

TOPOLOGICAL WIENER-WINTNER ERGODIC THEOREM WITH POLYNOMIAL WEIGHTS

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ABSTRACT. For a totally uniquely ergodic dynamical system, we prove a topological Wiener-Wintner ergodic theorem with polynomial weights under the coincidence of the quasi discrete spectrums of the system in both senses of Abramov and of Hahn-Parry. The result applies to ergodic nilsystems. Fully oscillating sequences can then be constructed on nilmanifolds.

1. INTRODUCTION

Let (X, \mathcal{B}, μ, T) be an ergodic measure-preserving dynamical system. The Wiener-Wintner ergodic theorem states that for any $f \in L^1(\mu)$ and for almost all $x \in X$, the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i n \alpha} f(T^n x) \quad (1.1)$$

exists for all $\alpha \in \mathbb{R}$ [65]. Since the work [65], several different proofs have appeared [27, 7, 52]. Bourgain [9] (see also [3]) proved that the above limit is uniform on α and the limit is zero if $f \in E_1(T)^\perp$ where $E_1(T)$ is the set of eigenfunctions.

Lesigne [52, 54] proved a generalized Wiener-Wintner theorem which states that for almost all x the limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i P(n)} f(T^n x) \quad (1.2)$$

exists for all real polynomials P . Under the further total ergodicity, the necessary and sufficient condition on f was found for the limit in (1.2) to be zero [54]. The notion of Abramov's quasi discrete spectrum is used to describe that condition. See [1] for this spectrum theory, which finds its origin in Halmos and von Neumann [37]. Recall that for the above ergodic system, one defines inductively the group of k -th (measurable) quasi eigenfunctions by

$$E_k(T) = \{f \in L^2(\mu) : |f| = 1, Tf \cdot \bar{f} \in E_{k-1}(T)\}, \quad \forall k \geq 1$$

where $E_0(T)$ denotes the group of eigenvalues. Lesigne proved that if we assume that (X, \mathcal{B}, μ, T) is totally ergodic, then $f \in E_k(T)^\perp$ if and only if for a.e. $x \in X$, the limit (1.2) is equal to zero for all $P \in \mathbb{R}_k[t]$, where $\mathbb{R}_k[t]$ denotes the set of polynomials of degree at most k with real coefficients. Later Frantzikinakis [26] proved that the limit in (1.2) is uniform in $P \in \mathbb{R}_k[t]$, answering a question of Lesigne [54] (pp. 771) and generalizing the result of Bourgain mentioned above. There is a version of Wiener-Wintner ergodic theorem with nilsequences as weights obtained by Host-Kra [40] (see also [16]).

Lesigne's result can be restated as follows. If $f \in E_k(T)^\perp$, the sequence $f(T^n x)$ is oscillating of order k for a.e. x . Recall that a sequence $(w_n)_{n \geq 0}$ of complex numbers is defined to be *oscillating of order* d ($d \geq 1$) if for any real polynomial $P \in \mathbb{R}_d[t]$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n e^{2\pi i P(n)} = 0.$$

A *fully oscillating sequence* is defined to be an oscillating sequence of all orders. These two notions of oscillation were introduced in [20]. The notion of oscillation of order 1 was defined in [23], in order to consider questions similar to Sarnak's conjecture ([61, 62]). Namely, for a given sequence (w_n) , we would like to find those topological dynamical systems (X, T) of zero entropy such that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} w_n f(T^n x) = 0 \tag{1.3}$$

for any $f \in C(X)$ and any $x \in X$. Sarnak's conjecture states that the limit in (1.3) is zero for all systems of zero entropy when (w_n) is the Möbius function. Sarnak's conjecture is proved for different systems [10, 11, 12, 14, 17, 18, 19, 20, 21, 23, 25, 30, 31, 41, 43, 42, 56, 58, 66, 64].

One motivation of the present work is to find topological dynamical systems (X, T) and continuous functions f such that $(f(T^n x))$ is fully oscillating or oscillating of order d for all $x \in X$ without exception. If T is an affine dynamics of zero entropy on a compact abelian group, there is no such function different from zero which gives fully oscillating sequences [21, 63]. But as we shall see, we can find such functions for some nilsystems, like ergodic nilsystems on Heisenberg homogeneous spaces.

There is already a topological version of Wiener-Wintner theorem due to Robinson [60] (see also Assani [3], Theorem 2.10). Let (X, T) be a uniquely ergodic topological dynamical system. Suppose that

$E_0(T) = G_0(T)$ where $E_0(T)$ (resp. $G_0(T)$) is the group of measurable (resp. continuous) eigenvalues. Then for any $f \in C(X)$ and any $x \in X$, the limit (1.1) exists. Furthermore, the limit is zero if $f \in E_1(T)^\perp$. The condition $E_0(T) = G_0(T)$ is necessary to some extent. In fact, Robinson constructed some strictly ergodic skew product on torus \mathbb{T}^2 such that $E_0(T) \setminus G_0(T) \neq \emptyset$, for which there exist $e^{2\pi i\alpha} \in E_0(T) \setminus G_0(T)$, $f \in C(X)$ and $x \in X$ such that the limit (1.1) fails to exist. See also [55].

We shall prove a topological version of Lesigne's Wiener-Wintner theorem, which generalizes to some extent Robinson's theorem. The condition we find will involve the quasi discrete spectrum of the system in the sense of Abramov [1] as well as the quasi discrete spectrum of the system in the sense of Hahn-Parry [33]. Recall that for a transitive topological dynamical system (X, T) , one defines inductively the group of k -th (continuous) quasi eigenfunctions by

$$G_k(T) = \{f \in C(X) : |f| = 1, Tf \cdot \bar{f} \in G_{k-1}(T)\}, \quad \forall k \geq 1.$$

The main result in this paper is the following.

Theorem A. *Let (X, T) be a topological dynamical system and let $k \geq 1$ be an integer. Suppose*

(H1) (X, T^j) for $1 \leq j < \infty$ are all strictly ergodic.

(H2) $E_j(T) = G_j(T)$ for all $0 \leq j \leq k$.

Then for any continuous function $f \in C(X)$, the following assertions are equivalent:

- (a) $f \in G_k(T)^\perp$;
- (b) for any $x \in X$, we have

$$\lim_{N \rightarrow \infty} \sup_{P \in \mathbb{R}_k[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i P(n)} f(T^n x) \right| = 0. \quad (1.4)$$

If a system satisfies (H1), we say it is *totally uniquely ergodic*. The condition (H2) is referred to as the *coincidence of spectrums* up to order k .

Some result similar to Theorem A was obtained by Eisner and Zorin-Kranch [16] where the sequence $e^{2\pi i P(n)\alpha}$ is replaced by nilsequences produced by Sobolev functions, but it was assumed that the projection f to some Host-Kra factor is zero, and consequently f is orthogonal to certain Host-Kra factor (Host-Kra factor being introduced in [39]), not only to the Abramov factor. However it was only assumed in [16] that (X, T) is uniquely ergodic.

An application of the main theorem to ergodic nilsystems leads to the following theorem.

Theorem B. *Let G be a connected and simply connected nilpotent Lie group, Γ a discrete cocompact subgroup of G and $g \in G$. Let $X = G/\Gamma$ be the nilmanifold and let $T : X \rightarrow X$ be defined by $x\Gamma \mapsto gx\Gamma$. Suppose that (X, T) is uniquely ergodic. Then for any $F \in C(X)$ such that $F \in G_\infty(T)^\perp$ and any $x \in G$, the sequence $F(g^n x\Gamma)$ is fully oscillating.*

Applied to Heisenberg groups, Theorem B gives us the following result, which was mentioned in [22].

Theorem C. *Let $m \geq 1$ and let $\alpha_1, \dots, \alpha_m; \beta_1, \dots, \beta_m$ be real numbers. Suppose $1, \alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m$ are \mathbb{Q} -independent. For any continuous function $\varphi \in C(\mathbb{T})$ such that $\int \varphi(x) dx = 0$, the sequence*

$$n \mapsto \varphi(n\alpha_1[n\beta_1] + \dots + n\alpha_m[n\beta_m])$$

is fully oscillating.

The function $n \mapsto \varphi(n\alpha_1[n\beta_1] + \dots + n\alpha_m[n\beta_m])$ is a special generalized polynomial. Generalized polynomials, especial their uniform distributions, have been well studied by Haland [34, 35, 36], Bergelson and Leibman [6], Leibman [48, 50]. Notice that the sequence $(e^{2\pi i n^{d+1} \alpha})$ with α irrational is uniformly distributed on $K := \{z \in \mathbb{C} : |z| = 1\}$, oscillating of order d but not oscillating of order $d + 1$. The oscillation is a notion relative to but different from the uniform distribution.

Here is the sketch of the proof of Theorem A, which is a long argument by induction on k . The main idea is inspired by Lesigne [54]. The proof of the case $k = 1$ is essentially a simple application of Van der Corput inequality and Krylov-Bogoliubov theorem, but the proof of the uniformity on α (see Theorem 2 (4)) follows an idea of Frantzikinakis [26] (this idea is also used in the proof of Proposition 4). The Van der Corput inequality also allows us to reduce the order k of the polynomial P to a polynomial of order $k - 1$, by induction (see Theorem 3). But in this way the result is only proved for all polynomials but some exceptions. To deal with these exceptional polynomials, we convert the problem to that of some unique ergodic extension of (X, T^j) in the sense of Furstenberg [28] (see Lemma 3).

We make preparations in Section 2 (recall of two notions of quasi-discrete spectrums) and Section 3 (extension of unique ergodicity) in order to prove Theorem A in Section 4. Theorem B and Theorem C are proved in Section 5.

2. QUASI DISCRETE SPECTRUMS

We recall here the two theories of spectrum, one measure-preserving and the other topological .

2.1. Definitions of two quasi-discrete spectra. Let (X, T) be a topological dynamical system. Assume that (X, T) is transitive, i.e. the orbit $O(x) := \{T^n x : n \geq 0\}$ of some $x \in X$ is dense in X . Let $C(X)$ be the Banach algebra of continuous complex valued functions on X and $G(X)$ be the subset of $C(X)$ consisting of f such that $|f(x)| = 1$ for all $x \in X$. It is clear that $G(X)$ is a group with multiplication as group operation. The quasi-discrete spectrum concerns the isometry $f \mapsto f \circ T$ on $C(X)$, which is still denoted by T , namely $Tf = f \circ T$. Now let us recall the notion of quasi-discrete spectrum of Hahn-Parry [33], a notion similar to Abramov's on measure-theoretic dynamics [1] which uses the concept of quasi-eigenfunction due to Halmos and von Neumann [37].

We say that $f \in C(X)$, $f \neq 0$, is an *eigenfunction* if there is a complex number $\lambda \in \mathbb{C}$ for which

$$f \circ T = \lambda f. \quad (2.1)$$

The number λ is called an *eigenvalue*. Let H_1 be the set of all eigenvalues. The eigenfunctions corresponding to the eigenvalue 1 are called *invariant functions*. The transitivity of T implies that invariant functions are constant functions, and $H_1 \subset K$ where K is the group $\{z \in \mathbb{C} : |z| = 1\}$ under multiplication, and eigenfunctions have constant modulus. Denote

$$G_1 := \{f \in G(X) : \exists \lambda \in \mathbb{C} \text{ such that } Tf = \lambda f\}.$$

It is the group of eigenfunctions. Let $G_0 = H_1$ and let us identify a constant with a constant function. Thus we have $H_1 = G_0 \subset G_1$.

Quasi-eigenvalues and quasi-eigenfunctions have different orders. They are inductively defined. Assume that subgroups H_n and G_n of $G(X)$ are defined in a such a way that

$$H_1 \subset H_2 \subset \cdots \subset H_n; \quad G_1 \subset G_2 \subset \cdots \subset G_n; \quad \forall i < n, H_{i+1} \subset G_i.$$

We define G_{n+1} to be the set of all $f_{n+1} \in G(X)$ such that there is a $g_n \in G_n$ with

$$f_{n+1} \circ T = g_n f_{n+1}. \quad (2.2)$$

Then we define H_{n+1} to be the set of all $g_n \in G_n$ for which there is an $f_{n+1} \in G_{n+1}$ satisfying (2.2). Let

$$G_\infty := \bigcup_{n=1}^{\infty} G_n, \quad H_\infty := \bigcup_{n=1}^{\infty} H_n.$$

The elements in the group H_∞ are called *quasi-eigenvalues* and the elements in the group G_∞ are called *quasi-eigenfunctions*. For $n \geq 2$, the elements in $H_n \setminus H_{n-1}$ are called *n-th quasi-eigenvalues* and the elements in $G_n \setminus G_{n-1}$ are called *n-th quasi-eigenfunctions*. If necessary, we shall write $G_n(T)$ and $G_\infty(T)$ for G_n and G_∞ . The notations $H_n(T)$ and $H_\infty(T)$ are sometimes also useful.

A dynamical system (X, T) is said to have *quasi-discrete spectrum* if the algebra generated by the quasi-eigenfunctions is dense in $C(X)$, or equivalently the linear span of quasi-eigenvalues is dense in $C(X)$ because G is a multiplicative group. By using Stone-Weierstrass theorem we see that this is equivalent to say that quasi-eigenfunctions separate points of X . If, furthermore, $G_d = G_{d+1}$ and d_T is the least such integer, we say that (X, T) has *quasi-discrete spectrum of order d_T* .

2.2. Orthogonality of quasi-eigenfunctions.

Proposition 1. *Let (X, \mathcal{B}, μ, T) be an ergodic measure-preserving dynamical system. Suppose that there is no eigenvalue of finite order (except the eigenvalue 1). Then all quasi-eigenfunctions are orthogonal.*

Proof. Let $E(T)$ be the group of all $f \in L^\infty(\mu)$ such that $|f(x)| = 1$ a.e.. Let us first make a remark: suppose

$$\begin{aligned} Tf_2 &= f_1 f_2, & Tf_1 &= h f_1; \\ Tg_2 &= g_1 g_2, & Tg_1 &= h g_1 \end{aligned}$$

where $f_1, f_2, g_1, g_2, h \in E(T)$, then we have $f_1 = c g_1$ for some eigenvalue c . In fact, first observe that g_1/f_1 . By the ergodicity we get $g_1 = c f_1$ for some $c \in K$. Then from $Tf_2 = f_1 f_2$ and $Tg_2 = c f_1 g_2$, we get

$$\frac{Tg_2}{Tf_2} = c \frac{g_2}{f_2}.$$

So, c is an eigenvalue to which the eigenfunction g_2/f_2 is associated.

Let us consider two arbitrary different quasi-eigenfunctions f and g which are not proportional. We are going to show $\int f \bar{g} d\mu = 0$. More

precisely, let f be a quasi-eigenfunction of order k and g be a quasi-eigenfunction of order ℓ . Assume $1 \leq k \leq \ell < \infty$. In other words,

$$Tf = f_{k-1}f, \quad Tf_{k-1} = f_{k-2}f_{k-1}, \quad \dots, \quad Tf_1 = f_0f_1$$

$$Tg = g_{\ell-1}g, \quad Tg_{\ell-1} = g_{\ell-2}g_{\ell-1}, \quad \dots, \quad Tg_1 = g_0g_1$$

for some $f_j \in E(X)$ ($1 \leq j < k-1$) and $g_j \in E(X)$ ($1 \leq j < \ell-1$), where f_0 and g_0 are two eigenvalues. By an inductive argument, we deduce that

$$f(T^n x) = f(x)f_{k-1}(x)^{\binom{n}{1}}f_{k-2}(x)^{\binom{n}{2}} \dots f_1(x)^{\binom{n}{k-1}}f_0(x)^{\binom{n}{k}}.$$

Let $f_j(x) = e^{2\pi i \theta_j}$ ($0 \leq j \leq k-1$), where θ_j ($1 \leq j < k$) depends on x , but θ_0 doesn't. We get

$$f(T^n x) = f(x)e^{2\pi i p_x(n)}, \quad \text{with } p_x(n) = \theta_0 \binom{n}{k} + \theta_1 \binom{n}{k-1} + \dots + \theta_{k-1} \binom{n}{1}.$$

Similarly we have

$$g(T^n x) = g(x)e^{2\pi i q_x(n)}, \quad \text{with } q_x(n) = \phi_0 \binom{n}{\ell} + \phi_1 \binom{n}{\ell-1} + \dots + \phi_{\ell-1} \binom{n}{1}$$

where $\phi_j \in [0, 1)$ is the argument of $g_j(x) = e^{2\pi i \phi_j}$. By the invariance we get

$$\int f \bar{g} d\mu = \int f \bar{g} e^{2\pi i (p_x(n) - q_x(n))} d\mu$$

which holds for all n , so that we have

$$\begin{aligned} \int f \bar{g} d\mu &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \int f(x) \overline{g(x)} e^{2\pi i (p_x(n) - q_x(n))} d\mu(x) \\ &= \int f(x) \overline{g(x)} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i (p_x(n) - q_x(n))} d\mu(x) \end{aligned}$$

when the last limit exists for all x (for the last equality we use the Lebesgue dominated convergence theorem). This limit does exist and is equal to zero. Thus we finish the proof.

We prove that the limit is equal to zero by induction on ℓ . Assume $\ell = 1$. Then $k = 1$ and f and g are eigenfunctions. If $f_0 \neq g_0$, it is well known that f and g are orthogonal. The case $f_0 = g_0$ is not possible, because otherwise, f and g are proportional as eigenfunctions associated to the same eigenvalue.

Now we suppose the conclusion holds for $\ell - 1$. We prove the case $\ell \geq 2$ by distinguishing several cases.

Case I. $k = \ell$, $f_0 \neq g_0$: In this case, $p_x - q_x$ is a real polynomial of degree ℓ with leading coefficient $(\theta_0 - \phi_0)/\ell!$ which is irrational,

because $\theta_0 - \varphi_0 \neq 0$ is the argument of the eigenvalues $f_0 \bar{g}_0$. Without use of the induction hypothesis we conclude by Weyl theorem, because $p_x(n) - q_x(n)$ is uniformly distributed.

Case II. $k = \ell$, $f_0 = g_0$: In this case, we first apply the above remark to get $g_1 = cf_1$ where $c = e^{2\pi i \xi}$ is an eigenvalue. If $c \neq 1$, then ξ is irrational and

$$p_x(n) - q_x(n) = \xi \binom{n}{k-1} + (\theta_2 - \phi_2) \binom{n}{k-2} + \dots$$

is a polynomial of degree $k-1$ with leading irrational coefficient $\xi \binom{n}{k-1}$. We conclude with Weyl theorem. If $c = 1$, we get $g_1 = f_1$ and fall into the following situation

$$\begin{aligned} Tf_3 &= f_2 f_3, & Tf_2 &= f_1 f_2, & (Tf_1 &= f_0 f_1) \\ Tg_3 &= g_2 g_3, & Tg_2 &= f_1 g_2, & (Tf_1 &= f_0 f_1). \end{aligned}$$

Again we apply the above remark to get $g_2 = df_2$ for some eigenvalue d . If $d \neq 1$, we conclude. Otherwise we get $g_2 = f_2$. In this inductive way, we can conclude otherwise we get $g = f$, a contradiction.

Case III. $k < \ell$: If $g_0 = 1$, then g_1 is an invariant function then constant. Thus we can forget the trivial equality $Tg_1 = g_0 g_1$ and just start with $Tg_2 = g_1 g_2$. In other words, we have reduced ℓ to $\ell - 1$. Therefore we can apply the induction hypothesis to conclude. If $g_0 \neq 1$, then $p_x - q_x$ is a real polynomial of degree ℓ with leading coefficient $-\varphi_0/\ell!$ which is irrational. We conclude by Weyl theorem. \square

2.3. Quasi-discrete spectra of T and of T^m . The following lemma has a version for measure-preserving dynamical systems, due to Lesigne [54], for which the ergodicity and the total ergodicity replace the transitivity and the total minimality.

Lemma 1. *Let T_1 and T_2 be two transitive maps on a compact metric space X . Suppose that T_1 and T_2 are commutative. Then $G_k(T_1) = G_k(T_2)$ for all $k \geq 1$. In particular, if both T and T^m are transitive ($m \geq 2$), then*

$$G_k(T) = G_k(T^m)$$

for all $k \geq 1$.

Proof. The proof is the same as in [54]. It suffices to replace the ergodicity by the transitivity which ensures that a continuous invariant function is constant. We include the proof here for completeness. We prove it by induction on k . We assume $T_1 f = \lambda f$ with $f \in G_1(T_1)$ and $\lambda \in K$. By the commutativity, we have

$$T_1(T_2 f \cdot \bar{f}) = T_2 T_1 f \cdot T_1 \bar{f} = |\lambda|^2 T_2 f \cdot \bar{f} = T_2 f \cdot \bar{f}.$$

By the transitivity of T_1 , we deduce that the T_1 -invariant function $T_2f \cdot \bar{f}$ is a constant, so $f \in G_1(T_2)$. Thus $G_1(T_1) \subset G_1(T_2)$. By the symmetry, we get $G_1(T_1) = G_1(T_2)$.

Now assume that $G_j(T_1) = G_j(T_2)$ for all $1 \leq j < k$ ($k \geq 2$). Let $f \in G_k(T_1)$. Then there exists $g \in G_{k-1}(T_1)$ such that

$$T_1f = gf.$$

By the induction hypothesis, $g \in G_{k-1}(T_2)$. Then there exists $h \in G_{k-2}(T_2)$ such that

$$T_2g = hg.$$

Thus, by the commutativity, we have

$$T_1(T_2f) = T_2(T_1f) = T_2(gf) = T_2g \cdot T_2f = hgT_2f.$$

It follows that

$$T_1(T_2f \cdot \bar{f}) = hgT_2f \cdot \overline{gf} = h(T_2f \cdot \bar{f}).$$

Since $h \in G_{k-2}(T_2)$, we have $h \in G_{k-2}(T_1)$ by the induction hypothesis. Therefore

$$T_2f \cdot \bar{f} \in E_{k-1}(T_1).$$

Again, by the induction hypothesis,

$$T_2f \cdot \bar{f} \in G_{k-1}(T_2).$$

So, $f \in G_k(T_2)$. By the symmetry, we have $G_k(T_1) = G_k(T_2)$. \square

3. EXTENSION OF UNIQUE ERGODICITY

Assume that (X, T) is a uniquely ergodic topological dynamical system with μ as the only invariant measure. Let G be a compact abelian group with normalized Haar measure m and $\phi : X \rightarrow G$ be a continuous map. Define the map $S := S_\phi : X \times G \rightarrow X \times G$ by

$$S(x, z) = (Tx, \phi(x)z). \quad (3.1)$$

The dynamical system $(X \times G, S)$ is called a group extension of (X, T) . The product measure $\mu \times m$ is S -invariant. The following lemma of Furstenberg [28] (p. 579) gives the condition for a group extension to be still uniquely ergodic (the "only if" part is obvious).

Lemma 2 ([28]). *Suppose that (X, T) is unique ergodic with μ as its invariant measure. Then the above defined extension $(X \times G, S)$ is uniquely ergodic if (and only if) the S -invariant measure $\mu \times m$ is ergodic. It is the case iff the following equation*

$$\phi(x)^k = \frac{h(Tx)}{h(x)}$$

has no solution for $k \neq 0$ an integer and h a measurable function.

Let (X, T) be a topological dynamical system and let $p \geq 1$ be an integer. Suppose

$$\phi_1 : X \rightarrow K, \quad \phi_2 : X \times K \rightarrow K, \quad \dots, \quad \phi_p : X \times K^{p-1} \rightarrow K$$

are given continuous maps. Then $S_1 : X \times K \rightarrow X \times K$ defined by

$$S_1(x, z_1) = (Tx, \phi_1(x)z_1)$$

is a group extension of (X, T) , and $S_2 : X \times^2 \rightarrow X \times^2$ defined by

$$S_2(x, z_1, z_2) = (Tx, \phi_1(x)z_1, \phi(x, z_1)z_2)$$

is a group extension of $(X \times K, S_1)$. Inductively, we define S_3, \dots, S_p such that S_{j+1} is a group extension of S_j . In particular, the map $S_p : X \times K^p \rightarrow X \times K^p$ is defined by

$$S_p(x, z_1, \dots, z_p) = (Tx, \phi_1(x)z_1, \phi(x, z_1)z_2, \dots, \phi_p(x, z_1, \dots, z_{p-1})z_p).$$

We call it a p -th group extension of (X, T) .

Let us consider the following special case

$$\phi_1(x) = \gamma(x), \quad \phi(x, z_1) = z_1, \quad \dots, \quad \phi_p(x, z_1, \dots, z_{p-1}) = z_{p-1}$$

where $\gamma : X \rightarrow K$ is a given continuous map.

Let us consider the following special case where $G = K^p$ ($p \geq 1$) and

$$S(x, z_1, \dots, z_p) = (Tx, \gamma(x)z_1, z_1z_2, \dots, z_{p-1}z_p). \quad (3.2)$$

with $\gamma : X \rightarrow K$ a continuous map.

For any given eigenfunction $\tilde{\gamma} \in G_1(T)$, we will find a number $\lambda \in K$ such that the p -th extension S defined (3.2) with $\gamma = \lambda\tilde{\gamma}$ have the following property: whenever T^n is uniquely ergodic, so is S^n . By Lemma 2, it suffices to deduce the ergodicity of $\mu \times m$ relative to S^n from the ergodicity of μ relative to T^n .

Lemma 3. *Let (X, T) be uniquely ergodic with invariant measure μ and let $p \geq 1$ be integer. For any eigenfunction $\tilde{\gamma} \in G_1(T)$, there exists a number $\lambda \in K$ such that the extension S on $X \times K^p$ of (X, T) defined by (3.2) with $\gamma := \lambda\tilde{\gamma} \in G_1(T)$ has the following properties:*

- (1) S is uniquely ergodic.
- (2) for any integer $n \geq 1$, S^n is uniquely ergodic if T^n is uniquely ergodic.

Proof. (1) The proof of this part is contained in [54] (pp. 779-780) and we repeat it here for completeness. We first discuss the following cocycle equation (3.3), which is also useful for part (2). In [54], only the case $k = 0$ was discussed. The general case with arbitrary k would have been discussed [54], because the powers of the extension were used.

Let $\xi \in H_1(T)$ be the eigenvalue associated to the eigenfunction $\tilde{\gamma}$. We introduce a parameter $\lambda \in K$, to be determined later, and consider the equation

$$\xi^k(\lambda\tilde{\gamma}(x))^j = \frac{h(Tx)}{h(x)} \quad \mu\text{-a.e.} \quad (3.3)$$

where the unknown is the triple (k, j, h) with $k \in \mathbb{Z}$, $j \in \mathbb{Z} \setminus \{0\}$ and $h : X \rightarrow K$ a Borel function. We claim that there exists $\lambda \in K$ such that (3.3) has no solution. Since K is uncountable, it suffices to show that for any fixed couple $(k, j) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$, there are at most countably many (λ, h) such that (3.3) is solvable. That is really the case. In fact, if (λ_1, h_1) and (λ_2, h_2) are distinct solutions, then

$$\lambda_1^j h_1(x) \overline{h_2(x)} = \lambda_2^j h_1(Tx) \overline{h_2(Tx)}.$$

If furthermore $\lambda_1^j \neq \lambda_2^j$, then h_1 and h_2 are orthogonal. But any family of orthogonal functions is countable because $L^2(\mu)$ is separable. So, there are at most countable many possibilities λ^j . To finish the argument, we just remark that $\lambda_1^j = \lambda_2^j$ means $\lambda_2 = \lambda_1 e^{2\pi il/j}$ ($0 \leq l < j$).

In the following we fix a number $\lambda \in K$ such that (3.3) has no solution. Notice that this λ depends on $\tilde{\gamma}$. Let $\gamma := \lambda\tilde{\gamma} \in G_1(T)$. Consider the extension S defined by

$$S(x, z_1, z_2, \dots, z_p) = (Tx, \gamma(x)z_1, z_1z_2, \dots, z_{p-1}z_p).$$

According to Lemma 2, in order to prove that S is uniquely ergodic, it suffices to prove that $\mu \times m$ is S -ergodic. Suppose that f is a bounded S -invariant function. By a Fourier method, we are going to prove that f is constant. For $J := (j_1, \dots, j_p) \in \mathbb{Z}^p$, define

$$f_J(x) = \int_{K^p} f(x, z_1, \dots, z_p) z_1^{j_1} \dots z_p^{j_p} dz_1 \dots dz_p.$$

It is a Fourier coefficient of the function $z \mapsto f(x, z)$. We make the change of variables $(Z_1, \dots, Z_p) := (\gamma(x)z_1, z_1z_2, \dots, z_{p-1}z_p)$, which preserves the Haar measure dz , to get

$$\begin{aligned} f_J(Tx) &= \int_{K^p} f(Tx, Z_1, \dots, Z_p) Z_1^{j_1} \dots Z_p^{j_p} dZ_1 \dots dZ_p \\ &= \int_{K^p} f(Tx, \gamma(x)z_1, z_1z_2, \dots, z_{p-1}z_p) \gamma(x)^{j_1} z_1^{j_1+j_2} z_2^{j_2+j_3} \dots z_{p-1}^{j_{p-1}+j_p} z_p^{j_p} dz_1 \dots dz_p. \end{aligned}$$

Thus

$$f_{j_1, \dots, j_p}(Tx) = \gamma(x)^{j_1} f_{j_1+j_2, \dots, j_{p-1}+j_p, j_p}(x). \quad (3.4)$$

Then, by the T -invariance of μ , we get

$$\int_X |f_{j_1, j_2, \dots, j_p, j_p}(x)|^2 d\mu(x) = \int_X |f_{j_1+j_2, \dots, j_{p-1}+j_p, j_p}(x)|^2 d\mu(x). \quad (3.5)$$

On the other hand, notice that

$$\sum_{J \in \mathbb{Z}^p} \int_X |f_J(x)|^2 d\mu(x) = \int_X \int_{K^p} |f(x, z)|^2 d\mu(x) dz < \infty.$$

It follows that

$$\lim_{|J| \rightarrow \infty} \int_X |f_{j_1, j_2, \dots, j_p}(x)|^2 d\mu(x) = 0. \quad (3.6)$$

From (3.5) and (3.6), we deduce that for μ -almost every x , $f_J(x) = 0$ when at least one of j_2, \dots, j_p is non zero. That is to say, f depends only on x and z_1 . Write $f_j(x) = f_{j_1, 0, \dots, 0}(x)$. Then (3.4) becomes

$$f_j(Tx) = \gamma(x)^j f_j(x). \quad (3.7)$$

So, $|f_j|$ is T -invariant by the ergodicity. We claim that $|f_j| = 0$. Otherwise, we get a contradiction to the non-solvability of the equation (3.3). Therefore f depends only on x . But μ is T -ergodic, so f is almost everywhere constant. Thus we have proved the S -ergodicity of $\mu \times m$, then the unique ergodicity of S .

(2) Recall $T\gamma = \xi\gamma$. We have the following formula for the powers of S :

$$S^n(x, z_1, \dots, z_p) = (T^n x, Z_1, \dots, Z_p) \quad (3.8)$$

where

$$\begin{aligned} Z_1 &= \xi^{(2)} \gamma(x) \binom{n}{1} z_1; \\ Z_2 &= \xi^{(3)} \gamma(x) \binom{n}{2} z_1^{(n)} z_2; \\ &\vdots \\ Z_p &= \xi^{(p+1)} \gamma(x) \binom{n}{p} z_1^{(p-1)} \dots z_{p-1}^{(1)} z_p. \end{aligned}$$

We can prove the formula (3.8) by induction on n using the Pascal formula. Here $\binom{n}{k} = 0$ for $k > n$. Notice that S^n is a p -th extension of T^n . We have assumed that T^n is uniquely ergodic. Again, according to Lemma 2, in order to prove that S^n is uniquely ergodic, we have only to prove that $\mu \times m$ is S^n -ergodic. Suppose that f is a bounded S^n -invariant function. For $J := (j_1, \dots, j_p) \in \mathbb{Z}^p$, we also define

$$f_J(x) = \int_{K^p} f(x, z_1, \dots, z_p) z_1^{j_1} \dots z_p^{j_p} dz_1 \dots dz_p.$$

Consider $(z_1, \dots, z_p) \mapsto (Z_1, \dots, Z_p)$ as a change of variable, which preserves the Haar measure dz . We have

$$\begin{aligned} f_J(T^n x) &= \int_{K^p} f(T^n x, Z_1, \dots, Z_p) Z_1^{j_1} \dots Z_p^{j_p} dZ_1 \dots dZ_p \\ &= \int_{K^p} f(S^n(x, x, \gamma(x)z)) \xi^a \gamma(x)^b z_1^{c_1} z_2^{c_2} \dots z_p^{c_p} dz_1 \dots dz_p \end{aligned}$$

where

$$\begin{aligned} a &= j_1 \binom{n}{2} + j_2 \binom{n}{3} + \dots + j_p \binom{n}{p+1} \\ b &= j_1 \binom{n}{1} + j_2 \binom{n}{2} + \dots + j_p \binom{n}{p} \\ c_1 &= j_1 \binom{n}{0} + j_2 \binom{n}{1} + \dots + j_p \binom{n}{p-1} \\ c_2 &= j_2 \binom{n}{0} + j_3 \binom{n}{1} + \dots + j_p \binom{n}{p-2} \\ &\vdots \\ c_{p-1} &= j_{p-1} \binom{n}{0} + j_p \binom{n}{1} \\ c_p &= j_p. \end{aligned}$$

Thus we get a formula generalizing (3.4)

$$f_{j_1, \dots, j_p}(T^n x) = \xi^a \gamma(x)^b f_{c_1, \dots, c_p}(x). \quad (3.9)$$

As in the proof of (1), from (3.9) we can deduce that f only depends on x and z_1 . Let $f_j(x) = f_{j, 0, \dots, 0}(x)$. Then (3.9) becomes

$$f_j(T^n x) = \xi^{jn(n-1)/2} \gamma(x)^j f_j(x). \quad (3.10)$$

The non solvability of (3.3) implies that $f_j(x) = 0$ for $j \neq 0$. Thus f depends only on x . The S^n -invariance of f implies the T^n -invariance of f . Finally we conclude that f is constant by the T^n -ergodicity of μ . \square

Notice that the above choice λ is valid for all $n \geq 1$.

The quasi eigenfunctions of the extension S is simply related to those of the base dynamics T , as the following lemma shows.

Lemma 4 ([54], p.782). *Let S be the extension defined by $\gamma \in G_1(T)$. Let $1 \leq k \leq p$. If $g \in G_k(S)$, there exists $\tilde{g} \in G_{k+1}(T)$ and $d_1, \dots, d_j \in \mathbb{Z}$ such that*

$$g(x, z_1, \dots, z_p) = \tilde{g}(x) z_1^{d_1} \dots z_k^{d_k}.$$

4. TWWT-PROOF OF THEOREM A

The first two results below concerning Topological Wiener-Wintner Theorem (TWWT) constitute the first two steps towards the proof by induction of Theorem A. For their proofs, we don't need the results in Section 2 and Section 3. The proofs given here are adapted from Lesigne [54] who treated measure-preserving systems instead of topological systems. Another argument used in the proof of Theorem A is due to Frantzikinakis [26] (see Proposition 4).

4.1. Orthogonality to polynomials of degree 1. The following Theorem 2 is mainly due to Assani [2] (see also [3], p. 42). A particular case of Robinson's theorem [60] asserts that the limit in (4.2) is uniform on x for fixed α . As pointed out in [60], B. Weiss obtained some unpublished similar results. We first give a direct of the point-wise convergence of (4.2), based on the Krylov-Bogolioubov theorem and the inequality of Van der Corput and Herglotz theorem (through the spectral measure). Then we prove (4.3) as a Robinson's uniform consequence mentioned above and of a technique due to Frantzikinakis [26]. This technique of Frantzikinakis will be used once more in a more involved way in the proof of Proposition 4 where polynomials, instead of $n\alpha$, are concerned.

Theorem 2. *Let (X, T) be a uniquely ergodic system with the unique ergodic measure μ . Suppose that $E_0(T) = G_0(T)$. Let $f \in C(X)$ and $\alpha \in [0, 1)$.*

(1) *If $e^{2\pi i\alpha} \in E_0(T)$, we have*

$$\forall x \in X, \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i n \alpha} f(T^n x) = g(x)^{-1} \int f g d\mu \quad (4.1)$$

where g is an eigenfunction associated to $e^{2\pi i\alpha}$ (unique up to multiplicative constant).

(2) *If $e^{2\pi i\alpha} \notin E_0(T)$, the limit in (4.1) is zero.*

(3) *We have $f \in E_1(T)^\perp$ if and only if*

$$\forall \alpha \in [0, 1), \forall x \in X, \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i n \alpha} f(T^n x) = 0. \quad (4.2)$$

(4) *We have $f \in E_1(T)^\perp$ if and only if*

$$\lim_{N \rightarrow \infty} \sup_{\alpha \in \mathbb{R}} \left\| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i n \alpha} f(T^n \cdot) \right\|_{C(X)} = 0. \quad (4.3)$$

Proof. (1) Let $\lambda = e^{2\pi i\alpha}$. Assume $g(Tx) = \lambda g(x)$. Then by the unique ergodicity, the limit in (4.1) is uniform on x and is equal to

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} g(x)^{-1} g(T^n x) f(T^n x) = g(x)^{-1} \int f g d\mu.$$

Thus we have checked (1).

(2) We follow Lesigne [54] by using the inequality of Van der Corput. In the present topological case, Krilov-Bogoliubov theorem will be used in the place of Birkhoff ergodic theorem. First remark that the unique ergodicity implies that for any $x \in X$ and any $h \in \mathbb{N}$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^{n+h}x) \overline{f(T^n x)} = \int f \circ T^h \cdot \bar{f} d\mu = \int_{\mathbb{T}} e^{2\pi i h t} d\sigma(t) \quad (4.4)$$

where σ is the spectral measure associated to f . Let $0 \leq H < N$. By the inequality of Van der Corput, we have

$$\begin{aligned} & \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i n \alpha} f(T^n x) \right|^2 \\ & \leq \frac{N+H}{N(H+1)} \cdot \frac{1}{N} \sum_{n=0}^{N-1} |f(T^n x)|^2 + 2 \frac{N+H}{N(H+1)^2} \\ & \quad \times \left| \sum_{h=1}^H \frac{(H+1-h)(N-h)}{N} \cdot e^{2\pi i h \alpha} \cdot \frac{1}{N-h} \sum_{n=0}^{N-h-1} f(T^{n+h}x) \overline{f(T^n x)} \right| \end{aligned}$$

Then taking limit as N tends to infinity leads to

$$\begin{aligned} \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i n \alpha} f(T^n x) \right|^2 & \leq \frac{\langle f, f \rangle}{H+1} \\ & \quad + \left| \int_{\mathbb{T}} \left[\frac{2}{(H+1)^2} \sum_{h=1}^H (H+1-h) \cdot e^{2\pi i h(\alpha+t)} \right] d\sigma(t) \right|. \end{aligned}$$

Here we have used (4.4). Since

$$\lim_{H \rightarrow \infty} \frac{1}{H} \sum_{h=1}^H e^{2\pi i(\alpha+t)} = 0 \quad \text{if } \alpha+t \notin \mathbb{Z},$$

as the second order Césaro mean we have

$$\lim_{H \rightarrow \infty} \frac{1}{(H+1)^2} \sum_{h=1}^H (H+1-h) e^{2\pi i(\alpha+t)} = 0 \quad \text{if } \alpha+t \notin \mathbb{Z}. \quad (4.5)$$

On the other hand, since $e^{2\pi i\alpha} \notin E_0(T)$, we have $e^{-2\pi i\alpha} \notin E_0(T)$ too. So, the measure σ has no measure at $t = -\alpha$ and the limit in (4.5) is σ -almost everywhere equal to 0. Finally we can conclude for (2) by using the dominated convergence theorem of Lebesgue.

(3) is a direct consequence of (1) and (2).

(4) Because of (3), we have only to prove the uniform convergence (4.3) for $f \in E_1(T)^\perp$. Suppose that for any $\alpha \in [0, 1]$ we have

$$\lim_{M \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \left\| \frac{1}{M} \sum_{m=1}^M e^{2\pi i m \alpha} b_{Mn+m} \right\|_{\mathbb{B}} = 0. \quad (4.6)$$

Then

$$\lim_{N \rightarrow \infty} \sup_{\alpha \in [0, 1]} \frac{1}{N} \left\| \sum_{n=1}^N e^{2\pi i n \alpha} b_n \right\|_{\mathbb{B}} = 0. \quad (4.7)$$

This is Lemma 2.2 in [26], where b_n 's are complex numbers. But the proof is identical when b_n 's are in a Banach space \mathbb{B} . Now notice that

$$\left\| \frac{1}{M} \sum_{m=1}^M e^{2\pi i m \alpha} f(T^{Mn+m}x) \right\|_{C(X)} \leq \left\| \frac{1}{M} \sum_{m=1}^M e^{2\pi i m \alpha} f(T^m x) \right\|_{C(X)}.$$

The RHS in the above inequality tends to zero by Robinson's theorem. Thus the condition in (4.6) is satisfied by $b_n = f \circ T^n$ in the Banach space $(C(X), \|\cdot\|_{C(X)})$. Then we get (4.7) with $b_n = f \circ T^n$ and $\mathbb{B} = C(X)$. This is what we have to prove. \square

Theorem 2 asserts the existence of the limit for every $x \in X$, which is even uniform on x , but under the imposed condition $E_0(T) = G_0(T)$. This condition cannot be dropped. In fact, Robinson showed that there is a strictly ergodic analytic Anzai skew product T on the torus \mathbb{T}^2 , which has an essentially discontinuous eigenvalue (i.e. $E_0(T) \setminus G_0(T) \neq \emptyset$), and for such an eigenvalue $e^{2\pi i\alpha}$ and for some $f \in C(\mathbb{T}^2)$ the limit in (4.1) fails to exist for some point $x \in \mathbb{T}^2$ (Proposition 3.1 in [60]).

4.2. Orthogonality to polynomials of degree k . Let

$$D(T) := \{\alpha \in [0, 1) : \exists m \in \mathbb{Z} \setminus \{0\}, e^{2\pi i m \alpha} \in E_0(T)\}.$$

The set $D(T)$ represents the roots of eigenvalues of T .

Theorem 3. *Let (X, T) be a uniquely ergodic system with the unique ergodic measure μ . Suppose that $E_0(T) = G_0(T)$. Let $f \in C(X)$ and*

$\alpha \in [0, 1)$. If $\alpha \notin D(T)$, then for $k \geq 1$

$$\forall x \in X, \quad \lim_{N \rightarrow \infty} \sup_{P \in \mathbb{R}_{k-1}[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i(n^k \alpha + P(n))} f(T^n x) \right| = 0. \quad (4.8)$$

Proof. Following again Lesigne [54] (pp.773-774), we prove it by induction on k using again the inequality of Van der Corput. The case $k = 1$ is (2) of Theorem 2 (3). Assume the conclusion for $k \geq 1$. This hypothesis of induction applied to $f \circ T^h \cdot \bar{f}$ gives us

$$\lim_{N \rightarrow \infty} \sup_{Q \in \mathbb{R}_{k-1}[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i(n^k \alpha + Q(n))} f(T^{n+h} x) \overline{f(T^n x)} \right| = 0. \quad (4.9)$$

The unique ergodicity implies that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |f(T^n x)|^2 = \langle f, f \rangle. \quad (4.10)$$

Let $0 \leq H < N$. By the inequality of Van der Corput, we have

$$\begin{aligned} & \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i(n^{k+1} \alpha + P(n))} f(T^n x) \right|^2 \\ & \leq \frac{N+H}{N(H+1)} \cdot \frac{1}{N} \sum_{n=0}^{N-1} |f(T^n x)|^2 + 2 \frac{N+H}{N^2(H+1)^2} \sum_{h=1}^H (H+1-h) \\ & \quad \times \left| \sum_{n=0}^{N-h-1} e^{2\pi i((n+h)^{k+1} - n^{k+1}) \alpha + [P(n+h) - P(n)]} f(T^{n+h} x) \overline{f(T^n x)} \right|. \end{aligned}$$

Notice that $d^\circ(P(\cdot + h) - P(\cdot)) \leq k - 1$ and $(n+h)^{k+1} - n^{k+1} = (k+1)n^k + R$ with $d^\circ R \leq k - 1$. It follows that

$$\begin{aligned} & \sup_{P \in \mathbb{R}_k[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i(n^{k+1} \alpha + P(n))} f(T^n x) \right|^2 \\ & \leq \frac{N+H}{N(H+1)} \cdot \frac{1}{N} \sum_{n=0}^{N-1} |f(T^n x)|^2 + 2 \frac{N+H}{N^2(H+1)^2} \sum_{h=1}^H (H+1-h) \\ & \quad \times \sup_{Q \in \mathbb{R}_{k-1}[t]} \left| \sum_{n=0}^{N-h-1} e^{2\pi i((k+1)hn^k \alpha + Q(n))} f(T^{n+h} x) \overline{f(T^n x)} \right| \end{aligned}$$

Since $e^{2\pi i(k+1)h\alpha} \notin E_0(T)$, by (4.9) and (4.10) we get

$$\limsup_{N \rightarrow \infty} \sup_{P \in \mathbb{R}_k[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i(n^{k+1} \alpha + P(n))} f(T^n x) \right|^2 \leq \frac{\langle f, f \rangle}{H+1}.$$

Letting $H \rightarrow \infty$ finishes the proof by induction. \square

4.3. Frantzikinakis lemma. The following proposition is another ingredient for proving our Theorem A. It allows us to prove the uniformity on $P \in \mathbb{R}_k[t]$ of the convergence. There is a version for totally ergodic measure-preserving systems due to Frantzikinakis [26], and the convergence for individual polynomials is due to Lesigne [54]. For the proof for totally uniquely ergodic topological systems, we mimick [26]. Actually the proof for our topological systems is simpler.

Proposition 4. *Let (X, T) be a totally uniquely ergodic topological dynamical system and let $f \in C(X)$. Suppose that for any $\alpha \in \mathbb{R}$ and any $x \in X$ we have*

$$\lim_{N \rightarrow \infty} \sup_{P \in \mathbb{R}_{k-1}[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i(n^k \alpha + P(n))} f(T^n x) \right| = 0. \quad (4.11)$$

Then for any x we have

$$\lim_{N \rightarrow \infty} \sup_{P \in \mathbb{R}_k[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i P(n)} f(T^n x) \right| = 0. \quad (4.12)$$

Proof. We first claim that for any $\alpha \in \mathbb{R}$ and any $x \in X$ we have

$$\lim_{M \rightarrow \infty} \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{N} \sup_{P \in \mathbb{R}_{k-1}[t]} \left| \frac{1}{M} \sum_{m=0}^{M-1} e^{2\pi i(m^k \alpha + P(m))} f(T^{Mn+m} x) \right| = 0. \quad (4.13)$$

In fact, let

$$g_M(\alpha, x) = \sup_{P \in \mathbb{R}_{k-1}[t]} F_M(\alpha, P, x)$$

where

$$F_M(\alpha, P, x) := \left| \frac{1}{M} \sum_{m=0}^{M-1} e^{2\pi i(m^k \alpha + P(m))} f(T^{Mn+m} x) \right|.$$

The function F_M depends only on the fractional parts of the coefficients of P . So, each $P \in \mathbb{R}_{k-1}[t]$ can be identified as a point in $\mathbb{R}^k/\mathbb{Z}^k$. The function F_M depends only on the fractal part of α either. Thus F_M is a continuous function of $(\alpha, P, x) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}^k/\mathbb{Z}^k \times X$. It follows that g_M is a continuous function of α and x . By applying Krilov-Bogoliubov theorem to the system (X, T^M) and to the function $g_M(\alpha, \cdot)$, we get

$$\lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{N} \sup_{P \in \mathbb{R}_{k-1}[t]} \left| \frac{1}{M} \sum_{m=0}^{M-1} e^{2\pi i(m^k \alpha + P(m))} f(T^{Mn+m} x) \right| = \int g_M(\alpha, x) d\mu(x)$$

where μ is the unique invariant measure. The hypothesis (4.11) means that $g_M(\alpha, x)$ converges to zero for every point x . This and Lebesgue's bounded convergence theorem allow us to conclude for (4.13) if we take limit as $M \rightarrow \infty$.

Fix α and x . Secondly we claim that for any $\epsilon > 0$, there exist an integer N_α and an open neighborhood V_α of α (both N_α and V_α depending on ϵ and x too) such that

$$\forall N > N_\alpha, \quad \sup_{\beta \in N_\alpha} \sup_{P \in \mathbb{R}_{k-1}[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i(n^k \beta + P(n))} f(T^n x) \right| \leq \epsilon. \quad (4.14)$$

In fact, consider the general term $e^{2\pi i(n^k \beta + P(n))} f(T^n x)$ as a function of α and we denote it by $a_n(\alpha)$. For any β , we have

$$\frac{1}{N} \sum_{n=0}^{N-1} a_n(\beta) = \frac{1}{[N/M]} \sum_{n=1}^{[N/M]} \frac{1}{M} \sum_{m=0}^{M-1} a_{nM+m}(\beta) + O(M/N).$$

where the constant involved in $O(\cdot)$ is $2\|f\|_\infty$. Observe that

$$(Mn + m)^k \beta = m^k \beta + P_{M,n,\beta}(m)$$

with $P_{M,n,\beta} \in \mathbb{R}_{k-1}[t]$. For any β and α , we can write

$$e^{2\pi i(Mn+m)^k \beta} = \left(e^{2\pi i m^k \beta} - e^{2\pi i m^k \alpha} \right) e^{2\pi i P_{M,n,\beta}(m)} + e^{2\pi i(m^k \alpha + P_{M,n,\beta}(m))}$$

Thus

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} a_n(\beta) \right| \leq A_{M,N}(\alpha, \beta) + B_{M,N}(\alpha, \beta) + O(M/N) \quad (4.15)$$

where

$$A_{M,N}(\alpha, \beta) = \frac{1}{[N/M]} \sum_{n=1}^{[N/M]} \frac{1}{M} \sum_{m=0}^{M-1} \left| e^{2\pi i m^k \beta} - e^{2\pi i m^k \alpha} \right| |f(T^{Mn+m} x)|$$

$$B_{M,N}(\alpha, \beta) = \frac{1}{[N/M]} \left| \sum_{n=1}^{[N/M]} \frac{1}{M} \sum_{m=0}^{M-1} e^{2\pi i(m^k \alpha + P_{M,n,\beta}(m) + P(Mn+m))} f(T^{Mn+m} x) \right|.$$

Now fix α . By our first claim (see (4.13)), there exists an integer M_α such that for N large enough and for all β we have

$$|B_{M_\alpha, N}(\alpha, \beta)| \leq \frac{1}{[N/M_\alpha]} \sum_{n=1}^{[N/M_\alpha]} \sup_{Q \in \mathbb{R}_{k-1}[t]} \left| \frac{1}{M_\alpha} \sum_{m=0}^{M_\alpha-1} e^{2\pi i(m^k \alpha + Q(m))} f(T^{M_\alpha n+m} x) \right| \leq \frac{\epsilon}{3}.$$

Now we deal with $A_{M_\alpha, N}$. Choose a neighborhood V_α of α such that

$$\sup_{\beta \in V_\alpha} \sup_{1 \leq m \leq M_\alpha} \left| e^{2\pi i m^k \beta} - e^{2\pi i m^k \alpha} \right| \leq \frac{\epsilon}{3\|f\|}.$$

Then for all N and all $\beta \in V_\alpha$ we have

$$A_{M, N}(\alpha, \beta) \leq \frac{\epsilon}{3}.$$

For N large enough we have $2\|f\|M_\alpha/N \leq \frac{\epsilon}{3}$. Thus we have proved (4.14).

We conclude for (4.12) from (4.14) by using a finite covering argument for the compact set $[0, 1]$ where α varies. \square

4.4. TWWT with polynomial weights. We restate Theorem A as follows.

Theorem 5. *Let (X, T) be a uniquely ergodic topological dynamical system and let $k \geq 1$ be an integer. Suppose that the invariant measure has X as support and*

(H1) (X, T^j) for $1 \leq j < \infty$ are all uniquely ergodic.

(H2) $E_j(T) = G_j(T)$ for all $0 \leq j \leq k$.

For any continuous function $f \in C(T)$, the following assertions are equivalent

- (a) $f \in G_k(T)^\perp$;
- (b) for $x \in X$, we have

$$\lim_{N \rightarrow \infty} \sup_{P \in \mathbb{R}_k[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i P(n)} f(T^n x) \right| = 0. \quad (4.16)$$

Proof. (b) implies (a): Let $g \in G_k(T)$. Then there are $g_j \in G_j(T)$ with $g_k = g$ such that

$$g_j(Tx) = g_{j-1}(x)g_j(x) \quad (1 \leq j \leq k).$$

Then $g(T^n x) = e^{2\pi i P(n)}$ with

$$P(t) = \sum_{j=0}^k \theta_j \binom{t}{k-j} \in \mathbb{R}_k[t]$$

where $e^{2\pi i \theta_j} = g_j(x)$. Therefore, by the Krylov-Bogoliubov theorem and (4.16), we get

$$\int g f d\mu = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} g(T^n x) f(T^n x) = 0.$$

(a) *implies* (b): We prove that (H1), (H2) and (a) imply (b), by induction on k . The case $k = 1$ was already proved (see Theorem 2 (3)). Let $k \geq 2$ and assume that the result is true for $k - 1$. We are going to prove that (H1), (H2), (a) and the induction hypothesis imply (b). By Theorem 3 and Proposition 4, it suffices to prove that for $\alpha \in D(T)$ and any $f \in G_k(T)^\perp \cap C(X)$ we have

$$\forall x \in X, \quad \lim_{N \rightarrow \infty} \sup_{Q \in \mathbb{R}_{k-1}[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i(n^k \alpha + Q(n))} f(T^n x) \right| = 0. \quad (4.17)$$

That $\alpha \in D(T)$ means $\eta := e^{2\pi i \ell \alpha} \in G_0(T)$ for some integer $\ell \in \mathbb{Z} \setminus \{0\}$. Notice that $\xi := \eta^{k!} \in G_0(T)$. Let $\tilde{\gamma} \in G_1(T)$ be an eigenfunction associated to ξ , i.e.

$$T\tilde{\gamma} = \xi\tilde{\gamma}.$$

Let us consider an extension $(X \times K^{k-1}, S)$ of (X, T) , as that in Lemma 3 with $p = k - 1$, where

$$S(x; z_1, \dots, z_{k-1}) = (Tx; \gamma(x)z_1, z_1 z_2, \dots, z_{k-2} z_{k-1}), \quad \gamma = \lambda \tilde{\gamma}.$$

By Lemma 3, we can choose $\lambda \in K$ such that S^j ($j \geq 1$) are all uniquely ergodic because T^j ($j \geq 1$) are all assumed uniquely ergodic. That is to say S verifies (H1). It is clear that the invariant measure of S has full support. By lemma 4, S verifies (H2) with k replaced by $k - 1$, because T verifies (H2) with k . Then we can apply the induction hypothesis to S to obtain: for any $F \in G_{k-1}(S)^\perp$, we have

$$\forall \omega \in X \times K^{k-1}, \quad \lim_{N \rightarrow \infty} \sup_{P \in \mathbb{R}_{k-1}[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i P(n)} F(S^n \omega) \right| = 0. \quad (4.18)$$

Choose $F(x, z_1, \dots, z_{k-1}) = f(x)z_{k-1}$. For $\omega = (x, 1, \dots, 1)$, we have

$$F(S^n \omega) = f(T^n x) \xi^{\binom{n}{k}} \gamma(x)^{\binom{n}{k-1}}$$

where we have used the formula (3.8) for the expression of S^n . Since $f \in G_k(T)^\perp$, we have $F \in G_{k-1}(S)^\perp$ by Lemma 4. So, we can apply (4.18) to the function $F(x, z_1, \dots, z_{k-1}) = f(x)z_{k-1}$ and the point $\omega = (x, 1, \dots, 1)$. This gives

$$\forall x \in X, \quad \lim_{N \rightarrow \infty} \sup_{Q \in \mathbb{R}_{k-1}[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i Q(n)} \xi^{\binom{n}{k}} f(T^n x) \right| = 0.$$

Here we have used the facts that $|\gamma(x)| = 1$ and $\binom{n}{k-1}$ is a polynomial of degree $k - 1$. Now observe that

$$\xi^{\binom{n}{k}} = \eta^{k! \binom{n}{k}} = e^{2\pi i n(n-1)\dots(n-k+1)\ell\alpha} = e^{2\pi i(n^k \ell\alpha + R(n))}$$

with $R \in \mathbb{R}_{k-1}[t]$. Thus we can conclude that if $e^{2\pi i \ell \alpha} \in G_0(T)$ we have

$$\forall x \in X, \quad \lim_{N \rightarrow \infty} \sup_{Q \in \mathbb{R}_{k-1}[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i(Q(n) + n^k \ell \alpha)} f(T^n x) \right| = 0. \quad (4.19)$$

Now we are going to take off ℓ in (4.19) in order to finish the proof. Since $\eta = e^{2\pi i \ell \alpha} \in G_0(T)$, we have $\eta^\ell \in G_0(T^\ell)$. Then $e^{2\pi i \ell^k \alpha} \in G_0(T^\ell)$ because $G_0(T^\ell)$ is a group. We are going to apply (4.19) to the system (X, T^ℓ) . First remark that, by the transitivity of T^j and Lemma 1, the system (X, T^ℓ) has the properties (H1) and (H2). On the other hand, as $f \in G_k(T)^\perp$ and $G_k(T)$ is stable under T , we have $f \circ T^j \in G_k(T)^\perp$ for all $j \geq 0$. By Lemma 1, $f \circ T^j \in G_k(T^\ell)^\perp$ for all $j \geq 0$. So, we can apply (4.19) to the system (X, T^ℓ) , the function $f \circ T^j$ and $\ell^k \alpha$ (replacing $\ell \alpha$) in order to get

$$\forall x \in X, \quad \lim_{N \rightarrow \infty} \sup_{Q \in \mathbb{R}_{k-1}[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i(Q(n) + n^k \ell^k \alpha)} f \circ T^j(T^{\ell n} x) \right| = 0$$

which is equivalent to

$$\forall x \in X, \quad \lim_{N \rightarrow \infty} \sup_{Q \in \mathbb{R}_{k-1}[t]} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i(Q(n\ell+j) + (n\ell+j)^k \ell \alpha)} f(T^{n\ell+j} x) \right| = 0.$$

This allows us to deduce (4.17) by taking average over $0 \leq j < \ell$. \square

5. NILSYSTEMS

Let $s \geq 1$ be an integer. Let N be a s -step, simply connected nilpotent Lie group, Γ a discrete subgroup of N such that N/Γ is compact. The s -step nilpotence means that we have the following lower central series

$$G_1 \triangleright G_2 \triangleright \cdots \triangleright G_s \triangleright G_{s+1} = \{e\}$$

where $G_{i+1} = [N, G_i]$ for $i \geq 1$ with $G_0 = G_1 = N$. Recall that the commutator group $[N, G_i]$ is the group generated by all commutators $hgh^{-1}g^{-1}$ with $h \in N, g \in G_i$. Then the quotient space N/Γ is called an s -step nilmanifold. Any $g \in N$ acts on N/Γ by left multiplication $x\Gamma \mapsto gx\Gamma$. This left translation will be denoted by $T_g : N/\Gamma \rightarrow N/\Gamma$, called a s -step nilsystem.

A (basic) s -step nilsequence is a sequence of the form $(f(T_g^n x))$ i.e. $(f(g^n x))$ where x is a point of N/Γ and $f : N/\Gamma \rightarrow \mathbb{C}$ a continuous function.

The additive group \mathbb{R}^d is 1-step nilpotent and the torus $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ is a 1-step nilmanifold.

5.1. Fully oscillating nilsequences. The following is the restatement of Theorem B in Introduction.

Theorem 6. *Let G be a connected and simply connected nilpotent Lie group, Γ a discrete cocompact subgroup of G and $g \in G$. Let $X = G/\Gamma$ be the nilmanifold and let $T : X \rightarrow X$ be defined by $x\Gamma \mapsto gx\Gamma$. Suppose that (X, T) is uniquely ergodic. Then for any $F \in C(X)$ such that $F \in G_1(T)^\perp$ and any $x \in G$, the sequence $F(g^n x\Gamma)$ is fully oscillating.*

This is a direct consequence of Theorem 5, because there is no other quasi-eigenfunctions than eigenfunctions, which are all continuous, for any ergodic nilsystem associated to a connected and simply connected nilpotent Lie groups. In fact, if the system had higher order eigenfunctions, then it would have second order eigenfunctions. These second order eigenfunctions live on a 2-step factor which would also be associated to connected and simply connected nilpotent Lie group, but which supports no second order eigenfunctions other than true eigenfunctions.

It is the moment to give some comments. First recall the following theorem due to Lesigne.

Theorem 7 (Lesigne [51, 53]). *Let N/Γ be a nilmanifold and let $a \in N$. For any continuous function $f \in C(N/\Gamma)$ and any point $x \in N/\Gamma$, the following limit exists*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(a^n x).$$

The essential point of this theorem is the everywhere existence of the limit of the ergodic averages (with constant weights). The almost every convergence of the following multiple ergodic limit

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{k=1}^{\ell} f_k(a^{kn} x), \quad (f_1, \dots, f_\ell \in L^\infty(m_{N/\Gamma})) \quad (5.1)$$

was proved by Lesigne [51, 53]. Under the assumption of ergodicity, an explicit formula for the limit was found by Lesigne [53] for 2-step nilsystems, by Ziegler [67] for all nilsystems. This formula was later generalized to the case where $n, 2n, \dots, \ell n$ are replaced by polynomials by Leibman [49].

As pointed by B. Host [38] (personal communication), it could be possible to deduce Theorem 6 from Theorem 7. The reason is as follows. Let U be a $d \times d$ unipotent matrix with integer entries and $b \in \mathbb{T}^d$. Then the affine map $Sy = Uy + b$ defines an affine d -step nilsystem. If $P \in \mathbb{R}_d[t]$, then the sequence $e^{2\pi i P(n)}$ is produced by an affine d -step nilsystem, namely there exists an affine d -step nilsystem (\mathbb{T}^d, S) and a

point $y_0 \in \mathbb{T}^d$ such that $e^{2\pi iP(n)} = f(S^n y_0)$ for every n . Let $(N/\Gamma, T_g)$ be an ergodic nilsystem (hence minimal and uniquely ergodic), a continuous function F on N/Γ and $x_0 \in N/\Gamma$. Let P, S, y_0 and f be as above. Then the sequence of general term $F(g^n x_0)e^{2\pi iP(n)}$ is produced by the nilsystem $(X \times Y, T \times S)$:

$$F(g^n x_0)e^{2\pi iP(n)} = (F \otimes f)(T_g \times S)^n(x_0, y_0).$$

It follows from Theorem 7 that the averages of this sequence converge. More precisely, by Leibman [49] there exists a sub-nilmanifold W of $X \times Y$ such that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} F(g^n x_0)e^{2\pi iP(n)} = \int_W F(x)f(y)dm_W(x, y) \quad (5.2)$$

where m_W is the Haar measure on W . Note that $(W, T \times S)$ is a joining of (X, T_g) and (Y, S) . However, to complete the argument, we need to prove that the integral on the RHS of (5.2) is equal to zero.

5.2. 3-dimensional Heisenberg group. Before proving Theorem C, we discuss a special Heisenberg group. The 3-dimensional Heisenberg group

$$H := \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ & 1 & \mathbb{R} \\ & & 1 \end{pmatrix}$$

is a 2-step simply connected nilpotent Lie group. The group operation is the matrix multiplication. If we simply write $\langle x, y, z \rangle$ for an element of H , then the group operation in H is defined by

$$\langle a, b, c \rangle \langle x, y, z \rangle = \langle a + x, b + y, c + z + ay \rangle. \quad (5.3)$$

If we take the subgroup

$$\Gamma := \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ & 1 & \mathbb{Z} \\ & & 1 \end{pmatrix},$$

we get a 2-step nilmanifold H/Γ . Let $g = \langle \alpha_1, \alpha_2, \alpha_3 \rangle \in H$. Then

$$T_g \langle x, y, z \rangle = \langle x + \alpha_1, y + \alpha_2, z + \alpha_3 + \alpha_1 y \rangle \pmod{\Gamma}.$$

Take $\mathcal{F} = [0, 1) \times [0, 1) \times [0, 1)$ as fundamental domain of H/Γ . For $x = \langle x_1, x_2, x_3 \rangle \in N$, let

$$\gamma_x = \langle -[x_1], -[x_2], -[x_3 - x_1[x_2]] \rangle \in \Gamma.$$

Then $\tau(x) := x\gamma_x \in \mathcal{F}$. Such a γ_x is unique. Actually we have

$$\tau(x) = \langle \{x_1\}, \{x_2\}, \{x_3 - x_1[x_2]\} \rangle. \quad (5.4)$$

It can be inductively proved that

$$g^n = \langle n\alpha_1, n\alpha_2, n\alpha_3 + C_n^2\alpha_1\alpha_2 \rangle.$$

Then for any $x = \langle x_1, x_2, x_3 \rangle \in N$, we have

$$g^n x = \langle n\alpha_1 + x_1, n\alpha_2 + x_2, n\alpha_3 + x_3 + C_n^2\alpha_1\alpha_2 + n\alpha_1x_2 \rangle.$$

Then for the map $T_g : N/\Gamma \rightarrow N/\Gamma$ with N/Γ represented by the fundamental domain \mathcal{F} , by (5.4) we have

$$T_g^n x = \langle n\alpha_1 + x_1, n\alpha_2 + x_2, n\alpha_3 + x_3 + C_n^2\alpha_1\alpha_2 + n\alpha_1x_2 - (n\alpha_1 + x_1)[n\alpha_2 + x_2] \rangle \quad (5.5)$$

where the coordinates on the RHS are considered $\pmod{\Gamma}$. In particular,

$$T_g^n 0 = \langle \{n\alpha_1\}, \{n\alpha_2\}, \{n\alpha_3 + C_n^2\alpha_1\alpha_2 - n\alpha_1[n\alpha_2]\} \rangle. \quad (5.6)$$

Let us give here a direct proof of no second order quasi-functions in the case of Heisenberg ergodic translation. Let $F(x_1, x_2, x_3)$ be a second order quasi-eigenfunction, i.e. $T_g F = hF$ with h an eigenfunction, which is of the form $ae^{2\pi i(kx+jy)}$ with $|a| = 1$, $(k, j) \in \mathbb{Z}^2$. Since F is orthogonal to all eigenfunctions, F is independent of x_1 and x_2 . So $F(x_1, x_2, x_3) = f(x_3)$ for some function f . Then, using (5.4) and (5.5), we can write the equation $T_g F = hF$ as

$$f(\alpha_3 + x_3 + \alpha_1x_2 - [\alpha_1 + x_1](\alpha_2 + x_2)) = h(x_1, x_2)f(x_3).$$

For almost all (x_1, x_2) we develop the function of x_3 into Fourier series

$$\sum_n \widehat{f}(n) e^{2\pi in(\alpha_3 + \alpha_1x_2 - (\alpha_1 + x_1)(\alpha_2 + x_2))} e^{2\pi inx_3} = h(x_1, x_2) \sum_n \widehat{f}(n) e^{2\pi inx_3}$$

So, by comparing the Fourier coefficients, we obtain that, for a fixed n , either $\widehat{f}(n) = 0$ or for almost all (x_1, x_2)

$$e^{2\pi in(\alpha_3 + \alpha_1x_2 - (\alpha_1 + x_1)(\alpha_2 + x_2))} = h(x_1, x_2).$$

In other words, for $0 \leq x_2 < 1 - \alpha_1$ we have

$$e^{2\pi in(\alpha_3 + \alpha_1x_2)} = h(x_1, x_2)$$

which is impossible; and for $1 - \alpha_1 \leq x_2 < 1$ we have

$$e^{2\pi in(\alpha_3 - \alpha_1 + \alpha_1x_2 - x_1)} = h(x_1, x_2),$$

which is impossible too. Thus F must be constant and h must be 1.

Let us consider a Mal'cev basis of H , consisting of

$$e_1 = \begin{pmatrix} 1 & 1 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 1 \\ & & 1 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 1 & 0 & 1 \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

See [13] for Mal'cev bases. These elements determine three one-parameter subgroups $(e_i^t)_{t \in \mathbb{R}}$ ($i = 1, 2, 3$):

$$e_1^t = \begin{pmatrix} 1 & t & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}, \quad e_2^t = \begin{pmatrix} 1 & 0 & 0 \\ & 1 & t \\ & & 1 \end{pmatrix}, \quad e_3^t = \begin{pmatrix} 1 & 0 & t \\ & 1 & 0 \\ & & 1 \end{pmatrix}.$$

Any element $g \in H$ has a unique representation as follows

$$g = e_1^{t_1} e_2^{t_2} e_3^{t_3} = \begin{pmatrix} 1 & t_1 & t_3 + t_1 t_2 \\ & 1 & t_2 \\ & & 1 \end{pmatrix}.$$

The triple t_1, t_2, t_3 will be denoted $\langle t_1, t_2, t_3 \rangle_{\text{II}}$, called the Mal'cev coordinates of second kind of g . We write $g = \phi_{\text{II}}(t_1, t_2, t_3)$, or simply $g = \langle t_1, t_2, t_3 \rangle_{\text{II}}$. Notice that $\phi_{\text{II}} : \mathbb{R}^3 \rightarrow H$ is a diffeomorphism and that $\Gamma = \phi_{\text{II}}(\mathbb{Z}^3)$. Also notice that

$$\langle t_1, t_2, t_3 \rangle_{\text{II}} = \langle t_1, t_2, t_3 + t_1 t_2 \rangle, \quad \langle a, b, c \rangle = \langle a, b, c - ab \rangle_{\text{II}}.$$

The group law is expressed by the Mal'cev coordinates as follows

$$\langle t_1, t_2, t_3 \rangle_{\text{II}} * \langle s_1, s_2, s_3 \rangle_{\text{II}} = \langle t_1 + s_1, t_2 + s_2, t_3 + s_3 - t_2 s_1 \rangle_{\text{II}}. \quad (5.7)$$

In fact

$$\begin{aligned} & \langle t_1, t_2, t_3 \rangle_{\text{II}} * \langle s_1, s_2, s_3 \rangle_{\text{II}} \\ &= \langle t_1, t_2, t_3 + t_1 t_2 \rangle \langle s_1, s_2, s_3 + s_1 s_2 \rangle \\ &= \langle t_1 + s_1, t_2 + s_2, (t_3 + t_1 t_2) + (s_3 + s_1 s_2) + t_1 s_2 \rangle \\ &= \langle t_1 + s_1, t_2 + s_2, (t_3 + t_1 t_2) + (s_3 + s_1 s_2) + t_1 s_2 - (t_1 + s_1)(t_2 + s_2) \rangle_{\text{II}} \\ &= \langle t_1 + s_1, t_2 + s_2, t_3 + s_3 - t_2 s_1 \rangle_{\text{II}} \end{aligned}$$

Let $x = \langle x_1, x_2, x_3 \rangle_{\text{II}}$. Let

$$\gamma_x = \langle -[x_1], -[x_2], -[x_3 + [x_1]x_2] \rangle_{\text{II}} \in \Gamma.$$

Then $\tau_2(x) := x\gamma_x \in \mathcal{F}$. We have

$$\tau(x) = \langle \{x_1\}, \{x_2\}, \{x_3 + [x_1]x_2\} \rangle_{\text{II}}$$

Let $g = \langle \alpha_1, \alpha_2, \alpha_3 \rangle_{\text{II}}$. Inductively we get

$$g^n = \langle n\alpha_1, n\alpha_2, n\alpha_3 - C_n^2 \alpha_1 \alpha_2 \rangle_{\text{II}}$$

then for any $x = \langle x_1, x_2, x_3 \rangle_{\text{II}}$, we have

$$g^n x = \langle n\alpha_1 + x_1, n\alpha_2 + x_2, n\alpha_3 + x_3 - C_n^2 \alpha_1 \alpha_2 - n\alpha_2 x_1 \rangle_{\text{II}}$$

Then for $T_g : N/\Gamma \rightarrow N/\Gamma \rightarrow$ with N/Γ represented by the fundamental domain \mathcal{F} , mod 1 we have

$$T_g^n x = \langle n\alpha_1 + x_1, n\alpha_2 + x_2, n\alpha_3 + x_3 - C_n^2 \alpha_1 \alpha_2 - n\alpha_2 x_1 - [n\alpha_1 + x_1](n\alpha_2 + x_2) \rangle_{\Pi}$$

In particular

$$T_g^n 0 = \langle n\alpha_1, n\alpha_2, n\alpha_3 - C_n^2 \alpha_1 \alpha_2 - [n\alpha_1]n\alpha_2 \rangle_{\Pi} \pmod{1}. \quad (5.8)$$

5.3. Proof of Theorem C. Consider the $(2m+1)$ -dimensional Heisenberg group H_m ($m \geq 1$), which is the space $\mathbb{R}^{2m+1} = \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ equipped with the group law defined by

$$\langle a, b, c \rangle \langle x, y, z \rangle = \langle a + x, b + y, c + z + B(a, y) \rangle \quad (5.9)$$

for $(a, b, c) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$ and $(x, y, z) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$, where $B : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ is the bilinear form

$$B(a, y) = \sum_{i=1}^m a_i y_i.$$

Let $g = \langle \alpha, \beta, \gamma \rangle \in H_n$ with $\alpha = (\alpha_1, \dots, \alpha_m)$, $\beta = (\beta_1, \dots, \beta_m)$ and $\gamma \in \mathbb{R}$ (any choice for γ). We consider the translation T_g defined by g :

$$T_g \langle x, y, z \rangle = \langle \alpha + x, \beta + y, \gamma + z + B(\alpha, y) \rangle. \quad (5.10)$$

Take $\Gamma_m = \mathbb{Z}^{2m+1}$. For $\bar{x} = \langle x, y, z \rangle \in H_m$. Let

$$\gamma_{\bar{x}} = \langle -[x], -[y], -[z - B(x, [y])] \rangle \in \Gamma_m.$$

Then $\tau(\bar{x}) := \bar{x}\gamma_{\bar{x}} \in \mathcal{F}_m := [0, 1]^{2m+1}$. Such a $\gamma_{\bar{x}}$ is unique. We have

$$\tau(\bar{x}) = \langle \{x\}, \{y\}, \{z - B(x, [y])\} \rangle \quad (5.11)$$

We have

$$g^n = \langle n\alpha, n\beta, n\gamma + C_n^2 B(\alpha, \beta) \rangle.$$

Then for any $\bar{x} = \langle x, y, z \rangle \in N := H_m/\Gamma_m$, we have

$$g^n \bar{x} = \langle n\alpha + x, n\beta + y, n\gamma + z + C_n^2 B(\alpha, \beta) + nB(\alpha, y) \rangle$$

Then for the map $T_g : H_m/\Gamma_m \rightarrow H_m/\Gamma_m$ with H_m/Γ_m represented by the fundamental domain \mathcal{F}_m , by (5.11) we have

$$T_g^n 0 = \langle \{n\alpha\}, \{n\beta\}, \{n\gamma + C_n^2 B(\alpha, \beta) - B(n\alpha, [n\beta])\} \rangle. \quad (5.12)$$

The condition on α and β implies that T_g is totally ergodic, by a theorem of Green [32] (see a simpler proof in [59]) and that there is no quasi eigenfunctions other than true eigenfunctions.

Let

$$\omega_n = n\gamma + C_n^2 B(\alpha, \beta) - B(n\alpha, [n\beta]) \pmod{1}.$$

Fix an integer $m \in \mathbb{Z} \setminus \{0\}$. We can apply Theorem 6 to $F(x, y, z) = e^{2\pi i m z}$, which is orthogonal to all eigenfunctions, and we get that the

sequence $(e^{2\pi i m \omega_n})$ is orthogonal to all polynomial sequences $e^{2\pi i P(n)}$ with $P \in \mathbb{R}[t]$. In other words, $(e^{2\pi i m \omega_n})$ is fully oscillating. Since $Q(n) = n\gamma + C_n^2 B(\alpha, \beta)$ is a real polynomial of n , so the sequence $(e^{-2\pi i m B(n\alpha, [n\beta])})$ then the sequence $(e^{2\pi i m B(n\alpha, [n\beta])})$ is fully oscillating.

Let $\varphi \in C(\mathbb{T})$ with $\int \varphi(x) dx = 0$. Now we claim that the sequence $\varphi(B(n\alpha, [n\beta]))$ is fully oscillating. In fact, for any $\epsilon > 0$ there exists a trigonometric polynomial g without the constant term such that $|\varphi(x) - g(x)| < \epsilon$. Then for any $P \in \mathbb{R}[t]$ we have

$$\begin{aligned} & \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i P(n)} \varphi(B(n\alpha, [n\beta])) \right| \\ & \leq \limsup_{N \rightarrow \infty} \left| \frac{1}{N} \sum_{n=0}^{N-1} e^{2\pi i P(n)} g(B(n\alpha, [n\beta])) \right| + \epsilon. \end{aligned}$$

But the last limit is equal to zero by the full oscillation of $(e^{2\pi i m B(n\alpha, [n\beta])})$ that we have already proved.

Similarly we can treat other generalized polynomial sequences by looking at different nilmanifolds.

Bergelson [5] pointed out to us that Lemma 5.1 of Haland [34] stated a result similar to Theorem C for polynomials of degree 2. Konieczny [45] observed that it could be possible to give an alternative proof of Theorem C, replacing the application of Theorem B by the application of the equidistribution developed by Leibman [50].

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REFERENCES

- [1] L. M. Abramov, *Metric automorphisms with quasi-discrete spectrum*, Izv. Akad. Nauk. U. S. S. R., 26 (1962), 513-530.
- [2] I. Assani, *Uniform Wiener-Wintner theorems for weakly mixing dynamical systems*, unpublished preprint, 1993.
- [3] I. Assani, *Wiener Wintner Ergodic theorems*, World Scientific Publishing Company, 2003.
- [4] L. Auslander, L. Green and F. Hahn, *Flows on homogeneous spaces*, With the assistance of L. Markus and W. Massey, and an appendix by L. Greenberg.

Annals of Mathematics Studies, No. 53, Princeton University Press, Princeton, N.J., 1963, vii+107 pages.

- [5] V. Bergelson, Personal communication.
- [6] V. Bergelson and A. Leibman, *Distribution of values of bounded generalized polynomials*, Acta. Math. 198 (2007), no. 2, 155-230.
- [7] A. Bellow and V. Losert, *The weighted pointwise ergodic theorem and the individual ergodic theorem along subsequences*, Trans. Amer.Math. Soc., 288 (1985), 307-345.
- [8] N. Bourbaki, *Lie groups and Lie algebras.*, Chapters 1-3, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation, xviii+450 pages.
- [9] J. Bourgain, *Double recurrence and almost sure convergence*, J. Reine Angew. Math. 404 (1990), 140-161.
- [10] J. Bourgain, *On the correlation of the Moebius function with rank-one systems*, J. Anal. Math. 120 (2013), 105-130.
- [11] J. Bourgain, *Möbius-Walsh correlation bounds and an estimate of Mauduit and Rivat*, J. Anal. Math. 119 (2013), 147-163.
- [12] J. Bourgain, P. Sarnak, and Ziegler, *Disjointness of Möbius from horocycle flows*. From Fourier analysis and number theory to Radon transforms and geometry, 67-83, Dev. Math. 28, Springer, New York, 2013, pp. 67-83.
- [13] L. J. Corwin and F. P. Greenleaf, *Representations of nilpotent Lie groups and their applications* Part I, Cambridge Studies in Advanced Mathematics, vol. 18, Cambridge University Press, Cambridge, 1990, Basic theory and examples, viii+269 pages.
- [14] T. Downarowicz and S. Kasjan, *Odometers and Toeplitz systems revisited in the context of Sarnak's conjecture*, Studia Math. 229 (2015), no. 1, 45-72.
- [15] T. Eisner, *A polynomial version of Sarnak's conjecture*, C. R. Math. Acad. Sci. Paris 353 (2015), 569-572.
- [16] T. Eisner and P. Zorin-Kranich, *Uniformity in the Wiener-Wintner theorem for nilsequences*, Discrete Contin. Dyn. Syst. 33 (2013) 3497-3516.
- [17] E. H. El Abdalaoui, M. Lemańczyk, and T. de la Rue, *On spectral disjointness of powers for rank-one transformations and Möbius orthogonality*, J. Funct. Anal. 266 (2014), no. 1, 284-317.
- [18] E. H. El Abdalaoui, M. Lemańczyk, and T. de la Rue, *Automorphisms with quasi-discrete spectrum, multiplicative functions and average orthogonality along short intervals*, Int.Math. Res. Not. IMRN, Volume 2017, Issue 14, 4350-4368
- [19] El H. El Abdalaoui, J. Kulaga-Przymus, M. Lemanczyk and Th. De La Rue, *The Chowla and the Sarnak conjectures from ergodic theory point of view*, Discrete Contin. Dyn. Syst. 37 (2017), no.6, 2899-2944.
- [20] A. H. Fan, *Oscillating Sequences of higher orders and topological systems of quasi-discrete spectrum*, preprint
- [21] A. H. Fan, *Fully oscillating sequences and weighted multiple ergodic limit*, C. R. Acad. Sci. Paris, Ser.I, 355 (2017) 866-870.
- [22] A. H. Fan, *Weighted Birkhoff ergodic theorem with oscillating weights*, Ergod. Th. & Dynam. Syst., Published online: 25 September 2017, pp. 1-15. <https://doi.org/10.1017/etds.2017.81> See also arXiv:1705.02501

- [23] A. H. Fan and Y. P. Jiang, *Oscillating sequences, minimal mean attractability and minimal mean-Lyapunov-stability*, Erg. Th. Dynam. Syst. Published online: 14 March 2017, pp. 1-36 doi:10.1017/etds.2016.121 See also arXiv:1511.05022v2
- [24] A. H. Fan and D. Schneider, *Recurrence properties of sequences of integers*, Sci. China Math. No. 3, Vol. 53 (2010), 641-656.
- [25] S Ferenzi, J. Kulaga-Przymus, M. Lemańczyk, and C Mauduit, *Substitutions and Möbius disjointness*, preprint (2015).
- [26] N. Frantzikinakis, *Uniformity in the polynomial Wiener-Wintner theorem*, Ergod. Th. & Dynam. Sys. 26 (2006), 1061-1071.
- [27] H. Furstenberg, *Stationary processes and prediction theory*, Ann. Math. Studies 44, Princeton University Press, Princeton 1960.
- [28] H. Furstenberg, *Strict ergodicity and transformations of the torus*, Amer. J. Math. 83 (1961), 573-601.
- [29] H. Furstenberg, *Nonconventional ergodic averages*, in The legacy of John von Neumann (Hempstead, NY, 1988), Proc. Sympos. Pure Math., vol. 50, Amer. Math. Soc., Providence, RI, 1990, p. 43-56.
- [30] B. Green and T. Tao, *Quadratic uniformity of the Möbius function*, Ann. Inst. Fourier, Grenoble 58, 6 (2008) 1863-1935.
- [31] B. Green and T. Tao, *The Möbius function is strongly orthogonal to nilsequences*, Ann. of Math. (2) 175 (2012), no. 2, 541-566.
- [32] L. W. Green, *Spectra of nilflows*, Bull. Amer. Math. Soc. 67 (1961), 414-415.
- [33] F. Hahn and W. Parry, *Minimal dynamical systems with quasi-discrete spectrum*. J. London Math. Soc., **40** (1965), 309-323.
- [34] I. J. Haland, *Uniform distribution of generalized polynomials*, Journal of Number Theory 45 (1993), 327-366.
- [35] I. J. Haland, *Uniform distribution of generalized polynomials of the product type*, Acta Arithmetica 67 (1994), 13-27.
- [36] I. J. Haland and D. E. Knuth, *Polynomials involving the floor function*, Mathematica Scandinavica 76 (1995), 194-200.
- [37] P. R. Halmos and J. von Neumann, *Operator methods in classical mechanics. II*, Ann. of Math. (2) 43 (1942), 332-350.
- [38] B. Host, Personal communication.
- [39] B. Host and B. Kra, *Nonconventional ergodic averages and nilmanifolds*, Ann. Math. (2) 161 (2005), no. 1, 397-488.
- [40] B. Host and B. Kra, *Uniformity seminorms on ℓ^∞ and applications*, J. Anal. Math. 108 (2009), 219-276.
- [41] W. Huang, Z. Lian, S. Shao, and X. Ye, *Sequences from zero entropy noncommutative toral automorphisms and Sarnak Conjecture*, J. Differential Equations 263 (2017), no.1, 779-810.
- [42] W. Huang, Z. R. Wang and X. D. Ye, *Measure complexity and Möbius disjointness*, arXiv:1707.06345.
- [43] W. Huang, Z. Wang and G. Zhang, *Möbius disjointness for topological model of any ergodic system with discrete spectrum*, preprint (2016), arXiv:1608.08289v2.
- [44] J. Kulaga-Przymus and M. Lemańczyk, *The Möbius function and continuous extensions of rotations*, Monatsh. Math. 178 (2015), no. 4, 553-582.
- [45] J. Konieczny, Personal communication.

- [46] U. Krengel, *Ergodic theorems*, Walter de Gruyter, Berlin, New York, 1982.
- [47] J. Kulaga-Przymus and M. Lemańczyk, *The Möbius function and continuous extensions of rotations*, Monatsh. Math. 178 (2015), no. 4, 553-582.
- [48] A. Leibman, *Polynomial sequences in groups*, J. Algebra 201 (1998), 189-206.
- [49] A. Leibman, *Pointwise convergence of ergodic averages for polynomial sequences of rotations of a nilmanifold*, Ergod. Th. & Dynam. Sys. 25 (1) (2005), 201-213.
- [50] A. Leibman, *A canonical form and the distribution of values of generalized polynomials*, Israel J. Math., 188 (2012), 131-176.
- [51] E. Lesigne, *Théorèmes ergodiques pour une translation sur une nilvariété*, Ergod. Th. & Dynam. Sys. 9 (1) (1989), 115-126.
- [52] E. Lesigne, *Un théorème de disjonction de systèmes dynamiques et une généralisation du théorème ergodique de Wiener-Wintner*, Ergod. Th. & Dynam. Syst., 10 (1990), 513-521.
- [53] E. Lesigne, *Sur une nilvariété, les parties minimales associées à une translation sont uniquement ergodiques*, Ergod. Th. & Dynam. Sys. 11(2) (1991), 379-391.
- [54] E. Lesigne, *Spectre quasi-discret et théorème ergodique de Wiener-Wintner pour les polymômes*, Ergod. Th. & Dynam. Syst., 13 (1993), 676-684.
- [55] D. Lenz, *Continuity of eigenfunctions of uniquely ergodic dynamical systems and intensity of Bragg peaks*, Comm. Math. Phys. 287 (2009), no. 1, 225-258.
- [56] J. Y. Liu and P. Sarnak, *The Möbius function and distal flows*, Duke Math. J. 164 (2015), no. 7, 1353-1399.
- [57] A. I. Mal'cev, *On a class of homogeneous spaces*, Izvestiya Akad. Nauk. SSSR. Ser. Mat. 13 (1949), 9-32.
- [58] C. Mauduit and J. Rivat, *Sur un problème de Gelfond: la somme des chiffres des nombres premiers*. Ann. Math. (2) 171 (2010), 1591-1646.
- [59] W. Parry, *Dynamical systems on nilmanifolds*, Bull. London Math. Soc. 2 (1970), 37-40.
- [60] E. A. Robinson, *On uniform convergence in the Wiener-Wintner theorem*, J. London Math. Soc. (2) 49 (1994), 493-501.
- [61] P. Sarnak, *Three lectures on the Möbius function, randomness and dynamics*, IAS Lecture Notes, 2009;
[http://publications.ias.edu/sites/default/files/MobiusFunctionsLectures\(2\).pdf](http://publications.ias.edu/sites/default/files/MobiusFunctionsLectures(2).pdf).
- [62] P. Sarnak, *Möbius randomness and dynamics*, Not. S. Afr. Math. Soc. 43 (2012), 89-97.
- [63] R. X. Shi, *Equivalent definitions of oscillating sequences of higher orders*, preprint.
- [64] Z. Wang, *Möbius disjointness for analytic skew products*, Invent. Math., Volume 209, Issue 1 (2017), 175-196.
- [65] N. Wiener and A. Wintner, *Harmonic analysis and ergodic theory*, Amer. J. Math., 63 (1941), 415-426.
- [66] W.A. Veech, *Möbius orthogonality for generalized Morse-Kakutani flows*, Amer. J. Math. (2016), to appear.
- [67] T. Ziegler, *A non-conventional ergodic theorem for nilsystems*, Ergod. Th. & Dynam. Sys. (2005), 25, 1357-1370.

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