SELF-CONFORMAL SETS WITH POSITIVE HAUSDORFF MEASURE

ANTTI KÄENMÄKI AND SASCHA TROSCHEIT

Abstract. We investigate the Hausdorff measure and content on a class of quasi self-similar sets that include, for example, graph-directed and sub self-similar and self-conformal sets. We show that any Hausdorff measurable subset of such sets has comparable Hausdorff measure and Hausdorff content. In particular, this proves that graph-directed and sub self-conformal sets with positive Hausdorff measure are Ahlfors regular, irrespective of separation conditions. When restricting to the real line and self-conformal sets with Hausdorff dimension strictly less than one, we additionally show that Ahlfors regularity is equivalent to the weak separation condition. In fact, we resolve a self-conformal extension of the dimension drop conjecture for self-conformal sets with positive Hausdorff measure by showing that its Hausdorff dimension falls below the expected value if and only if there are exact overlaps.

1. Introduction

Quasi self-similar sets appear in many branches of dynamical systems and metric geometry. The prime examples are self-conformal sets such as the boundaries of Julia sets of hyperbolic rational functions on $\mathbb{C}$. Notably, the boundary of the Julia set for the mappings $z \mapsto z^2 + c$ for certain $c \in \mathbb{C}$ such as those satisfying $|c| < 1/4$; see [13, §7]. Apart from self-conformal sets there are related constructions such as graph-directed self-conformal sets and sub self-conformal sets that satisfy the conditions of quasi self-similarity. Motivated by recent results in the self-similar case, our goal is to uncover new properties for the Hausdorff measure on this large class of sets.

While the Hausdorff measure is an important and widely known concept, its value is often difficult to establish. The related notion of the Hausdorff content is easier to compute and it shares the same critical exponent, the Hausdorff dimension. Under a mild assumption, we show that in the quasi self-similar case the Hausdorff measure and Hausdorff content are uniformly comparable for all measurable subsets of the quasi self-similar set. As a consequence, we show that quasi self-similar sets with positive Hausdorff measure are Ahlfors regular. Also, and perhaps more notably, we resolve the self-conformal generalisation of the dimension drop conjecture for self-conformal subsets of the real line with positive Hausdorff measure.

Our research is motivated by recent results of Farkas and Fraser [5] who showed that all Hausdorff measurable subsets of self-similar sets have equal Hausdorff measure and Hausdorff content. Their method relies on two main components. The first is an abstract lemma, see also [4, Lemma 2.2], that states: If F is a set with equal and finite Hausdorff measure and content, then every measurable subset of F has equal Hausdorff measure and content at the same exponent. The second component is a Vitali-covering argument which guarantees that the Hausdorff measure and content are equal for all self-similar sets. They give an example of a self-conformal set, the semicircle, which breaks the equality of measure and content. While it is straightforward to show via the Vitali-covering argument that self-conformal sets have comparable Hausdorff measure and content, their approach cannot be generalised as the abstract lemma completely fails for sets where one does not have equality.

In Theorem 2.1, we generalise the result of Farkas and Fraser for a large class of quasi self-similar sets, with a new and concise proof that does not rely on either Vitali-coverings nor decomposition into measurable hulls.

Date: March 23, 2018.

2000 *Mathematics Subject Classification.* Primary 28A78; Secondary 28A80.

Key words and phrases. Hausdorff measure, Hausdorff content, Ahlfors regularity, self-conformal sets.

2. Results for quasi self-similar sets

Recall that the *s-dimensional Hausdorff measure* $\mathcal{H}^s$ of a set $A \subset \mathbb{R}^d$ is defined by

$$\mathcal{H}^s(A) = \lim_{\delta \downarrow 0} \mathcal{H}^s_\delta(A) = \sup_{\delta > 0} \mathcal{H}^s_\delta(A),$$

where

$$\mathcal{H}^s_\delta(A) = \inf\biggl\{ \sum_i \operatorname{diam}(U_i)^s : A \subset \bigcup_i U_i \text{ and } \operatorname{diam}(U_i) \le \delta \biggr\}$$

is the *s-dimensional Hausdorff δ-content* of A. The Hausdorff measure is Borel regular and the Hausdorff content is an outer measure – usually highly non-additive and not a Borel measure. However, the Hausdorff content is slightly easier to compute, and is always finite for bounded sets, irrespective of s. It is straightforward to see that $\mathcal{H}^s(A) = 0$ if and only if $\mathcal{H}^s_\infty(A) = 0$ and so Hausdorff measure and content share the same critical exponent, the *Hausdorff dimension* $\dim_{\mathrm{H}}$ of A which is defined by $\dim_{\mathrm{H}}(A) = \inf\{s : \mathcal{H}^s(A) = 0\}$.

Our main result is the following theorem. We postpone its proof until §4.

Theorem 2.1. *Let $F \subset \mathbb{R}^d$ be a non-empty compact set and $s = \dim_{\mathrm{H}}(F)$. Suppose that there is a constant $D \ge 1$ such that for each $x \in F$ and $0 < r \le \operatorname{diam}(F)$ there exists a mapping $g \colon F \to F \cap B(x, r)$ for which*

$$D^{-1} r|y - z| \le |g(y) - g(z)| \le Dr|y - z| \tag{2.1}$$

for all $y, z \in F$. Then there exists a constant $C \ge 1$ such that

$$\mathcal{H}^s(F \cap A) \le C\mathcal{H}^s_\infty(F \cap A)$$

for all $\mathcal{H}^s$-measurable sets $A \subset \mathbb{R}^d$.

Observe that the assumptions of Theorem 2.1 are stronger than those that define quasi self-similar sets; see [2] and [3, §3.1]. Quasi self-similar sets differ to the sets we consider by only requiring the lower bound in (2.1) to hold. The upper bound is crucial in (4.6) and it seems unlikely that our assumptions are satisfied by quasi self-similarity alone.

The following result is a straightforward corollary of Theorem 2.1. We say that a set $A \subset \mathbb{R}^d$ is *Ahlfors s-regular* if there exists a non-trivial Radon measure μ supported on A and a constant $C \ge 1$ such that

$$C^{-1} r^s \le \mu(B(x, r)) \le Cr^s \tag{2.2}$$

for all $x \in A$ and $0 < r < \operatorname{diam}(A)$.

Theorem 2.2. *Let $F \subset \mathbb{R}^d$ be a set satisfying the assumptions of Theorem 2.1. If $s = \dim_{\mathrm{H}}(F)$, then $\mathcal{H}^s(F) > 0$ if and only if F is Ahlfors s-regular.*

Proof. Assuming F to be Ahlfors s-regular, let μ be a measure satisfying (2.2). Since $\mu(F) \le \sum_i \mu(U_i) \le C \sum_i \operatorname{diam}(U_i)^s$ for all δ-covers $\{U_i\}_i$ of F, we get $\mathcal{H}^s_\delta(F) \ge \mu(F) > 0$ for all $\delta > 0$ and, consequently, $\mathcal{H}^s(F) > 0$. To show the necessity of the Ahlfors regularity, suppose that $\mathcal{H}^s(F) > 0$. By Theorem 2.1, there is a constant $C \ge 1$ such that

$$\mathcal{H}^s|_F(B(x, r)) \le C\mathcal{H}^s_\infty(F \cap B(x, r)) \le C \operatorname{diam}(F \cap B(x, r))^s \le 2^s C r^s$$

for all $x \in F$ and $r > 0$. For each $x \in F$ and $0 < r < \operatorname{diam}(F)$, let $g_{x,r} \colon F \to F \cap B(x, r)$ be as in (2.1). The existence of such mappings implies

$$\mathcal{H}^s|_F(B(x, r)) \ge \mathcal{H}^s(g_{x,r}(F)) \ge D^{-s}\mathcal{H}^s(F) r^s$$

for all $x \in F$ and $0 < r < \operatorname{diam}(F)$. In Lemma 4.1, we shall see that $\mathcal{H}^s(F) < \infty$ and $\mathcal{H}^s|_F$ is therefore a Radon measure. We have thus finished the proof. □

3. Results for self-conformal sets

Let $N \geq 2$ and consider the family of N contractions $\{\varphi_1, \dots, \varphi_N\}$ on $\mathbb{R}^d$. We call this family an *iterated function system*. If all the mappings $\varphi_i \colon \mathbb{R}^d \to \mathbb{R}^d$ are strict contractions, then there exists a unique non-empty compact set F, called the *attractor* of the iterated function system, satisfying

$$F = \bigcup_{i=1}^{N} \varphi_i(F).$$

When all the mappings φ_i are similarities the attractor is known as a *self-similar set*. In this paper, we consider the larger class of iterated function systems where all the mappings are conformal contractions and in this case, we refer to F as a *self-conformal set*.

Let us next give a precise definition for a conformal iterated function system. Fix an open set $V \subset \mathbb{R}^d$. A C^1-mapping $\varphi \colon V \to \mathbb{R}^d$ is *conformal* if the differential $\varphi'(x) \colon \mathbb{R}^d \to \mathbb{R}^d$ is a similarity, i.e. satisfies $|\varphi'(x)y| = |\varphi'(x)||y| \neq 0$ for all $x \in V$ and $y \in \mathbb{R}^d \setminus \{0\}$. We assume that the differentials are Hölder continuous, that is, there exist $\alpha, c > 0$ such that

$$||\varphi_i'(x)| - |\varphi_i'(y)|| \leq c|x-y|^\alpha$$

for all $x, y \in V$ and $i \in \{1, \dots, N\}$. For $d \geq 2$, the Hölder continuity of $x \mapsto |\varphi_i'(x)|$ follows from conformality and injectivity. In fact, conformal mappings in the plane correspond to the holomorphic functions on $\mathbb{C}$ with non-zero derivative on their respective domain, and in higher dimensions, by Liouville's theorem, conformal mappings are either homotheties, isometries, or compositions of reflections and inversions of a sphere. In the one dimensional case, conformal mappings are simply the $C^{1+\alpha}$-functions with non-vanishing derivative. We say that $\{\varphi_i \colon X \to X\}_{i=1}^N$ is a *conformal iterated function system* on a compact set $X \subset \mathbb{R}^d$ if each φ_i extend to an injective conformal mapping $\varphi_i \colon V \to V$ on an open convex set $V \supset X$ and $\|\varphi_i'\| := \sup_{x \in V} |\varphi_i'(x)| < 1$. Self-conformal sets are a natural generalisation of self-similar sets.

In §5, we shall verify that self-conformal sets satisfy the assumptions of Theorem 2.1. We thus obtain the following result as an immediate corollary of Theorems 2.1 and 2.2.

Theorem 3.1. *Let $F \subset \mathbb{R}^d$ be a self-conformal set and $s = \dim_{\mathrm{H}}(F)$. Then there exists a constant $C \geq 1$ such that*

$$\mathcal{H}^s(F \cap A) \leq C\mathcal{H}^s_\infty(F \cap A)$$

for all $\mathcal{H}^s$-measurable sets $A \subset \mathbb{R}^d$. Furthermore, $\mathcal{H}^s(F) > 0$ if and only if F is Ahlfors s-regular.

The above theorem extends to graph-directed and sub self-conformal sets in a straightforward manner; see Remark 5.2. It is pointed out in [5, §4] that the constant C above cannot be chosen to be 1. We may thus consider that the theorem generalises the results of Farkas and Fraser [5, Theorem 2.1 and Corollary 3.1] for graph-directed self-similar sets.

To exhibit further results, let us introduce more definitions and notation. The *Assouad dimension* of a set $A \subset \mathbb{R}^d$, denoted by $\dim_{\mathrm{A}}(A)$, is the infimum of all s satisfying the following: There exists a constant $C \geq 1$ such that each set $A \cap B(x, R)$ can be covered by at most $C(R/r)^s$ balls of radius r centered at A for all $0 < r < R$. It is easy to see that $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{A}}(A)$ for all sets $A \subset \mathbb{R}^d$.

Let $\{\varphi_i\}_{i=1}^N$ be a conformal iterated function system and F be the associated self-conformal set. Let $\Sigma = \{1, \dots, N\}^{\mathbb{N}}$ be the collection of all infinite words constructed from integers $\{1, \dots, N\}$. If $\mathtt{i} = i_1 i_2 \cdots \in \Sigma$, then we define $\mathtt{i}|_n = i_1 \cdots i_n$ for all $n \in \mathbb{N}$. The empty word $\mathtt{i}|_0$ is denoted by $\varnothing$. Observe that $\Sigma_* = \bigcup_{n=0}^\infty \Sigma_n$, where $\Sigma_n = \{\mathtt{i}|_n : \mathtt{i} \in \Sigma\}$ for all $n \in \mathbb{N}$, is the free monoid on $\Sigma_1 = \{1, \dots, N\}$. If $n \in \mathbb{N}$ and $\mathtt{i} = i_1 \cdots i_n \in \Sigma_n$, then we write $\varphi_{\mathtt{i}} = \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}$. For $\mathtt{i} \in \Sigma_* \setminus \{\varnothing\}$ we set $\mathtt{i}^- = \mathtt{i}|_{|\mathtt{i}|-1}$, where $|\mathtt{i}|$ is the length of $\mathtt{i}$.

We say that F satisfies the *weak separation condition* if

$$\sup\{\#\Phi(x, r) : x \in F \text{ and } r > 0\} < \infty,$$

where

$$\Phi(x, r) = \{\varphi_{\mathtt{i}} : \operatorname{diam}(\varphi_{\mathtt{i}}(F)) \leq r < \operatorname{diam}(\varphi_{\mathtt{i}^-}(F)) \text{ and } \varphi_{\mathtt{i}}(F) \cap B(x, r) \neq \emptyset\}$$

for all $x \in \mathbb{R}^d$ and $r > 0$. The following theorem generalises the corresponding result of Farkas and Fraser [5, Corollary 3.2] for self-similar sets in the real line.

Theorem 3.2. *Let $F \subset \mathbb{R}$ be a self-conformal set and $s = \dim_{\mathrm{H}}(F) < 1$. Then the following four conditions are equivalent:*

(1) F satisfies the weak separation condition,
(2) $\mathcal{H}^s(F) > 0$,
(3) F is Ahlfors s-regular,
(4) $\dim_{\mathrm{A}}(F) = s$,
(5) $\inf\{\|\varphi'_{\mathtt{i}}\|^{-1}\|\varphi_{\mathtt{j}} - \varphi_{\mathtt{i}}\| : \mathtt{i}, \mathtt{j} \in \Sigma_*$ such that $\varphi_{\mathtt{i}} \ne \varphi_{\mathtt{j}}\} > 0$.

Proof. The fact that (1) implies (2) follows from [8, Propositions 3.8 and 3.5]. Theorem 3.1 guarantees that (2) and (3) are equivalent. It is more or less a triviality that (3) implies (4); see, for example, [7, §3]. Furthermore, [1, Theorem A] shows that (4) implies (1), which, by [1, Lemma 3.3] and [8, Remark 3.7(1)], is equivalent to $\{\varphi_{\mathtt{i}}^{-1} \circ \varphi_{\mathtt{j}} : \mathtt{i}, \mathtt{j} \in \Sigma_*\}$ not accumulating to the identity. Here we assume that the inverses are extended to the whole real line in a natural way; see [1, §3]. Therefore, by the mean value theorem,

$$\|\varphi'_{\mathtt{i}}\|^{-1}\|\varphi_{\mathtt{j}} - \varphi_{\mathtt{i}}\| \le \|\varphi_{\mathtt{i}}^{-1} \circ \varphi_{\mathtt{j}} - \varphi_{\mathtt{i}}^{-1} \circ \varphi_{\mathtt{i}}\| \le \|(\varphi_{\mathtt{i}}^{-1})'\|\|\varphi_{\mathtt{j}} - \varphi_{\mathtt{i}}\|.$$

By [11, Lemma 2.2], the Hölder continuity of the differentials implies the existence of a constant $K \ge 1$ such that $|\varphi'_{\mathtt{i}}(y)| \le K|\varphi'_{\mathtt{i}}(x)|$ for all $x, y \in V$ and $\mathtt{i} \in \Sigma_*$. This implies $\|(\varphi_{\mathtt{i}}^{-1})'\| \le K\|\varphi'_{\mathtt{i}}\|^{-1}$. Therefore, (5) is equivalent to $\{\varphi_{\mathtt{i}}^{-1} \circ \varphi_{\mathtt{j}} : \mathtt{i}, \mathtt{j} \in \Sigma_*\}$ not accumulating to the identity. □

A self-conformal set F satisfies the *open set condition* if there exists a non-empty open set $U \subset V$ such that $\varphi_i(U) \subset U$ for all i and $\varphi_i(U) \cap \varphi_j(U) = \emptyset$ whenever $i \ne j$. Recall that, by [9, Corollary 5.8 and Theorem 3.9], the open set condition is equivalent to

$$\sup\{\#\Sigma(x, r) : x \in F \text{ and } r > 0\} < \infty,$$

where

$$\Sigma(x, r) = \{\mathtt{i} \in \Sigma_* : \mathrm{diam}(\varphi_{\mathtt{i}}(F)) \le r < \mathrm{diam}(\varphi_{\mathtt{i}^-}(F)) \text{ and } \varphi_{\mathtt{i}}(F) \cap B(x, r) \ne \emptyset\}$$

for all $x \in \mathbb{R}^d$ and $r > 0$. Therefore, the open set condition is stronger than the weak separation condition. The *pressure* $P\colon [0, \infty) \to \mathbb{R}$, defined by

$$P(s) = \lim_{n \to \infty} \tfrac{1}{n} \log \sum_{\mathtt{i} \in \Sigma_n} \|\varphi'_{\mathtt{i}}\|^s,$$

is well-defined, convex, continuous, and strictly decreasing. In fact, there exists unique $s \ge 0$ for which $P(s) = 0$. It is a classical result that if F satisfies the open set condition, then $\dim_{\mathrm{H}}(F) = P^{-1}(0)$; for the latest incarnation of this observation, see [8, Proposition 3.5].

We say that a self-conformal set F has an *exact overlap* if there exist $\mathtt{i}, \mathtt{j} \in \Sigma_*$ such that $\mathtt{i} \ne \mathtt{j}$ and $\varphi_{\mathtt{i}} = \varphi_{\mathtt{j}}$. Observe that if F satisfies the open set condition, then it cannot have exact overlaps. For a self-similar set F in the real line, according to a folklore conjecture, $\dim_{\mathrm{H}}(F) = \min\{1, P^{-1}(0)\}$ if and only if there are no exact overlaps. Hochman [6, Corollary 1.2] has verified the conjecture under a mild assumption which is satisfied for example when the associated iterated function system is defined by algebraic parameters; see [6, Theorem 1.5]. To generalise Hochman's proof for self-conformal sets in the real line seems difficult since the semigroup generated by $C^{1+\alpha}$ maps is simply too large: there are no invariant metric and dimension $d \in \mathbb{N}$ for which there is a smooth injection to $\mathbb{R}^d$, which is bi-Lipschitz to its image in any compact neighbourhood of the identity.

However, the following result verifies the conjecture for self-conformal sets in the real line having positive Hausdorff measure.

Theorem 3.3. *Let $F \subset \mathbb{R}$ be a self-conformal set with $\mathcal{H}^s(F) > 0$ for $s = \dim_{\mathrm{H}}(F) < 1$. Then $s = P^{-1}(0)$ if and only if there are no exact overlaps.*

Proof. If $s = P^{-1}(0)$, then the assumption that $\mathcal{H}^s(F) > 0$ together with [12, Theorem 1.1], implies that F satisfies the open set condition and hence, cannot have exact overlaps. If there are no exact overlaps, then, by Theorem 3.2, the assumption $\mathcal{H}^s(F) > 0$ implies that F satisfies the weak separation condition. Therefore, by [8, Remark 3.7(2)], the lack of exact overlaps implies the open set condition and we have $s = P^{-1}(0)$. □

Note that Theorem 3.2(5) gives, at least in principle, a checkable condition for Theorem 3.3 to hold.

4. Proof of Theorem 2.1

For a bounded set $A \subset \mathbb{R}^d$ we let

$$N_r(A) = \min\{k : A \subset \bigcup_{i=1}^{k} B(x_i, r) \text{ for some } x \in \mathbb{R}^d\}$$

be the least number of balls of radius $r > 0$ needed to cover A.

Lemma 4.1. *Let $F \subset \mathbb{R}^d$ be a set satisfying the assumptions of Theorem 2.1. If $s = \dim_{\mathrm{H}}(F)$, then*

$$N_r(F) \le D^s r^{-s}$$

for all $r > 0$ and $\mathcal{H}^s(F) < (2D)^s$. In particular, $\mathcal{H}^s(F) > 0$ for $s = \dim_{\mathrm{H}}(F)$ if and only if $0 < \mathcal{H}^s(F) < \infty$. Furthermore, if $0 < \mathcal{H}^s(F) < \infty$, then there exists a constant $c > 0$ depending only on d such that

$$N_r(F) \ge c\mathcal{H}^s_\infty(F) r^{-s}$$

for all $r > 0$.

Proof. The first claim follows from the existence of mappings $g \colon F \to F \cap B(x, r)$ satisfying (2.1) and [3, Theorem 3.2]. The second claim follows immediately from the first one. To prove the third claim, let μ be a Radon measure supported on F and $c > 0$ be a constant depending only on d such that

$$\mu(F) \ge c\mathcal{H}^s_\infty(F) > 0 \tag{4.1}$$

and

$$\mu(B(x, r)) \le r^s \tag{4.2}$$

for all $x \in \mathbb{R}^d$ and $r > 0$; recall [10, Theorem 8.8]. If $\mathcal{B}$ is any cover of F consisting of balls of radius $r > 0$, then (4.2) implies

$$\mu(F) \le \sum_{B \in \mathcal{B}} \mu(B) \le \#\mathcal{B} r^s.$$

This gives $N_r(F) \ge \mu(F) r^{-s}$ and the claim follows now from (4.1). □

We are now ready to prove the main theorem.

Proof of Theorem 2.1. We may assume that $\mathcal{H}^s(F) > 0$ since otherwise there is nothing to prove. This of course implies that $\mathcal{H}^s_\infty(F) > 0$. Write $C = 2 \cdot D^{3s} c^{-1} 2^{2s} \mathcal{H}^s_\infty(F)^{-1}$, where $c > 0$ is as in Lemma 4.1. Let us contrarily assume that there exists an $\mathcal{H}^s$-measurable set $A \subset \mathbb{R}^d$ such that

$$\mathcal{H}^s(F \cap A) > C\mathcal{H}^s_\infty(F \cap A). \tag{4.3}$$

Again, this implies that $\mathcal{H}^s_\infty(F \cap A) > 0$. Fix $n \in \mathbb{N}$ and let $\mathcal{B}$ be a maximal collection of pairwise disjoint closed balls of radius 2^{-n} centered in F. Note that, by [10, Equation (5.4)] and Lemma 4.1, we have

$$c2^{-s}\mathcal{H}^s_\infty(F) 2^{ns} \le \#\mathcal{B} \le 2^s D^s 2^{ns}. \tag{4.4}$$

For each $B \in \mathcal{B}$, let $g_B \colon F \to F \cap B$ be as in (2.1). It follows that each ball B in the packing $\mathcal{B}$ contains $g_B(F \cap A)$, a scaled copy of $F \cap A$. Since the mapping g_B is bi-Lipschitz, the set $g_B(F \cap A)$ is $\mathcal{H}^s$-measurable and

$$\mathcal{H}^s(g_B(F \cap A)) \ge D^{-s} 2^{-ns} \mathcal{H}^s(F \cap A) \tag{4.5}$$

for all $B \in \mathcal{B}$. Furthermore, since $\operatorname{diam}(g_B(F \cap A)) \le D2^{-n} \operatorname{diam}(F \cap A) =: \delta 2^{-n}$, we also have

$$\mathcal{H}^s_{\delta 2^{-n}}(g_B(F \cap A)) = \mathcal{H}^s_\infty(g_B(F \cap A)) \le D^s 2^{-ns} \mathcal{H}^s_\infty(F \cap A) \tag{4.6}$$

for all $B \in \mathcal{B}$. Now (4.5), (4.4), and (4.3) imply

$$\begin{aligned}\sum_{B \in \mathcal{B}} \mathcal{H}^s(g_B(F \cap A)) &\ge \#\mathcal{B} D^{-s} 2^{-ns} \mathcal{H}^s(F \cap A) \\ &\ge D^{-s} c 2^{-s} \mathcal{H}^s_\infty(F) \mathcal{H}^s(F \cap A) \ge C D^{-s} c 2^{-s} \mathcal{H}^s_\infty(F) \mathcal{H}^s_\infty(F \cap A),\end{aligned} \tag{4.7}$$

and (4.6) and (4.4) give

$$\sum_{B \in \mathcal{B}} \mathcal{H}^s_{\delta 2^{-n}}(g_B(F \cap A)) \le \#\mathcal{B} D^s 2^{-ns} \mathcal{H}^s_\infty(F \cap A) \le D^{2s} 2^s \mathcal{H}^s_\infty(F \cap A). \tag{4.8}$$

Since, by the fact that the sets $g_B(F \cap A)$ are $\mathcal{H}^s$-measurable and (4.7),

$$\begin{aligned}\mathcal{H}^s(F) &= \mathcal{H}^s\Big(F \setminus \bigcup_{B \in \mathcal{B}} g_B(F \cap A)\Big) + \sum_{B \in \mathcal{B}} \mathcal{H}^s(g_B(F \cap A)) \\ &\ge \mathcal{H}^s\Big(F \setminus \bigcup_{B \in \mathcal{B}} g_B(F \cap A)\Big) + C D^{-s} c 2^{-s} \mathcal{H}^s_\infty(F) \mathcal{H}^s_\infty(F \cap A)\end{aligned}$$

and, by (4.8),

$$\begin{aligned}\mathcal{H}^s_{\delta 2^{-n}}(F) &\le \mathcal{H}^s_{\delta 2^{-n}}\Big(F \setminus \bigcup_{B \in \mathcal{B}} g_B(F \cap A)\Big) + \sum_{B \in \mathcal{B}} \mathcal{H}^s_{\delta 2^{-n}}(g_B(F \cap A)) \\ &\le \mathcal{H}^s\Big(F \setminus \bigcup_{B \in \mathcal{B}} g_B(F \cap A)\Big) + D^{2s} 2^s \mathcal{H}^s_\infty(F \cap A)\end{aligned}$$

we have, by recalling the definition of C, that

$$\begin{aligned}\mathcal{H}^s(F) - \mathcal{H}^s_{\delta 2^{-n}}(F) &\ge C D^{-s} c 2^{-s} \mathcal{H}^s_\infty(F) \mathcal{H}^s_\infty(F \cap A) - D^{2s} 2^s \mathcal{H}^s_\infty(F \cap A) \\ &= D^{2s} 2^s \mathcal{H}^s_\infty(F \cap A) > 0.\end{aligned}$$

This is a contradiction since the lower bound is independent of n. □

5. Proof of Theorem 3.1

Lemma 5.1. *Let $\{\varphi_i\}_{i=1}^N$ be a conformal iterated function system and F the associated self-conformal set such that* $\operatorname{diam}(F) > 0$. *Then there exist constants $K \ge 1$ and $0 < L_{\mathtt{i}} < 1$, $\mathtt{i} \in \Sigma_*$, such that*

$$K^{-1} L_{\mathtt{i}} |x - y| \le |\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \le L_{\mathtt{i}} |x - y| \tag{5.1}$$

for all $x, y \in V$ and $\mathtt{i} \in \Sigma_*$ and

$$\frac{1}{\operatorname{diam}(F)} \operatorname{diam}(\varphi_{\mathtt{i}}(F)) \le L_{\mathtt{i}} \le \frac{K}{\operatorname{diam}(F)} \operatorname{diam}(\varphi_{\mathtt{i}}(F)) \tag{5.2}$$

for all $\mathtt{i} \in \Sigma_*$.

Proof. By [11, Lemma 2.2], the Hölder continuity of the differentials implies the existence of a constant $K \ge 1$ for which

$$|\varphi'_{\mathtt{i}}(y)| \le K |\varphi'_{\mathtt{i}}(x)|$$

for all $x, y \in V$ and $\mathtt{i} \in \Sigma_*$. By the convexity of V and the mean value theorem, we have

$$\|(\varphi_{\mathtt{i}}^{-1})'\|^{-1} |x - y| \le |\varphi_{\mathtt{i}}(x) - \varphi_{\mathtt{i}}(y)| \le \|\varphi'_{\mathtt{i}}\| |x - y|$$

for all $x, y \in V$ and $\mathtt{i} \in \Sigma_*$. Noting that $\|(\varphi_{\mathtt{i}}^{-1})'\|^{-1} \ge K^{-1} \|\varphi'_{\mathtt{i}}\|$, we have shown (5.1). Since (5.2) follows from (5.1), we have finished the proof. □

Iterated function systems $\{\varphi_i\}_{i=1}^N$ satisfying (5.1) are studied in [9, §5] and [14, §4].

Remark 5.2. Let M be an $N \times N$-matrix with entries in $\{0,1\}$. We say that a word $\mathtt{i} = i_1 i_2 \cdots \in \Sigma$ is *M-admissible* if $M_{i_k,i_{k+1}} = 1$ for all k. The collection of M-admissible infinite words starting with $i \in \{1,\ldots,N\}$ defines a set when projected onto $\mathbb{R}^d$ by $\mathtt{i} \mapsto \lim_{n\to\infty} \varphi_{\mathtt{i}|_n}(0)$ and the resulting attractor F_i is known as the *graph-directed self-conformal set* of i. It is well-known that if M is irreducible and the iterated function system consists of conformal contractions that the resulting sets F_i are also quasi self-similar and satisfy $\dim_{\mathrm{H}}(F_i) = \dim_{\mathrm{H}}(F_j)$ for all $i,j \in \{1,\ldots,N\}$. It is also easy to show that there exists $C > 0$ such that $\mathcal{H}^s(F_i) \le C\mathcal{H}^s(F_j)$ for all $i,j \in \{1,\ldots,N\}$. If M is irreducible, then it is not too difficult to see that Lemma 5.1, as well as the proof of Theorem 3.1 below, hold for graph-directed self-conformal sets.

A *sub self-conformal set* is a non-empty compact subset $E \subset F$ such that $E \subset \bigcup_{i=1}^N \varphi_i(E)$. Note that sub self-conformal sets are closed under $\varphi_{\mathtt{i}}$ and it is a straightforward calculation to check that Lemma 5.1 and Theorem 3.1 hold for sub self-similar sets. Generally, graph-directed self-conformal sets are not closed unde $\varphi_{\mathtt{i}}$ and it is rather easy to find examples such that the sets F_i are not sub self-conformal. However, some authors prefer to define a single graph-directed set using subshifts of finite type. In our notation this amounts to considering $F = \bigcup_{i=1}^N F_i$. This set is closed under application of contractions and thus is a sub self-conformal set.

For both cases above we have omitted detailed proofs to avoid cumbersome notation of M-admissible words and arbitrary subsets.

Proof of Theorem 3.1. Since we consider a fixed self-conformal set F, we may assume that its defining conformal iterated function system $\{\varphi_i\}_{i=1}^N$ is given. Thus we assume that the constants $K \ge 1$ and $L_{\mathtt{i}}$, $\mathtt{i} \in \Sigma_*$, are as in Lemma 5.1.

Let $x \in F$ and $0 < r < \mathrm{diam}(F)$. Pick $\mathtt{i} \in \Sigma$ such that $\lim_{n\to\infty} \varphi_{\mathtt{i}|_n}(0) = x$ and choose $n \in \mathbb{N}$ for which $\varphi_{\mathtt{i}|_n}(F) \subset B(x,r)$ but $\varphi_{\mathtt{i}|_{n-1}}(F) \setminus B(x,r) \ne \emptyset$. Note that the latter property implies $\mathrm{diam}(\varphi_{\mathtt{i}|_{n-1}}(F)) \ge r$. By (5.1) and (5.2), we have

$$\begin{aligned} |\varphi_{\mathtt{i}|_n}(y) - \varphi_{\mathtt{i}|_n}(z)| &\ge K^{-2} L_{\mathtt{i}|_{n-1}} \min_{i\in\{1,\ldots,N\}} L_i |y-z| \\ &\ge \frac{\min_{i\in\{1,\ldots,N\}} L_i}{K^2\,\mathrm{diam}(F)} \mathrm{diam}(\varphi_{\mathtt{i}|_{n-1}}(F))|y-z| \ge \frac{\min_{i\in\{1,\ldots,N\}} L_i}{K^2\,\mathrm{diam}(F)} r|y-z| \end{aligned}$$

and

$$|\varphi_{\mathtt{i}|_n}(y) - \varphi_{\mathtt{i}|_n}(z)| \le \frac{K}{\mathrm{diam}(F)} \mathrm{diam}(\varphi_{\mathtt{i}|_n}(F))|y-z| \le \frac{2K}{\mathrm{diam}(F)} r|y-z|$$

for all $y,z \in F$. By setting

$$D = \max\left\{1, \frac{K^2\,\mathrm{diam}(F)}{\min_{i\in\{1,\ldots,N\}} L_i}, \frac{2K}{\mathrm{diam}(F)}\right\},$$

we have thus shown that for each $x \in F$ and $0 < r < \mathrm{diam}(F)$ there exist $\mathtt{i} \in \Sigma$ and $n \in \mathbb{N}$ such that $\varphi_{\mathtt{i}|_n}(F) \subset F \cap B(x,r)$ and

$$D^{-1} r|y-z| \le |\varphi_{\mathtt{i}|_n}(y) - \varphi_{\mathtt{i}|_n}(z)| \le Dr|y-z|$$

for all $y,z \in F$. Theorem 3.1 follows now immediately from Theorems 2.1 and 2.2. □

Acknowledgements. Antti Käenmäki was supported by the Finnish Center of Excellence in Analysis and Dynamics Research, the Finnish Academy of Science and Letters, and the Väisälä Foundation. Sascha Troscheit was supported by EPSRC Grant DTG/EP/K503162/1, NSERC Grants 2016-03719 and 2014-03154, and the University of Waterloo. Both authors thank the Institut Mittag-Leffler, where the paper was finished, for an excellent working environment and Balázs Bárány for helpful comments.

References

[1] J. Angelevska and S. Troscheit. A dichotomy of self-conformal subsets of $\mathbb{R}$ with overlaps. Preprint, available at arXiv:1602.05821, 2016.

[2] K. J. Falconer. Dimensions and measures of quasi self-similar sets. *Proc. Amer. Math. Soc.*, 106(2):543–554, 1989.

[3] K. J. Falconer. *Techniques in fractal geometry.* John Wiley & Sons Ltd., Chichester, 1997.

[4] Á. Farkas. Dimension and measure theory of self-similar structures with no separation condition, 2015. PhD thesis, University of St Andrews.

[5] Á. Farkas and J. M. Fraser. On the equality of Hausdorff measure and Hausdorff content. *J. Fractal Geom.*, 2(4):403–429, 2015.

[6] M. Hochman. On self-similar sets with overlaps and inverse theorems for entropy. *Ann. of Math. (2)*, 180(2):773–822, 2014.

[7] A. Käenmäki, J. Lehrbäck, and M. Vuorinen. Dimensions, Whitney covers, and tubular neighborhoods. *Indiana Univ. Math. J.*, 62(6):1861–1889, 2013.

[8] A. Käenmäki and E. Rossi. Weak separation condition, Assouad dimension, and Furstenberg homogeneity. *Ann. Acad. Sci. Fenn. Math.*, 41(1):465–490, 2016.

[9] A. Käenmäki and M. Vilppolainen. Separation conditions on controlled Moran constructions. *Fund. Math.*, 200(1):69–100, 2008.

[10] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[11] R. D. Mauldin and M. Urbański. Dimensions and measures in infinite iterated function systems. *Proc. London Math. Soc. (3)*, 73(1):105–154, 1996.

[12] Y. Peres, M. Rams, K. Simon, and B. Solomyak. Equivalence of positive Hausdorff measure and the open set condition for self-conformal sets. *Proc. Amer. Math. Soc.*, 129(9):2689–2699, 2001.

[13] Y. B. Pesin. *Dimension theory in dynamical systems.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1997. Contemporary views and applications.

[14] T. Rajala and M. Vilppolainen. Weakly controlled Moran constructions and iterated functions systems in metric spaces. *Illinois J. Math.*, 55(3):1015–1051 (2013), 2011.

(Antti Käenmäki) Department of Mathematics and Statistics, P.O. Box 68 (Gustaf Hällströmin katu 2b), FI-00014 University of Helsinki, Finland

E-mail address: antti.kaenmaki@helsinki.fi

(Sascha Troscheit) Department of Pure Mathematics, University of Waterloo, Waterloo, Ont., N2L 3G1, Canada

E-mail address: stroscheit@uwaterloo.ca