$ satisfy the measure lower bounds (a) and (b) from the statement of the lemma.

Under the conditions (i)-(ii), I claim that it is possible to find subsets $K_{k+1} \subset K_k$ and $E_{k+1} \subset E_k$, satisfying (ii) at level $k + 1$, and also the non-concentration condition (2.5) at level $k + 1$. This is why (2.5) is only claimed to hold for $k > k_0$, and no one is indeed claiming that it holds for the sets $K_0$ and $E_0$. These sets satisfy (i), however, which should be viewed as a weaker substitute for (2.5) at level $k$, which is just strong enough to guarantee (2.5) at level $k + 1$. There is one obvious question at this point: if (i) at level $k$ gives (2.5) at level $k + 1$, then where does one get (i) back at level $k + 1$?

If $k + 1 \in \{k_0, \ldots, k_0 + \Gamma\}$, the condition (i) is simply guaranteed by the choice of $k_0$ (one does not even need to assume that $T(\ell, \delta_{k-\Gamma}) \cap E_{k-\Gamma} \neq \emptyset$). For $k + 1 > k_0 + \Gamma$, this is no longer true. However, for $k + 1 > \Gamma + k_0$, one has $k + 1 - \Gamma > k_0$, and thus $K_{k+1-\Gamma}$ and $E_{k+1-\Gamma}$ have already been constructed to satisfy (2.5). In particular, if $E_{k+1-\Gamma} \cap T(\ell, \delta_{k+1-\Gamma}) \neq \emptyset$, then
\[
\mu(K_{k+1} \cap T(\ell, \delta_{k+1-\Gamma})) \leq \mu(K_{k+1-\Gamma} \cap T(\ell, \delta_{k+1-\Gamma})) \leq \delta_{k+1-\Gamma}^\eta \leq (\frac{1}{4})^{(k+1)-k_0+1}
\] (2.10)
by (2.5) and (2.8). This means that (i) is satisfied at level \( k + 1 \), and the induction may proceed.

So, it remains to prove that (i)–(ii) at level \( k \) imply (ii) and (2.5) at level \( k + 1 \). To avoid clutter, I write

\[
\delta := \delta_{k+1}.
\]

Assume that the sets \( K_k, E_k \) have been constructed for some \( k \geq k_0 \), satisfying (i)–(ii). The main task is to understand the structure of the set of points \( p \in E_k \) for which (2.5) fails, and these points are denoted by \( \text{Bad}_k \). More precisely, \( p \in \text{Bad}_k \), if and only if \( p \in E_k \), and there exist \( N \leq \delta^{-T} \) tubes \( T(\ell_1, \delta), \ldots, T(\ell_N, \delta) \), each containing \( p \), such that

\[
\mu \left( K_k \cap \bigcup_{j=1}^{N} T(\ell_j, \delta) \right) > \delta^n. \tag{2.11}
\]

Note that if \( \text{Bad}_k = \emptyset \), then one can simply define \( E_{k+1} := E_k \) and \( K_{k+1} := K_k \), and (ii) and (2.5) (at level \( k + 1 \)) are clearly satisfied.

Instead of analysing \( \text{Bad}_k \) directly, it is useful to split it up into "directed" pieces, and digest the pieces individually. To make this precise, let \( S \) be the "space of directions"; for concreteness, I identify \( S \) with the upper half of the unit circle. Then, if \( T = (\ell, \delta) \subset \mathbb{R}^2 \) is a tube, I denote by \( \text{dir}(T) \) the unique vector \( e \in S \) such that \( \ell \parallel e \).

Recall the small parameter \( \eta > 0 \), and partition \( S \) into \( D = \delta^{-\eta} \) arcs \( J_1, \ldots, J_D \) of length \( \sim \delta^{\eta}. \)

For \( d \in \{1, \ldots, D\} \) fixed ("d" for "direction"), consider the set \( \text{Bad}_d^k \): it consists of those points \( p \in E_k \) such that there exist \( N \leq \delta^{-T} \) tubes \( T(\ell_1, \delta), \ldots, T(\ell_N, \delta) \), each containing \( p \), with \( \text{dir}(T(\ell_i, \delta)) \in J_d \) and satisfying

\[
\mu \left( K_k \cap \bigcup_{j=1}^{N} T(\ell_j, \delta) \right) > \delta^{2n}. \tag{2.12}
\]

Since the direction of every possible tube in \( \mathbb{R}^2 \) belongs to one of the arcs \( J_d \), and there are only \( D = \delta^{-\eta} \) arcs in total, one has

\[
\text{Bad}_k \subset \bigcup_{d=1}^{D} \text{Bad}_d^k. \tag{2.13}
\]

The next task is to understand the structure of \( \text{Bad}_d^k \) for a fixed direction \( d \in \{1, \ldots, D\} \).

I claim that \( \text{Bad}_d^k \) looks like a garden of flowers, with all the petals pointing in direction \( J_d \), see Figure 2 for a rough idea. To make the statement more precise, I introduce an additional piece of notation. Fox \( X \subset K_k \), let \( B_d(X) \) consist of those points \( p \in E_k \) such that \( X \) can be covered by \( N \leq \delta^{-T} \) tubes \( T(\ell_1, \delta), \ldots, T(\ell_N, \delta) \), with directions \( \text{dir}(T(\ell_i, \delta)) \in J_d \), and each containing \( p \). Then, note that

\[
\text{Bad}_d^k = \{ p \in E_k : \exists X \subset K_k \text{ with } \mu(X) > \delta^{2n} \text{ and } p \in B_d(X) \}. \tag{2.13}
\]

The sets \( B_d(X) \) also has the trivial but useful property that

\[
X \subset X' \subset K_k \implies B_d(X') \subset B_d(X). \]

\[\hfill^{1}\text{Here, it might be better style to pick another letter, say } \alpha > 0, \text{ in place of } \eta, \text{ since the two parameters play slightly different roles in the proof. Eventually, however, one would end up considering } \min\{\eta, \alpha\}, \text{ and it seems a bit cleaner to let } \eta > 0 \text{ be a "jack of all trades" from the start.}\]
There are two steps in establishing the “garden” structure of $\text{Bad}_d^k$: first, one needs to find the “flowers”, and second, one needs to check that the sets obtained actually look like flowers in a non-trivial sense. I start with the former task. Assuming that $\text{Bad}_d^k \neq \emptyset$, pick any point $p_1 \in \text{Bad}_d^k$, and an associated subset $X_1 \subset K_k$ with $\mu(X_1) > \delta^{2\eta}$ and $p_1 \in B_d(X_1)$.

Then, assume that $p_1, \ldots, p_m \in \text{Bad}_d^k$ and $X_1, \ldots, X_m$ have already been chosen with the properties above, and further satisfying

$$\mu(X_i \cap X_j) \leq \delta^{4\eta}/2, \quad 1 \leq i < j \leq m. \quad (2.14)$$

Then, see if there still exists a subset $X_{m+1} \subset K_k$ with the following three properties: $\mu(X_{m+1}) > \delta^{2\eta}$, $B_d(X_{m+1}) \neq \emptyset$, and $\mu(X_{m+1} \cap X_i) \leq \delta^{4\eta}/2$ for all $1 \leq i \leq m$. If such a set no longer exists, stop; if it does, pick $p_{m+1} \in B_d(X_{m+1})$, and add $X_{m+1}$ to the list.

It follows from the “competing” conditions $\mu(X_i) > \delta^{2\eta}$, and (2.14), that the algorithm needs to terminate in at most

$$M \leq 2\delta^{-4\eta} \quad (2.15)$$

Indeed, assume that the sets $X_1, \ldots, X_M$ have already been constructed, and consider the following chain of inequalities:

$$\frac{1}{M} + \frac{1}{M(M-1)} \sum_{i_1 \neq i_2} \mu(X_{i_1} \cap X_{i_2}) \geq \frac{1}{M^2} \sum_{i_1, i_2=1}^{M} \mu(X_{i_1} \cap X_{i_2})$$

$$= \frac{1}{M^2} \int \sum_{i_1, i_2=1}^{M} 1_{X_{i_1} \cap X_{i_2}}(x) \, d\mu(x)$$

$$= \frac{1}{M^2} \int [\text{card}\{1 \leq i \leq M : x \in X_i\}]^2 \, d\mu(x)$$

$$\geq \frac{1}{M^2} \left( \int \text{card}\{1 \leq i \leq M : x \in X_i\} \, d\mu(x) \right)^2$$

$$= \frac{1}{M^2} \left( \sum_{i=1}^{M} \mu(X_i) \right)^2 > \delta^{4\eta}. \quad (2.16)$$
Thus, if $M > 2\delta^{-4\eta}$, there exists a pair $X_{i_1}, X_{i_2}$ with $i_1 \neq i_2$ such that $\mu(X_{i_1} \cap X_{i_2}) > \delta^{4\eta}/2$, and the algorithm has already terminated earlier. This proves (2.15).

With the sets $X_1, \ldots, X_M$ now defined, write

$$B'_d(X_j) := \{ p \in E_k : \exists X' \subset X_j \text{ with } \mu(X') > \delta^{4\eta}/2 \text{ and } p \in B_d(X') \}.$$  

I claim that

$$\text{Bad}^d_k \subset \bigcup_{j=1}^M B'_d(X_j). \tag{2.16}$$

Indeed, if $p \in \text{Bad}^d_k$, then $p \in B_d(X)$ for some $X \subset K_k$ with $\mu(X) > \delta^{2\eta}$ by (2.13).

It follows that

$$\mu(X \cap X_j) > \delta^{4\eta}/2 \tag{2.17}$$

for one of the sets $X_j$, $1 \leq j \leq M$, because either $X \in \{X_1, \ldots, X_M\}$, and (2.17) is clear (all the sets $X_j$ even satisfy $\mu(X_j) > \delta^{2\eta}$), or else (2.17) must hold by virtue of $X$ not having been added to the list $X_1, \ldots, X_M$ in the algorithm. But (2.17) implies that $p \in B'_d(X_j)$, since $X' = X \cap X_j \subset X_j$ satisfies $\mu(X') > \delta^{4\eta}/2$ and $p \in B_d(X) \subset B_d(X')$.

According to (2.15) and (2.16) the set $\text{Bad}^d_k$ can be covered by $M \leq 2\delta^{-4\eta}$ sets of the form $B'_d(X_j)$, see Figure 2. These sets are the "flowers", and their structure is explored in the next lemma:

**Lemma 2.18.** The following holds, if $\delta = \delta_{k+1}$ is small enough. For $1 \leq d \leq D$ and $1 \leq j \leq M$ fixed, the set $B'_d(X_j)$ can be covered by $\leq 4\delta^{-8\eta}$ tubes of the form $T = T(\ell_t, \delta^\rho)$, where $\text{dir}(T) \in J_d$, and $\rho = \rho(\kappa, \tau) > 0$. The tubes can be chosen to contain the point $p_j \in B_d(X_j)$.

**Proof.** Fix $1 \leq j \leq M$ and $p \in B'_d(X_j)$. Recall the point $p_j \in B_d(X_j)$ from the definition of $X_j$. By definition of $p \in B'_d(X_j)$, there exists a set $X' \subset X_j$ with $\mu(X') > \delta^{4\eta}/2$ and $p \in B_d(X')$. Unwrapping the definitions further, there exist $N \leq \delta^{-\tau}$ tubes $T(\ell_1, \delta), \ldots, T(\ell_N, \delta)$, the union of which covers $X'$, and each satisfies $\text{dir}(T(\ell_t, \delta)) \in J_d$ and $p \in T(\ell_t, \delta)$. In particular, one of these tubes, say $T_p = T(\ell_t, \delta)$, has

$$\mu(X_j \cap T_p) \geq \mu(X' \cap T_p) \geq \mu(X') \cdot \delta^\tau \geq \delta^{4\eta+\tau}/2 \geq \delta^{8\eta+\tau}/4. \tag{2.19}$$

(The final inequality is for just a triviality at this point, but is useful for later technical purposes later.) Here comes perhaps the most basic geometric observation in the proof: if the measure lower bound (2.19) holds for some $\delta$-tube $T$ – this time $T_p$ – and a sufficiently small $\eta > 0$ (crucially so small that $8\eta + \tau < \kappa/2$), then the whole set $B_d(X_j)$ is actually contained in a neighbourhood of $T$, called $T^*$, because $X_j \cap T$ is so difficult to cover by $\delta$-tubes centred at points outside $T^*$, see Figure 3. In particular, in the present case,

$$p_j \in B_d(X_j) \subset T(\ell_t, \delta^{4\rho}) =: \text{T}_p^* \tag{2.20}$$

for a suitable constant $\rho = \rho(\kappa, \tau) > 0$, specified in (2.22). To see this formally, pick $q \in B(0, 1) \setminus \text{T}_p^*$, and argue as follows to show that $q \notin B_d(X_j)$. First, any $\delta$-tube $T$ containing $q$, and intersecting $T_p \cap B(0, 1)$, makes an angle of at least $\geq \delta^4\rho$ with $T_p$. It follows that

$$\text{diam}(T \cap T_p \cap B(0, 1)) \lesssim \delta^{1-4\rho},$$
and consequently $\mu(T \cap T_p \cap B(0, 1)) \lesssim C_\mu \delta^{\kappa_\mu(1-4\rho)}$. So, in order to cover $X_j \cap T_p$ (let alone the whole set $X_j$) it takes by (2.19) at least

$$\mu(X_j \cap T_p) \gtrsim \frac{\delta^{8\eta+\tau-\kappa_\mu(1-4\rho)}}{C_\mu} \gtrsim \frac{\delta^{8\eta-\kappa_\mu/2+8\rho}}{C_\mu}$$

(2.21)
tubes $T$ containing $q$. But if

$$0 < 8\eta < \frac{\kappa_\mu - \tau}{2} \quad \text{and} \quad 8\rho = \frac{\kappa_\mu - \tau}{2},$$

(2.22)
then the number on the right hand side of (2.21) is far larger than $\delta^{-\tau}$, which means that $q \notin B_d(X_j)$, and proves (2.20).

Recall the statement of the Lemma 2.18, and compare it with the previous accomplishment: (2.20) states that whenever $p \in B'_d(X_j)$, then $p$ lies in a certain tube of width $\delta^{4\rho}$ (namely $T_p$), which has direction in $J_{d_1}$ and also contains $p_j$. This sounds a bit like the statement of the lemma, but there is a problem: in principle, every point $p \in B'(X_j)$ could give rise to a different tube $T_p$. So, it essentially remains to show that all these $\delta^{4\rho}$-tubes $T_p$ can be covered by a small number of tubes of width $\delta^\rho$. To begin with, note that the ball $B_j := B(p_j, \delta^{2\rho})$ can be covered by a single tube of width $\delta^\rho$, in any direction desired. So, to prove the lemma, it remains to cover $B'_d(X_j) \setminus B_j$.

Note that if $p, q$ satisfy $|p - q| \geq \delta^{2\rho}$, then the direction of any $\delta^{4\rho}$-tube containing both $p, q$ lies in a fixed arc $J(p, q) \subset S$ of length $|J(p, q)| \lesssim \delta^{4\rho}/\delta^{2\rho} = \delta^{2\rho}$. As a corollary, the union of all $\delta^{4\rho}$-tubes containing $p, q$, intersected with $B(0, 1)$, is contained in a single tube of width $\sim \delta^{2\rho}$. In particular, this union (still intersected with $B(0, 1)$) is contained in a single $\delta^\rho$-tube, assuming that $\delta > 0$ is small; this tube can be chosen to be a $\delta^\rho$-tube around an arbitrary $\delta^{4\rho}$-tube containing both $p$ and $q$.

The tube-cover of $B'_d(X_j) \setminus B_j$ can now be constructed by adding one tube at a time. First, assume that there is a point $q_1 \in B'_d(X_j) \setminus B_j$, and find a tube $T(\ell_1, \delta^{4\rho})$ containing both $q$ and $p_j$, with direction in $J_{d_1}$; existence follows from (2.20). Add the tube $T(\ell_1, \delta^\rho)$ to the the tube-cover of $B'_d(X_j) \setminus B_j$, and recall from the previous paragraph that $T(\ell_1, \delta^\rho)$ now contains $T \cap B(0, 1)$ for any $\delta^{4\rho}$-tube $T \supset \{q_1, p_j\}$ (of which $T = T(\ell_1, \delta^{4\rho})$ is just one example). Finally, by definition of $q_1 \in B'_d(X_j)$, associate to $q_1$ a subset $X'_1 \subset X_j$ with

$$\mu(X'_1) > \delta^{8\eta}/2 \quad \text{and} \quad q_1 \in B_d(X'_1).$$

(2.23)
Assume that the points \(q_1, \ldots, q_H \in B'_d(X_j) \setminus B_{j_d}\), along with the associated tubes \(\{q_i, p_j\} \subset T(\ell_i, \delta^{\ell \rho}) \subset T(\ell_i, \delta^\rho)\), and subsets \(X'_i \subset X_j\), as in (2.23), have already been constructed. Assume inductively that
\[
\mu(X'_{i_1} \cap X'_{i_2}) \leq \delta^{8\eta}/4, \quad 1 \leq i_1 < i_2 \leq H.
\] (2.24)

To proceed, pick any point \(q_{H+1} \in B'_d(X_j) \setminus B_{j_d}\), and associate to \(q_{H+1}\) a subset \(X'_{H+1} \subset X_j\) with \(\mu(X'_{H+1}) > \delta^{4\rho}/2\) and \(q_{H+1} \in B_d(X'_{H+1})\). Then, test whether (2.24) still holds, that is, whether \(\mu(X'_{H+1} \cap X'_i) \leq \delta^{8\eta}/4\) for all \(1 \leq i \leq H\). If such a point \(q_{H+1}\) can be chosen, run the argument from the previous paragraph, first locating a tube \(T(\ell_{H+1}, \delta^{4\rho})\) containing both \(q_{H+1}\) and \(p_j\), with direction in \(J_{d_H}\), and finally adding \(T(\ell_{H+1}, \delta^\rho)\) to the tube-cover under construction.

The "competing" conditions \(\mu(X'_i) > \delta^{4\eta}/2\), and (2.24), guarantee that the algorithm terminates in
\[
H \leq 4\delta^{-8\eta}
\] steps. The argument is precisely the same as used to prove (2.15), so I omit it. Once the algorithm has terminated, I claim that all points of \(B'_d(X_j) \setminus B_{j_d}\) are covered by the tubes \(T(\ell_i, \delta^\rho)\), with \(1 \leq i \leq H\). To see this, pick \(q \in B'_d(X_j) \setminus B_{j_d}\), and a subset \(X' \subset X_j\) with \(\mu(X') > \delta^{4\eta}/2\), and \(q \in B_d(X')\). Since the algorithm had already terminated, it must be the case that
\[
\mu(X' \cap X'_i) > \delta^{8\eta}/4
\] for some index \(1 \leq i \leq H\). Since \(X'' := X' \cap X'_i \subset X'\) and consequently \(q \in B_d(X'')\), one can find a tube \(T_q = T(\ell_q, \delta) \ni q\) with \(\text{dir}(T_q) \in J_{d_H}\), and satisfying
\[
\mu(X'_i \cap T_q) \geq \mu(X'' \cap T_q) \geq \mu(X'') \cdot \delta^\tau > \delta^{8\eta+\tau}/4.
\]

This lower bound is precisely the same as in (2.19). Hence, it follows from the same argument, which gave (2.20), that
\[
q_i \in B_d(X'_i) \subset T(\ell_q, \delta^{4\rho}).
\]
Since \(X'_i \subset X_j\), also \(p_j \in B_d(X_j) \subset B_d(X'_i) \subset T(\ell_q, \delta^{4\rho})\). So,
\[
\{q, q_i, p_j\} \subset B(0, 1) \cap T(\ell_q, \delta^{4\rho}).
\] (2.25)

In particular, \(T(\ell_q, \delta^{4\rho})\) is a \(\delta^{4\rho}\)-tube containing both \(q_i, p_j\), and hence
\[
B(0, 1) \cap T(\ell_q, \delta^{4\rho}) \subset T(\ell_q, \delta^\rho).
\]

Combined with (2.25), this yields \(q \in T(\ell_i, \delta^\rho)\), as claimed. This concludes the proof of Lemma 2.18.

Combining (2.15)-(2.16) with Lemma 2.18, the structural description of \(\text{Bad}^\delta_k\) is now complete: \(\text{Bad}^\delta_k\) is covered by
\[
\leq M \cdot 4\delta^{-8\eta} \leq 8\delta^{-12\eta}
\] (2.26)
tubes of width \(\delta^\rho\), with directions in \(J_{d_H}\). For non-adjacent \(d_1, d_2 \in \{1, \ldots, D\}\) (the ordering of indices corresponds to the ordering of the arcs \(J_{d} \subset S\)), the covering tubes are then fairly transversal. This is can be used to infer that most point in \(E_k\) do not lie in many different sets \(\text{Bad}^\delta_k\). Indeed, consider the set \(\text{BadBad}^\delta_k\) of those points in \(\mathbb{R}^2\), which lie in (at least) two sets \(\text{Bad}^\delta_k\) and \(\text{Bad}^\delta_k\) with \(|d_2 - d_1| > 1\). By Lemma 2.18, such points lie in
the intersection of some pair of tubes \( T_1 = T(\ell_1, \delta^\rho) \) and \( T_2 = T(\ell_2, \delta^\rho) \) with \( \text{dir}(T_i) \in J_d \).
The angle between these tubes is \( \gtrsim \delta^n \), whence
\[
\text{diam}(T_1 \cap T_2) \lesssim \delta^{\rho - \eta},
\]
and consequently
\[
\nu(T_1 \cap T_2) \lesssim C_\rho \delta^{\kappa_\nu(\rho - \eta)} \leq C_\rho \delta^{\kappa_\nu \rho - 2\eta}.
\] (2.27)
For \( d \in \{1, \ldots, D\} \) fixed, there correspond \( \lesssim \delta^{-12\eta} \) tubes in total, as pointed out in (2.26).
So, the number of pairs \( T_1, T_2 \), as above, is bounded by
\[
\lesssim D^2 \cdot \delta^{-24\eta} \leq \delta^{-26\eta}.
\]
Consequently, by (2.27),
\[
\nu(\text{BadBad}_k) \lesssim C_\rho \delta^{-28\eta + \kappa_\nu \rho}.
\]
This upper bound is far smaller than \( \nu(E_k)/2 \geq \delta_k^{\rho}/2 \), assuming that
\[
0 < \beta < \kappa_\nu \rho - 28\eta.
\] (2.28)
Given that \( 28\eta < \kappa_\nu \rho/2 \), one is free to make such an assumption on \( \beta \) (it holds for \( k = k_0 \), since \( \nu(E_{k_0}) = 1 \)), but the smaller \( \beta \) is, the more difficult it becomes be to ensure that
\[
\nu(E_{k+1}) \geq \delta_{k+1}^{\beta}.
\]
To see that this can be done, start by writing \( G_k := E_k \setminus \text{BadBad}_k \), so that
\[
\nu(G_k) \geq \nu(E_k)/2 \geq \delta_k^{\rho}/2
\]
by the choice of \( \beta \). Now, either
\[
\nu(G_k \cap \text{Bad}) \geq \nu(G_k)/2 \quad \text{or} \quad \nu(G_k \cap \text{Bad}) < \nu(G_k)/2.
\] (2.29)
The latter case is quick and easy: set \( E_{k+1} := G_k \setminus \text{Bad} \) and \( K_{k+1} := K_k \). Then \( \nu(E_{k+1}) \geq \nu(E_k)/4 \geq \delta_{k+1}^{\beta} \) (assuming that \( k \geq k_0 \) is large enough). Moreover, the set \( E_{k+1} \) no longer contains any points in \( \text{Bad} \), so (2.5) is satisfied at level \( k + 1 \), by the very definition of \( \text{Bad}_k \), see (2.11).
So, it remains to treat the first case in (2.29). Start by recalling from (2.12) that \( \text{Bad}_k \) is covered by the sets \( \text{Bad}_k^d \), \( 1 \leq d \leq D \), so
\[
\nu(G_k \cap \text{Bad}_k^d) \geq \nu(G_k)/2^D \geq \frac{\delta_k^{\rho} \delta_k^\beta}{4} = \frac{\delta_k^{\rho + \beta/(1 + \epsilon)}}{4}.
\] (2.30)
for some fixed \( d \in \{1, \ldots, D\} \). Then, recall from (2.26) that \( \text{Bad}_k^d \) can be covered by \( \leq 8\delta^{-12\eta} \) tubes of the form \( T(\ell, \delta^\rho) \), with directions in \( J_d \). It follows that there exists a fixed tube \( T_0 = T(\ell_0, \delta^\rho) \) such that
\[
\text{dir}(T_0) \in J_d \quad \text{and} \quad \nu(G_k \cap T_0 \cap \text{Bad}_k^d) \geq \frac{\delta_k^{13\eta + \beta/(1 + \epsilon)}}{32}.
\] (2.30)
So, to ensure \( \nu(G_k \cap T_0 \cap \text{Bad}_k^d) \geq \delta^\beta \), choose \( \eta > 0 \) so small that
\[
13\eta + \beta/(1 + \epsilon) < \beta.
\] (2.31)
To convince the reader that there is no circular reasoning at play, I gather here all the requirements for \( \beta \) and \( \eta \) (harvested from (2.22), (2.28), and (2.31)):
\[
0 < \beta < \frac{\kappa_\nu \rho}{2} \quad \text{and} \quad 0 < \eta < \min \left\{ \frac{\kappa_\mu/2 - \tau}{2}, \frac{\kappa_\nu \rho}{56}, \frac{\epsilon \beta}{1 + \epsilon} \right\}
\]
With such choices of \( \beta, \eta \), recalling (2.30), and assuming that \( \delta \) is small enough, the set

\[
E_{k+1} := G_k \cap T_0 \cap \text{Bad}^d_k.
\]

satisfies \( \nu(E_{k+1}) \geq \delta^{\beta} \), which is statement (b) from the lemma. It remains to define \( K_{k+1} \).

To this end, recall that \( T_0 \) is a tube around the line \( \ell_0 \subset \mathbb{R}^2 \). Define

\[
K_{k+1} := K_k \setminus T(\ell_0, \delta^{n/2}).
\]

Then, assuming that \( \eta/2 \) has the form \( \eta/2 = (1+\epsilon)^{-\Gamma-1} \) for an integer \( \Gamma = \Gamma(\epsilon, \kappa_\mu, \kappa_{\nu}, \tau) \in \mathbb{N} \) (this is finally the integer from (2.7)), one has

\[
\delta^{n/2} = \delta_{k-\Gamma}.
\]  

(2.32)

Since \( T(\ell_0, \delta_{k-\Gamma}) \cap E_{k-\Gamma} \neq \emptyset \), it follows from the induction hypothesis (i) that

\[
\mu(K_k \cap T(\ell, \delta_{k-\Gamma})) \leq \left( \frac{1}{3} \right)^{k-k_0+1}.
\]

Consequently,

\[
\mu(K_{k+1}) \geq \mu(K_k) - \left( \frac{1}{3} \right)^{k-k_0+1} \geq 1 - \sum_{k_0 \leq j < k+1} \left( \frac{1}{3} \right)^{j-k_0+1},
\]

which is the desired lower bound from (a) of the statement of the lemma. So, it remains to verify the non-concentration condition (2.5) for \( E_{k+1} \) and \( K_{k+1} \). To this end, pick \( p \in E_{k+1} \). First, observe that every tube \( T = T(\ell_0, \delta) \), which contains \( p \) and has non-empty intersection with \( K_{k+1} \subset B(0,1) \setminus T(\ell_0, \delta^{n/2}) \), forms an angle \( \gtrsim \delta^{n/2} \) with \( T_0 \). In particular, this angle is far larger than \( \delta^n \). Since \( \text{dir}(T_0) \in J_d \) by (2.30), this implies that \( \text{dir}(T) \in J_{d'} \) for some \( |d' - d| > 1 \).

Now, if the non-concentration condition (2.5) still failed for \( p \in E_{k+1} \), there would exist \( N \leq \delta^{-\tau} \) tubes \( T(\ell_1, \delta), \ldots, T(\ell_N, \delta) \), each containing \( p \), and with

\[
\mu \left( K_{k+1} \cap \bigcup_{i=1}^{N} T(\ell_i, \delta) \right) > \delta^n.
\]

By the pigeonhole principle, it follows that the tubes \( T(\ell_i, \delta) \) with \( \text{dir}(T_i) \in J_{d'} \), for some fixed arc \( J_{d'} \), cover a set \( X \subset K_{k+1} \subset K_k \) of measure \( \mu(X) > \delta^{2n} \). This means precisely that \( p \in \text{Bad}^{d'}_k \), and by the observation in the previous paragraph, \( |d' - d| > 1 \). But \( p \in E_{k+1} \subset \text{Bad}^{d'}_k \) by definition, so this would imply that \( p \in \text{BadBad}_k \), contradicting the fact that \( p \in E_{k+1} \subset G_k \). This completes the proof of (2.5), and the lemma.  

The proof of Theorem 1.5 is now quite standard:

Proof of Theorem 1.5. Write \( s := \dim_H K \), and assume that \( s > 0 \) and \( \dim_H E > 0 \). Make a counter assumption: \( E \) is not contained on a line, but \( \dim_H \pi_p(K) < s/2 \) for all \( p \in E \). Then, find \( t < s/2 \), and a positive-dimensional subset \( \tilde{E} \subset E \), not contained on any single line, with \( \dim_H \pi_p(K) \leq t \) for all \( p \in \tilde{E} \) (if your first attempt at \( \tilde{E} \) lies on some line \( \ell \), simply add a point \( p_0 \in \tilde{E} \setminus \ell \) to \( \tilde{E} \), and replace \( t \) by \( \max\{t, \dim_H \pi_{p_0}(K)\} < s/2 \). So, now \( \tilde{E} \) satisfies the same hypotheses as \( E \), but with "\( < s/2 \)" replaced by "\( \leq t < s/2 \)". Thus, without loss of generality, one may assume that

\[
\dim_H \pi_p(K) \leq t < s/2, \quad p \in E.
\]  

(2.33)
Using Frostman’s lemma, pick probability measures \( \mu, \nu \) with \( \text{spt} \mu \subset K \) and \( \text{spt} \nu \subset E \), and satisfying the growth bounds (2.3) with exponents \( 0 < \kappa_\mu < s \) and \( \kappa_\nu > 0 \). Pick, moreover, \( \kappa_\mu \) so close to \( s \) that
\[
\frac{\kappa_\mu}{2} > t. \tag{2.34}
\]
Observe that \( \mu(\ell) = 0 \) for all lines \( \ell \subset \mathbb{R}^2 \). Indeed, if \( \mu(\ell) > 0 \) for some line \( \ell \subset \mathbb{R}^2 \), then there exists \( p \in E \setminus \ell \) by assumption, and
\[
\dim_H \pi_\mu(K) \geq \dim_H \pi_\mu(\text{spt} \ell) \geq \kappa_\mu > t,
\]
violating (2.33) by (2.35). Then, apply Lemma 2.2 to find the parameters \( \beta, \eta > 0, k_0 \in \mathbb{N} \), and the sets \( \text{spt} \mu \supset K_{k_0} \supset K_{k_0+1} \supset \ldots \) and \( \text{spt} \nu \supset E_{k_0} \supset E_{k_0+1} \supset \ldots \) Write
\[
K' := \bigcap_{k \geq k_0} K_k \subset \text{spt} \mu \quad \text{and} \quad E' := \bigcap_{k \geq k_0} E_k \subset \text{spt} \nu.
\]
Both sets are non-empty and compact (being intersections of nested sequences of non-empty compact sets), \( \mu(K') \geq \frac{1}{2} \), and \( K' \cap E' = \emptyset \). Pick \( p \in E' \). I claim that
\[
\dim_H \pi_\mu(K') \geq \frac{\tau}{(1+\epsilon)^2}, \tag{2.36}
\]
which violates (2.33) by (2.35). If not, cover \( \pi_\mu(K) \) by efficiently by arcs \( J_1, J_2, \ldots \) of lengths restricted to the values \( \delta_k = 2^{-((1+\epsilon)k_0)} \), with \( k \geq k_0 \). More precisely: assuming that (2.36) fails, start with an arbitrary efficient cover \( \tilde{J}_1, \tilde{J}_2, \ldots \) by arcs of length \( |\tilde{J}_i| \leq \delta_{k_0} \), satisfying
\[
\sum_{j \geq 1} |\tilde{J}_j|^{\tau/(1+\epsilon)^2} \leq 1.
\]
Then, replace each \( \tilde{J}_j \) by the shortest concentric arc \( J_j \supset \tilde{J}_j \), whose length is of the form \( \delta_k \). Note that \( \ell(J_j) \leq \ell(\tilde{J}_j)^{(1+\epsilon)^{-1}} \), so that
\[
\sum_{j \geq 1} |J_j|^{\tau/(1+\epsilon)} \leq \sum_{j \geq 1} |\tilde{J}_j|^{\tau/(1+\epsilon)^2} \leq 1.
\]
The arcs \( J_1, J_2, \ldots \) now cover \( \pi_\mu(K') \), and there are \( \leq \delta_k^{-(1+\epsilon)} \) arcs of any fixed length \( \delta_k \). Since \( p \notin K' \), for every \( k \geq k_0 \) there exists a collection of tubes \( T_k \) of the form \( T(\ell, \delta_k) \ni p \), such that \( |T_k| \leq \delta_k^{-(1+\epsilon)} \) (the implicit constant depends on \( \text{dist}(p, K') \)), and
\[
K' \subset \bigcup_{k \geq k_0} \bigcup_{T \in T_k} T.
\]
In particular \( |T_k| \leq \delta_k^{1-\tau} \), assuming that \( \delta_k \) is small enough for all \( k \geq k_0 \). Recall that \( \mu(K') \geq \frac{1}{2} \). Hence, by the pigeonhole principle, one can find \( k \in \mathbb{N} \) such that the following holds: there is a subset \( K'_k \subset K' \) with \( \mu(K'_k) \geq \frac{1}{100k^2} \) such that \( K'_k \) is covered by the tubes in \( T_k \). But \( 1/(100k^2) \) is far larger than \( \delta_k^{\eta} \), so this is explicitly ruled out by
non-concentration estimate in Lemma 2.2, namely (2.5). This contradiction completes the proof.

\[\square\]

**REFERENCES**


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