LOCAL AND GLOBAL TRACE FORMULAE FOR $C^\infty$ HYPERBOLIC DIFFEOMORPHISMS

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ABSTRACT. We study local and global trace formulae for $C^\infty$ (a priori non-analytic) hyperbolic diffeomorphisms, that is we compare the sum of its Ruelle resonances with the flat trace of its transfer operator. We explicit the link between trace formulae and dynamical determinants and give examples of systems with explicit dynamical determinants exhibiting interesting behaviours (in particular, trace formulae do not always hold). We also establish a bound on the growth of the dynamical determinant for Gevrey dynamics from which we deduce a control on the number of resonances outside of a disc centred at 0.

INTRODUCTION

Let $T : M \to M$ be a $C^\infty$ diffeomorphism with hyperbolic basic set $\Lambda \subseteq M$ (see Definition 2.1, we could also consider an expanding map) and $g : M \to \mathbb{C}$ be a $C^\infty$ function. Then the formal power series

$$d_{T,g}(z) = \exp \left( - \sum_{n=1}^{+\infty} \frac{1}{n} \sum_{T^n x = x} \prod_{k=0}^{n-1} g(T^k x) \left| \det I - D_x T^n \right|^z \right)$$

defines an entire function, called a dynamical determinant (see [3] or [1], dynamical determinants first appeared in the work of Ruelle on dynamical zeta functions [19]). Moreover, the inverses of the zeroes of $d_{T,g}$ are the Ruelle resonances of the system $(T, g)$ (see [1, Definition 1.1] for a definition). When $g$ is strictly positive, these resonances describe accurately the decay of correlations, that is the asymptotics, when $n$ tends to infinity, of

$$\int_{\Lambda} \varphi \psi \circ T^n d\mu_g,$$

for some $T$-invariant measure $\mu_g$ which has a physical meaning and any smooth functions $\varphi$ and $\psi$ on $M$ (see [7]). The Ruelle resonances are the "relevant" eigenvalues of the transfer operator

$$L_g : \varphi \mapsto g \varphi \circ T$$

acting on suitable Banach spaces. Thus, the dynamical determinant $d_{T,g}$ plays the role of the Fredholm determinant $\det(I - z L_B)$ of $L_B$ (in some analytic settings, $d_{T,g}$ is actually the Fredholm determinant of $L_g$ acting on some Banach space, see for instance [4]). In particular, the formula (1) may be understood as an instance of the general formula valid for any nuclear operator $A$

$$\det(I - z A) = \exp \left( - \sum_{n=1}^{+\infty} \frac{1}{n} \text{tr} (A^n) z^n \right).$$

However, for general $T$ and $g$, we do not know any "good" Banach space on which $L_g$ acts as a nuclear operator (in fact, after Corollary 3.5 we shall give examples of $T$ and $g$ for which such a Banach space does not exist). Thus, the traces appearing in the expression (3) must be replaced by the "flat traces" of the iterates of $L_g$ defined by

$$\text{tr}^\flat (L_g^n) = \sum_{T^n x = x} \frac{\prod_{k=0}^{n-1} g(T^k x)}{|\det I - D_x T^n|} \text{ for all } n \geq 1.$$
to get (1). Since the resonances are the "relevant" eigenvalues of $L_g$, it seems natural to ask whether the relation

$$\text{tr}^b (L_g^n) = \sum_{\lambda \text{ resonance}} \lambda^n$$

holds in some sense for some values of $n \geq 1$. Our first main result, Theorem 1.4, gives conditions under which (4) holds.

The work presented here is motivated by the analogy with the continuous-time case (see [13] and references given there). Taking $f = d_{T,g}$ in Theorem 1.4, we may interpret (9) there as a local trace formula, that is a discrete-time analogue to the result of [13], while (ii) in the same theorem can be seen as a global trace formula (an analogue to Selberg trace formula, for instance). As pointed out in [13], global trace formulae for Anosov flows may be deduced from the finite order of some dynamical determinant. Theorem 1.4 asserts some kind of equivalence for discrete-time hyperbolic dynamical systems, which suggests that the problem is much simpler in this case. We point out that, under assumptions of finite order of some dynamical determinants, global trace formulae for continuous-time dynamical systems may also be deduced from pure complex analysis, see [14, Theorem 17] and [15, Theorem 8.1].

Since the dynamical determinant contains all the informations on both sides of (4), it seems natural that elementary arguments of basic complex analysis imply a relation between the validity of (4) (with absolute convergence of the right hand side) and the order of the dynamical determinant $d_{T,g}$: that is the point of Theorem 1.4 (see also Remark 1.5 below). For instance, combining [3, Theorem 1.5] and some well-known results [19, 20, 8, 10], one may show that (4) holds for all $n \geq 1$ under some assumption of analyticity. We shall see that (4) does not need to hold if $T$ and $g$ are only $C^\infty$ (even if $g$ is strictly positive). Indeed we will construct systems $(T, g)$ with explicit dynamical determinants in Proposition 3.3 for which (4) fails. This construction suggests that being a dynamical determinant is not a much stronger condition than being an entire function (e.g. it can be of infinite order, see Corollaries 3.4 and 3.5 and Remark 3.9). In particular, the set of integer for which (4) holds can be anything.

Thus, it is natural to wonder if the formula (4) holds for some smaller class of systems $(T, g)$, for instance for Gevrey functions. This would follow from some estimates on the growth of the dynamical determinant: we give such estimates in our second main result, Theorem 2.2. Unfortunately, the bounds in Theorem 2.2 are not sufficient to use Theorem 1.4 to prove that (4) holds for any $n \geq 1$. However, these estimates combined with Jensen's formula imply an upper bound on the growth of the number of resonances outside of a circle of center 0 and radius $r$ when $r$ tends to 0 (see Corollary 2.4 and compare with the bound given by Fried in finite differentiability in [9]).

In §1, we prove Theorem 1.4 which explains the link between the trace formula (4) and the order of the dynamical determinant $d_{T,g}$. We also give some counter-examples that justify the hypotheses of Theorem 1.4.

In §2, we prove Theorem 2.2 which gives a bound on the growth of the dynamical determinant when $T$ and $g$ are Gevrey. The proof is based on a quantitative investigation of the proof of [3, Theorem 1.5].

In §3, we construct systems $(T, g)$ with explicit dynamical determinants exhibiting interesting behaviours (see Proposition 3.3). The construction is based on ideas from [2] and [17, Example 1 p.163], the main tool is Whitney's extension Theorem [22] as in [6]. We use this construction to realize the counter-examples of §1 as dynamical determinants and show that (4) does not always hold (see Corollaries 3.4 and 3.5 and remarks below).

1. Trace formulae and order of the dynamical determinant

We shall now explain the link between trace formulae and finite order of the dynamical determinant. We will need the following definition.

**Definition 1.1.** If $f$ is an entire function, we say that $(z_m)_{m \geq 0}$ is an ordering of the zeroes of $f$ if $z_0, z_1, \ldots, z_m, \ldots$ are the zeroes of $f$ counted with multiplicity and the sequence $(|z_m|)_{m \geq 0}$ is non-decreasing.

We shall always order the zeroes of an entire function in this way. Recall the following definitions.

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1. It would be inappropriate to consider only the case $n = 1$, since the relation (4) holds for all $n \geq 2$ for nuclear operators (replacing "flat traces" by actual traces and resonances by eigenvalues) but not always for $n = 1$ (see [11, Theorem 4.1] and the remark preceding it, see also [12, Corollary 2 p.17, second part of the book] and the proof of Theorem 1.4).
2. Compare with the case of analytic expanding maps of the circle studied in [4].
3. It would also be relevant to study what happens when restricting to natural weights $g$ (such as $\det DT$ or $\det DT|_{E_u}$) or dynamics more geometrically constrained (e.g. Anosov diffeomorphisms).
Definition 1.2. Let $f$ be an entire function. The order of $f$ may be defined as

$$
\limsup_{r \to +\infty} \frac{\log \log \left( \sup_{|z| \leq r} |f(z)| \right)}{\log r}
$$

where $\log_+ x = \log \max(1, x)$. If $f$ is non-zero and has finite order, let $p$ be the smallest natural integer such that

$$
\sum_{m \geq 0} \frac{1}{|z_m|^{p+1}} < +\infty
$$

where $(z_m)_{m \geq 0}$ is an ordering of the zeroes of $f$ (the integer $p$ is well-defined thanks to Jensen’s formula). By Hadamard’s Factorization Theorem [5, 2.7.1] there is a polynomial $Q$ such that for all $z \in \mathbb{C}$

$$
f(z) = e^{Q(z)} \prod_{m \geq 0} E \left( \frac{z}{z_m}, p \right)
$$

where the function $E$ is defined by

$$
E(u, p) = (1 - u) \exp \left( \sum_{k=1}^{p} \frac{1}{k} u^k \right) = \exp \left( - \sum_{k=p+1}^{+\infty} \frac{1}{k} u^k \right).
$$

The genus of $f$ is then defined as $\max(\deg Q, p)$. We shall say that the genus of an entire function of infinite order is infinite.

Remark 1.3. It may be deduced from Hadamard’s Factorization theorem that if $\rho$ and $\gamma$ denote respectively the genus and the order of some entire function then $\gamma \leq \rho \leq \gamma + 1$ (see [5] for details).

As explained in Remark 1.5, the following theorem is an abstract way to express the link between the order of the dynamical determinant and trace formulae.

Theorem 1.4. Let $f$ be an entire function such that $f(0) = 1$. Let $G$ be a holomorphic function defined on a neighbourhood of 0 such that $G(0) = 0$ and $f(z) = e^{G(z)}$ for $z$ in a neighbourhood of zero. Write

$$
G(z) = -\sum_{n=1}^{+\infty} \frac{1}{n} a_n z^n
$$

and denote by $(z_m)_{m \geq 0}$ an ordering of the zeroes of $f$. Then for all $r > 0$ such that $f$ has no zero of modulus $r$, we have

$$
a_n = \sum_{n=r}^{+\infty} \frac{1}{z_m^n} + o \left( \frac{1}{r^n} \right).
$$

Furthermore, the following properties are equivalent:

(i) the order of $f$ is finite;

(ii) there is a natural integer $n_0$ such that for all integers $n \geq n_0 + 1$ the series

$$
\sum_{m \geq 0} \frac{1}{z_m^n}
$$

converges absolutely and its sum is $a_n$.

If (i) or (ii) holds then the minimal value of $n_0$ so that (ii) holds is the genus of $f$.

Remark 1.5. Taking $f = d_{T,g}$, we have $a_n = \text{tr}^T \left( E_n^g \right)$ and it appears that (4) holds with absolute convergence of the left hand side if $n \geq n_0 + 1$, where $n_0$ denotes the genus of $d_{T,g}$. In §3, we shall construct dynamical determinants with arbitrary (finite or infinite) genus, so that all the behaviours described in Theorem 1.4 may be realised by dynamical determinants.

Remark 1.6. As pointed out in the introduction, taking $f = d_{T,g}$ in Theorem 1.4, the formula (9) can be interpreted as a local trace formula (this formula was already known, see [9]), while (ii) may be interpreted as a global trace formula.
Remark 1.7. As we shall see in Proposition 1.8 below, the absoluteness of the convergence in (ii) is essential to get an equivalence. This is quite unfortunate especially as we shall realise the counter-examples of Proposition 1.8 as dynamical determinants in section 3. On the other hand, it is very easy essential to get an equivalence. This is quite unfortunate especially as we shall realise the coun-
	erAbel transform.

Remark 1.9. \( \sum \) values of \( n \) Remark 1.7.

• Proof of Theorem 1.4. To prove (9), one only needs to notice that the holomorphic function

\[
\frac{f(z)}{z \mapsto \prod_{i \in \mathbb{N}} (1 - \frac{z}{n})} = \exp \left( - \sum_{n=1}^{+\infty} \frac{1}{n} \left( a_n - \sum_{m=1 \atop |z_m| < |z|} \frac{1}{m} \right) z^n \right)
\]

does not vanish on a disc of center 0 and radius a little bigger than \( r \), and so admits a holomorphic logarithm there.

• Suppose (i). Recall \( p \) from Definition 1.2 and notice that the series (10) converges absolutely for \( n \geq p + 1 \). Let \( r \) be a positive real number such that \( r \leq |z_m| \) for all \( m \). Then define for \( |z| \leq \frac{r}{2} \) and \( m \geq 0 \)

\[
f_m(z) = - \sum_{k=p+1}^{+\infty} \frac{1}{k} \left( \frac{z}{z_m} \right)^k,
\]

and notice that |\( f_m(z) \)| \( \leq \frac{p+1}{2^r |z_m|^r} \). Then, recalling (5), the series \( \sum_{m \geq 0} f_m \) converges on the disc of center 0 and radius \( \frac{r}{2} \) to a holomorphic function \( F \) and, recalling (6), we have for \( z \) close enough to 0

\[
e^{G(z)} = f(z) = e^{Q(z)} + F(z).
\]

Thus we may identify the coefficients of order greater than \( \deg Q \) in the expansions in power series of \( F \) and \( G \), which ends the proof of (ii) recalling (8) and (11).

• Suppose (ii). Using the hypothesis for \( n = n_0 \), the infinite product

\[
P(z) = \prod_{m \geq 0} E \left( \frac{z}{z_m}, n_0 \right),
\]

converges on \( \mathbb{C} \) to a holomorphic function of finite order smaller than \( n_0 \) (see [5, Theorem 2.6.5]). But since \( a_n = \sum_{m \geq 0} \frac{1}{z_m^n} \) for \( n \geq n_0 \), we have, recalling the definition (7) of \( E \), for \( z \) close enough to 0,

\[
P(z) = \exp \left( - \sum_{n=n_0+1}^{+\infty} a_n z^n \right)
\]

and consequently

\[
f(z) = \exp \left( - \sum_{n=1 \atop n \neq n_0}^{n_0} \frac{1}{n} a_n z^n \right) P(z).
\]

Thus, \( f \) has finite order smaller than \( n_0 \) (and genus smaller than \( n_0 \) too, thanks to Remark 1.3).

We now give two counter-examples that highlight the necessity to ask for absolute convergence in (ii).

Proposition 1.8. (a) There exists an entire function \( f \) with \( f(0) = 1 \) such that if \( (z_m)_{m \geq 0} \) is an ordering of the zeroes of \( f \) (as defined in Definition 1.1) then for all \( n \geq 1 \) the series \( \sum_{m \geq 0} \frac{1}{z_m^n} \) converges with sum \( a_n \) (defined in (8)) but the convergence is not absolute.

(b) There exists an entire function \( f \) with \( f(0) = 1 \), an ordering \( (z_m)_{m \geq 0} \) of the zeroes of \( f \) and a permutation \( \sigma \) of \( \mathbb{N} \) such that \( (z_{\sigma(n)})_{n \in \mathbb{N}} \) is an ordering of the zeroes of \( f \) and, for all \( n \geq 1 \), the series \( \sum_{m \geq 0} \frac{1}{z_{\sigma(m)}^n} \) converges with sum \( a_n \), but \( \sum_{m \geq 0} \frac{1}{z_{\sigma(m)}^n} \) does not converge.

Remark 1.9. Theorem 1.4 implies that the functions constructed by Proposition 1.8 have infinite order, while the global trace formula holds in some weak sense.

To prove Proposition 1.8, we shall need the following lemma, whose proof is straightforward using an Abel transform.
Proposition 1.10. Let \((b_m)_{m \geq 0}\) be a sequence of complex numbers such that there is a constant \(M\) such that for all \(\ell \in \mathbb{N}\) we have \(\sum_{m=0}^{\ell} b_m \leq M\). Let \((c_m)_{m \geq 0}\) be a decreasing sequence of positive real numbers with null limit. Then the series \(\sum_{m \geq 0} b_m c_m\) converges, and we have the estimates
\[
\left| \sum_{m=0}^{\infty} b_m c_m \right| \leq 2Mc_0.
\]

Proof of Proposition 1.8. (a) Choose an irrational real number \(\theta\) for which there is a constant \(c\) such that for all \(n \in \mathbb{N}^*\) we have \(1 - e^{2i\pi n\theta} \geq \frac{c}{n^2}\) for some constant \(C > 0\) (almost any real number may be chosen thanks to Borel-Cantelli’s lemma). For every integer \(n\), set
\[
a_n = \sum_{m=2}^{\infty} \left( \frac{e^{2i\pi m\theta}}{\ln(m)} \right)^n,
\]
which is well-defined thanks to Lemma 1.10, but the convergence is clearly not absolute. Furthermore for all integers \(m_0 \geq 2\) we have
\[
\left| a_n - \sum_{m=2}^{m_0-1} \left( \frac{e^{2i\pi m\theta}}{\ln(m)} \right)^n \right| \leq \frac{4}{c} \frac{n^2}{\ln(m_0)}^n
\]
(take \(b_m = e^{2i\pi (m+m_0)}\) and \(c_m = (\ln(m+m_0))^{-1}\) in Lemma 1.10). Now (12) with
\[
\exp \left( \sum_{n=1}^{+\infty} \frac{1}{n} \left( \sum_{m=2}^{m_0-1} \left( \frac{e^{2i\pi m\theta}}{\ln(m)} \right)^n \right) z^n \right) = \prod_{m=2}^{m_0-1} \left( 1 - e^{2i\pi m\theta} \frac{1}{\ln(m)z} \right)
\]
implies that the function \(f\) defined by \(f(z) = \exp \left( -\sum_{n=1}^{+\infty} a_n z^n \right)\), for \(z\) in a neighbourhood of zero, extends to an entire function whose zeroes are exactly the \(\left( \frac{2i\pi m\theta}{\ln(m)} \right)^{-1}\). Since there is only one way to order the zeroes of \(f\) with increasing moduli, point (a) is proven.

(b) Choose \(\theta\) as in (a) and denote by \((n_k)_{k \geq 0}\) the sequence of integers defined by \(n_0 = 0\) and \(n_k = k!\) for \(k \geq 1\). Define \(I_0 = \{0\}\) and \(I_k = [n_k + 1, n_{k+1}]\) for \(k \geq 1\). For all \(n \in \mathbb{N}\), denote by \(k(n)\) the unique integer such that \(n \in I_k(n)\). Then set for all integers \(n \geq 1\)
\[
a_n = \sum_{m=0}^{+\infty} \left( \frac{e^{2i\pi m\theta}}{\ln(k(m) + 2)} \right)^n.
\]
Then, as in (a), we may use Lemma 1.10 to show that \(f(z) = \exp \left( -\sum_{n=1}^{+\infty} a_n z^n \right)\) extends to an entire function whose zeroes are exactly the \(z_m = \left( \frac{e^{2i\pi m\theta}}{\ln(k(m) + 2)} \right)^{-1}\) for \(m \in \mathbb{N}\). We shall see that there is another way to order the zeroes of \(f\), which breaks the convergence of the series (10) for all \(n \geq 1\), but preserves the monotony of the sequence of moduli.

Choose \(0 < \epsilon < 1\) such that for all \(x \in [0, \epsilon]\) we have \(\Re(e^{2i\pi x}) > \frac{x}{2}\). Then for all \(k \in \mathbb{N}\) and \(n \geq 1\), denote by \(N_k^{(n)}\) the number of those \(m \in I_k\) such that \(mn\theta \in [0, \epsilon]\) (mod 1), and choose a permutation \(\sigma_k^{(n)}\) of \(I_k\) which puts these elements first. Equidistribution of the \(mn\theta\), for \(n\) fixed and \(m \geq 0\), implies that
\[
\sum_{\ell=0}^{k} N_k^{(n)} \frac{n}{n_k} \rightarrow^\epsilon \epsilon,
\]
but \(\sum_{\ell=0}^{k} N_k^{(n)} \leq n_{k-1} + 1 = o(n_k)\), and thus
\[
\frac{N_k^{(n)}}{\ln(k+2)} \rightarrow^+ \infty.
\]

Now choose \(\varphi: \mathbb{N} \rightarrow \mathbb{N}^*\) such that for all \(n \geq 1\) the reciprocal image \(\varphi^{-1}(\{n\})\) is infinite (for instance \(1, 1, 2, 1, 2, 3, \ldots\)) and set
\[
\sigma = \bigcup_{k=0}^{+\infty} \sigma_k^{(\varphi(k))}.
\]
Now, if \( n \geq 1 \) the series \( \sum_{m \geq 0} \frac{1}{z^{N(m)}} \) does not converge. Indeed, for all \( k = n \), we have
\[
\Re \left( \frac{S_{nk+1-N(n)}}{k} \right) \geq \Re \left( S_{nk+1} \right) + \frac{N(n)}{2 \ln (k+2)},
\]
where \( (S_m)_{m \geq 0} \) is the sequence of partial sums of the series \( \sum_{m \geq 0} \frac{1}{z^m} \), and \( \left( \tilde{S}_m \right)_{m \geq 0} \) is the sequence of partial sums of the series \( \sum_{m \geq 0} \frac{1}{z^{N(m)}} \). We let \( k \) tend to \( +\infty \) with \( \varphi (k) = n \), which is possible thanks to our choice of \( \varphi \). By the first paragraph of part (b), the first term of the right hand side converges but the second one tends to \( +\infty \), and thus the left hand side does not converge. \( \square \)

In order to realise the counter-examples of Proposition 1.8 as dynamical determinants in \( \mathbb{C}^3 \), we shall need the two following, merely technical, lemmas.

**Lemma 1.11.** For all \( \epsilon > 0 \) and \( \rho > 0 \), the counter-examples of Proposition 1.8 may be realised as entire functions \( f \) of the form \( f : z \to 1 - 2z - z (1 - z) h (z) \), where \( h \) is an entire function such that for all \( z \in \mathbb{C} \) we have \( h (z) = \sum_{\ell=0}^{+\infty} \alpha \ell z^\ell \), with \( \alpha \in \left[ -\frac{\epsilon}{\rho}, \frac{\epsilon}{\rho} \right] \) for all integers \( \ell \).

**Proof.** For all \( k \geq 2 \) and \( n \geq 1 \) set either
\[
\alpha_n (k) = \sum_{m=k}^{+\infty} \left( \frac{e^{-i\pi m \theta}}{\ln (m)} \right)^n
\]
or
\[
\alpha_n (k) = \sum_{m=k-2}^{+\infty} \left( \frac{e^{-i\pi m \theta}}{\ln (k (m+2))} \right)^n,
\]
depending on whether you want to get a counter-example of type (a) or (b). Then set for all \( k \geq 1 \)
\[
\tilde{f}_k (z) = \exp \left( - \sum_{n=1}^{+\infty} \left( a_n (k) + \alpha_{n+1} (k) \right) z^n \right).
\]

Estimate (12) (and its analogue for the case (b) of Proposition 1.8) implies that \( \tilde{f}_k \) converges to 1 uniformly on all compact subsets of \( \mathbb{C} \) as \( k \) goes to \( +\infty \). Then set \( f_k : z \to \left( 1 - \frac{z}{\lambda_k} \right) \tilde{f}_k (z) \), where \( \lambda_k = \frac{f_k (1)}{1+f_k (1)} \).

Thus we have \( f_k (0) = 1 \), \( f_k (1) = -1 \), and it is easy to check that \( f_k \) is a counter-example of type (a) or (b), according to the way the \( \alpha_n (k) \) have been defined. We shall see that that for large enough \( k \) the function \( f_k \) satisfies the conditions of Lemma 1.11. Let \( h_k \) be the entire function defined by \( h_k (z) = \frac{f_k (z)^{1+2\ell}}{z (1-z)} \). We shall also need the auxiliary function \( H_k (z) = \frac{h_k (z)}{z (1-z)} - \left( \tilde{f}_k (1) - 1 \right) \) which vanishes at \( z = 1 \) and tends to 0 uniformly on all compact subsets of \( \mathbb{C} \) when \( k \) tends to \( +\infty \). Then notice that
\[
-h_k (z) = H_k (z) \frac{1-z}{1-z} \left( 1 - \frac{z}{\lambda_k} \right) + \frac{\tilde{f}_k (1)^2 - 1}{f_k (1)}
\]
and write
\[
H_k (z) = \frac{1}{1-z} \sum_{\ell=0}^{+\infty} \beta_{\ell} z^\ell
\]
then we have
\[
\alpha_0 = - \left( \beta_0 + \frac{\tilde{f}_k (1)^2 - 1}{f_k (1)} \right) \text{ and } \alpha_{\ell+1} = - \left( \beta_{\ell+1} + \frac{\beta_0}{\lambda_k} \right) \text{ if } \ell \geq 0.
\]
But the sequence \( \left( \frac{1}{\lambda_k} \right)_{k \geq 0} \) converges to \( \frac{1}{2} \) and (we may suppose \( \rho \geq 2 \))
\[
|\beta_\ell| = \frac{1}{2\pi} \left| \int_{|z|=r} H_k (z) \frac{dz}{(1-z)^{\ell+1}} \right| \leq \frac{2}{\rho} \sup_{|z| \leq r} \left| \tilde{f}_k (z) - 1 \right|
\]
which ends the proof, recalling that \( \tilde{f}_k \) converges to 1 uniformly on all compact subsets of \( \mathbb{C} \) as \( k \) goes to \( +\infty \). \( \square \)
Lemma 1.12. Let $f$ be an entire function such that $f(0) = 1$ and $(c_k)_{k \geq 0}$ be a sequence of positive real numbers such that $\sum_{k \geq 0} c_k < +\infty$. Then the infinite product

\begin{equation}
\prod_{k \geq 0} (z \mapsto f(c_k z))
\end{equation}

converges uniformly on all compact subsets of $\mathbb{C}$ to an entire function $d$ that has same genus\(^4\) than $f$. Furthermore, if $f$ is one of the counter-examples of type (a) or (b) constructed in Lemma 1.11, then $d$ also satisfies point (a) or (b) respectively of Proposition 1.8.

Proof. If $K$ is a compact subset of $\mathbb{C}$ then, since $f(0) = 1$, there is a constant $C > 0$ such that for all $z \in K$ we have $|f(z) - 1| \leq C|z|$. Thus for all $z \in K$ and $k \geq 0$, we have $|f(c_k z) - 1| \leq C|c_k||z|$. Thus the infinite product (13) does converge uniformly on all compact subset of $\mathbb{C}$ to an entire function $d$. That $d$ has same order and same genus than $f$ is straightforward from the Definition 1.2 and Hadamard’s factorization Theorem (we use the positivity of the $c_k$’s to ensure that no unwanted cancellation happens).

Let us point out that if $f(z) = \exp \left( - \sum_{n=1}^{+\infty} \frac{a_n}{n} z^n \right)$ then $d(z) = \exp \left( - \sum_{n=1}^{+\infty} \frac{\frac{1}{n} a_n}{1} \left( \sum_{k=0}^{+\infty} c_k^n \right) z^n \right)$.

Suppose now that $f$ is the counter-example of type (a) constructed in Lemma 1.11 and denote by $(z_m)_{m \geq 0}$ an ordering of its zeroes. Let $(w_m)_{m \geq 0}$ be an ordering of the zeroes of $d$, then there is a bijection $(\varphi, \psi) : \mathbb{N} \to \mathbb{N}^2$ such that for all $m \in \mathbb{N}$ we have $w_m = \frac{\varphi(m)}{\psi(m)}$, and for all $k \in \mathbb{N}$ the sequence $(\varphi(m))_{m \in \psi^{-1} (k)}$ is an ordering of the zeroes of $f$. Let $n \geq 1$. It is clear from our construction that $f$ has no more than two zeroes of a given modulus\(^5\), and so there is a constant $M$ such that for all $k \in \mathbb{N}$ and $m_0 \in \mathbb{N}$ we have

\begin{equation}
\left| \sum_{m \leq m_0 \atop \psi(m) = k} \frac{1}{z_\varphi(m)} \right| \leq M.
\end{equation}

Now, for all $k \in \mathbb{N}$, let $u_k$ be the sequence $\left( \sum_{m \leq m_0 \atop \psi(m) = k} \frac{1}{z_\varphi(m)} \right)_{m_0 \geq 0}$ whose limit is $c_k^0 a_n$ by construction of $f$. From (14), the sup norm of $u_k$ is smaller than $c_k^0 M$, and thus the series $\sum_{k \geq 0} u_k$ converges in the space of converging sequence equipped with the sup norm. But its sum is clearly the sequence of partial sums of $\sum_{m \geq 0} \frac{1}{w_m}$. Thus this series converges, and its sum is $a_n \sum_{k=0}^{+\infty} c_k^n$, as wanted.

We suppose that $f$ is a counter-example of type (b). There are two natural partitions of the zeroes of $d$ : the partition $Z_0, Z_1, \ldots, Z_k, \ldots$ by modulus ($Z_0$ contains the element of minimal modulus, the following are in $Z_1$ etc) and the partition $Z'_1, Z'_2, \ldots$ defined by

\[ Z'_k = \left\{ z \atop z \in Z_k : z \text{ is a zero of } f \right\}. \]

Both partitions are endowed with the natural notion of multiplicity. Now, we get an ordering for which the trace formulae hold in the following way : we put first the element of $Z_0 \cap Z'_0$ in the order which gave trace formulae for $f$, then we put the element of $Z_0 \cap Z'_1$ (according to the same order) then $Z_0 \cap Z'_2$, etc, when we are done with $Z_0$ (which happens in a finite number of steps), we do the same with $Z_1$, then $Z_2$, etc. The proof that trace formulae hold in this case is similar as in case (a) (in fact a bit easier). To get an ordering for which there is divergence of the inverse of the zeroes of $d$ at any power, we do exactly the same, except that at each step we put the elements of $Z'_0$ in the order which gave the divergence for $f$.

We end this section with the following lemma, which shall be used to prove Corollary 3.6.

Lemma 1.13. Let $E$ be a subset of $\mathbb{N}^\ast$. Then there is an entire function $Q$ such that $Q(0) = Q(1) = 0$ and if $Q : z \mapsto \sum_{n}^{+\infty} \beta_n z^n$ then $\beta_n = 0$ if and only if $n \in E$, and $\beta_n \in \mathbb{R}_+$ for all $n \in \mathbb{N}^\ast$. Moreover, for all $\epsilon > 0$ and $\rho > 0$, if $\alpha > 0$ is sufficiently small, then there is an entire function $h : z \mapsto \sum_{n=0}^{+\infty} \alpha_n z^n$ such that $\left( 1 - 2z \left( 1 - z \right) h(z) \right)$, for all $z \in \mathbb{C}$, and $\alpha_n \in \left[ \frac{\epsilon}{\rho^n}, \frac{\epsilon}{\rho^n} \right]$ for all $n \in \mathbb{N}$.

\(^4\)If $f$ has non integral order $\delta$ and $\sum c_k^0 < +\infty$, then one may show using [5, 2.9.1] that $d$ has also same order than $f$.

\(^5\)That’s where we use that $f$ is precisely the counter-example constructed above. We shall not need it for the case (b)
Proof. We shall construct $Q$ of the form $Q : z \mapsto z (1 - z) \sum_{n=0}^{+\infty} b_n z^n$. Then we have $b_1 = b_0$ and $b_{n+1} = b_n - b_{n-1}$, for all $n \geq 1$. If $E$ contains a final segment of $\mathbb{N}^*$ then it is easy to see that there is a polynomial $Q$ with real coefficients that satisfies the first part of Lemma 1.13. If $E$ does not contain a final segment of $\mathbb{N}^*$ then the sequence $(b_n)_{n \in \mathbb{N}}$ may be recursively defined by

- $b_0 = 1$ if $1 \in E$, $0$ otherwise;
- $b_n = b_{n-1}$ if $n \geq 1$ and $n + 1 \in E$;
- $b_n = 1 \min \{ \ell \geq n, \ell + 2 \notin E \}$ if $n \geq 1$ and $n + 1 \notin E$.

The second part of Lemma 1.13 may be proven in a similar way than Lemma 1.11. \qed

2. Growth of the dynamical determinant for hyperbolic dynamics

Let $M$ be a real analytic compact manifold\(^6\). Let $T : M \to M$ be a $C^\infty$ diffeomorphism of $M$. We recall the following definition and fix a hyperbolic basic set $\Lambda$ for $T$.

Definition 2.1 (Hyperbolic basic set). A $T$-invariant compact subset $\Lambda$ of $M$ is a hyperbolic basic set for $T$ if the following conditions are fulfilled:

- $\Lambda$ is hyperbolic, that is there is a Riemannian metric on a neighbourhood of $\Lambda$ and, for all $x \in \Lambda$ there exists a decomposition $T_x M = E_x^u \oplus E_x^s$ such that $D_T(E_x^u) = E_{T_x}^u$ for $i \in \{ u, s \}$, and there exists $\lambda < 1$ and $c > 0$ such that for all $n \in \mathbb{N}$ and $x \in \Lambda$ we have $\| D_T^n|_{E_x^u} \| \leq c \lambda^n$ and $\| D_T^{-n}|_{E_x^s} \| \leq c \lambda^n$;
- $\Lambda$ is isolated, i.e. there is a neighbourhood $V$ of $\Lambda$ in $M$ such that $\Lambda = \cap_{n \in \mathbb{Z}} T^n(V)$;
- $T|_\Lambda$ is transitive.

Recall also that a $C^\infty$ function $f : U \to \mathbb{R}^n$, where $U$ is an open subset of $\mathbb{R}^n$, is said to be Gevrey on a compact subset $K$ of $U$ if there are constants $C, R > 0$ and $\sigma > 1$ such that for all multi-indices $\alpha \in \mathbb{N}^n$ we have

$$\| \partial^\alpha f \|_\infty \leq CR^{|\alpha|} |\alpha|^\sigma |\alpha|.$$

Notice that if we can choose $\sigma = 1$ then $f$ is analytic on the interior of $K$. We shall say that a map between manifolds is Gevrey if it is Gevrey in coordinates (see footnote 6).

Let $g : M \to \mathbb{C}$ be a $C^\infty$ function. For all $m \in \mathbb{N}$ and $x \in M$ write

$$g^{(m)}(x) = \prod_{k=0}^{m-1} g(T^k x).$$

From [3, Theorem 1.5], we know that the expression (1) defines an entire function $d_{T,g}$. We can now state the main result of this section.

Theorem 2.2. If $T$ and $g$ are Gevrey, then there exists a constant $c$ such that

$$|d_{T,g}(z)| = O \exp \left( c \exp \left( c |z| \ln |\ln |z|| \right) \right).$$

Moreover, there exists an integer $m_0$ such that for all integers $m \geq m_0$ there is a constant $c'$ such that

$$|d_{T^m,g^{(m)}}(z)| = O \exp \left( c' |z| \ln |\ln |z|| \right).$$

Remark 2.3. Morally speaking, the reason why the bound (16) is much better than (15) is because the iterates of $T$ enjoy more hyperbolicity than $T$ itself. However, the bound (16) still narrowly misses finite order (see Definition 1.2) and thus is not enough to apply Theorem 1.4.

From Theorem 2.2 and Jensen’s formula (see [5, 1.2.1 p.2] for instance), we immediately get:

Corollary 2.4. Let $T$ and $g$ be Gevrey. For all $r > 0$, denote by $N(r)$ the number of resonances of $(T,g)$ outside of the disc of center 0 and radius $r$ then there is a constant $c > 0$ such that

$$N(r) = O \left( \frac{1}{r^c \ln |\ln r|} \right).$$

\(^6\)We do not really need $M$ to be real analytic. To define Gevrey objects on $M$, we only need to equip $M$ with an atlas whose change of charts are Gevrey, and then say that a map is Gevrey if it is in these coordinates. For sake of simplicity, we shall assume anyway that $M$ is real analytic, even if all our results remain true without this hypothesis.
Proof. From the definition of the resonances as eigenvalues of a transfer operator (see [1, Definition 1.1]) and the properties of dynamical determinants (see for instance [1, Theorem 6.2]), we have that \( N(r) \) is the number of zeroes of \( d_{T_n,g(m)} \) in the disc of center zero and radius \( r^{-m} \) (for any \( m \in \mathbb{N}^* \)). Thus, by Jensen’s formula, we have

\[
N(r) \leq \sup_{|z| \leq r^{-m}} \log |d_{T_n,g(m)}(z)|.
\]

Taking \( m \geq m_0 \), the estimate (17) is now an immediate consequence of bound (16).

The proof of Theorem 2.2 given below is based on a quantitative investigation of the proof of [3, Theorem 1.5]. Ignoring technical difficulties, the idea of the proof may be naively described as follows: we get a decomposition \( L_g = (L_g)_0 + (L_g)_1 \), of the transfer operator acting on a space \( B^s \), where \( s \in \mathbb{N}^* \). The term \((L_g)_0\) gets smaller when \( s \) goes to \( +\infty \) and the term \((L_g)_1\) is nuclear. Thus, using the generalized notion of determinant defined in §2.3, and choosing \( s = s(z) \) reasonably, we want to write

\[
d_{T,g}(z) = \det^s (I - zL_g) = \det^s (I - z (L_g)_0) \det^s \left( I - (I - z (L_g)_0)^{-1} z (L_g)_1 \right).
\]

Then the first factor in the right-hand side of (18) should be the exponential of a polynomial, while the second one may be estimated using a bound on the nuclear operator norm of \((L_g)_1\). Note that if the rigorous proof could be written in this way then we would always have the estimate (16) instead of (15).

2.1. Local spaces and transfer operators. We begin by recalling some notations and definitions from [3] with some minor adaptations.

Let \( d = d_+ + d_0 \) be an integer. Let \( \Theta = (C_+, C_-, \varphi_+, \varphi_-) \) be a Gevrey polarization, that is \( C_+ \) and \( C_- \) are closed cones in \( \mathbb{R}^d \) containing respectively a \( d_+ \)-dimensional and a \( d_0 \)-dimensional subspace, and such that \( C_+ \cap C_- = \{0\} \). Furthermore the maps \( \varphi_+ : S^{d-1} \to [0, 1] \) are Gevrey and satisfy \( \varphi_+ + \varphi_- = 1 \) and \( \varphi_+ (x) = 1 \) for all \( x \in \{+, -\} \) and \( x \in C_+ \cap S^{d-1} \).

Pick a Gevrey function \( \chi : \mathbb{R} \to [0, 1] \) such that \( \chi (s) = 1 \) for \( s \geq 1 \) and \( \chi (s) = 0 \) for \( s \geq 2 \). Then define \( \chi_n, \psi_n : \mathbb{R}^d \to [0, 1] \) for \( n \in \mathbb{N} \) by \( \chi_n (\xi) = \chi (2^{-n} \xi) \) for \( \xi \in \mathbb{R}^d \) and \( \psi_n = \chi_n - \chi_{n-1} \) (with \( \chi_{-1} = 0 \)). Set \( \Gamma = \{(n, \sigma) : n \in \mathbb{N}, \sigma \in \{+, -\}\} \), and define a partition of unity \((\psi_{\alpha,n,\sigma})_{(n,\sigma) \in \Gamma}\) by

\[
\psi_{\alpha,n,\sigma} (\xi) = \begin{cases} \psi_n (\xi) \varphi_\sigma \left( \frac{\xi}{\xi} \right) & \text{if } n \geq 1 \\ \varphi_\sigma \left( \frac{\xi}{\xi} \right) & \text{if } n = 0. \end{cases}
\]

If \( \psi \in \mathcal{S} \) we denote by \( \psi(D) \) the operator defined for \( \varphi \in \mathcal{S}' \) by

\[
\psi(D) \varphi = \mathcal{F}^{-1} (\psi, \mathcal{F}(\varphi)) \in \mathcal{S}',
\]

where \( \mathcal{S} \) is the Schwartz class, \( \mathcal{S}' \) is the space of tempered distribution and \( \mathcal{F} \) denotes the Fourier transform on \( \mathbb{R}^d \). Then, for all \( n \in \mathbb{N} \), we define

\[
B_n = \{ \varphi \in \mathcal{C}^\infty (\mathbb{R}^d) \cap L^1 (\mathbb{R}^d) : \chi_n (D) \varphi = \varphi \}.
\]

Equipped with the \( L^1 \) norm, \( B_n \) is a well-known Banach of analytic functions (see for instance [21, I.1]). Now for \( s > 0 \) we can define

\[
\mathcal{B}^{\Theta,s}_\Gamma = \left\{(\varphi_{n,\sigma})_{(n,\sigma) \in \Gamma} \in \prod_{(n,\sigma) \in \Gamma} B_{n+3} : \lim_{n \to +\infty} \max_{\sigma \in \{+, -\}} 2^{\sigma s} \|\varphi_{n,\sigma}\|_{L^1} = 0 \right\}
\]

equipped with the norm

\[
\left\|\left(\varphi_{n,\sigma}\right)_{(n,\sigma) \in \Gamma}\right\|_{\mathcal{B}^{\Theta,s}_\Gamma} = \sup_{(n,\sigma) \in \Gamma} 2^{\sigma s} \|\varphi_{n,\sigma}\|_{L^1}.
\]

Let \( \beta \) be defined for \( x \in \mathbb{R}^d \) by \( \beta (x) = 1 + |x|^2 \). We will also need the Banach spaces

\[
\hat{B}_n = \{ \varphi \in \mathcal{C}^\infty (\mathbb{R}^d) \cap L^1 (\mathbb{R}^d) : \chi_n (D) \varphi = \varphi \text{ and } \beta \cdot \varphi \in L^1 (\mathbb{R}^d) \}
\]
equipped with the norm \( \varphi \mapsto \|\beta \cdot \varphi\|_{L^1} \), for \( n \in \mathbb{N} \), and, for \( s > 0 \),

\[
\hat{\mathcal{B}}^{\Theta,s}_\Gamma = \left\{(\varphi_{n,\sigma})_{(n,\sigma) \in \Gamma} \in \prod_{(n,\sigma) \in \Gamma} \hat{B}_{n+2} : \lim_{n \to +\infty} \max_{\sigma \in \{+, -\}} 2^{\sigma s} \|\beta \cdot \varphi_{n,\sigma}\|_{L^1} = 0 \right\}
\]

\(^7\)In [3], the space \( B_n \) was defined differently, which allows to get a better bound on the essential spectral radius of the transfer operator. This is not of much interest here since we do not try to explicit the better constants in Theorem 2.2, and so we shall stick to this more common space.
equipped with the norm
\[ \left\| (\varphi_{n,\sigma})_{(n,\sigma)\in \Gamma} \right\|_{B_{\Gamma}^{\sigma}} = \sup_{(n,\sigma)\in \Gamma} 2^{\sigma \delta} \| \varphi_{n,\sigma} \|_{L^1}. \]

Now, we get to the action of a transfer operator on these spaces. Let \( \Theta' = (C'_+, C'_-, \varphi'_+, \varphi'_-) \) be another polarization and \( T : \mathbb{R}^d \to \mathbb{R}^d \) be a Gevrey bilipschitz diffeomorphism, cone-hyperbolic8 with respect to \( \Theta \) and \( \Theta' \), that is we have
\[ \forall x \in \mathbb{R}^d : t D_x T \left( \mathbb{R}^d \setminus C_+ \right) \subseteq \{0\} \cup \text{int } C'_+. \]

Let \( G : \mathbb{R}^d \to \mathbb{C} \) be a Gevrey function with compact support, and define the local transfer operator
\[ L : C^\infty (\mathbb{R}) \to C^\infty (\mathbb{R}), L \varphi = G. (\varphi \circ T). \]

Then define
\[ h^+_\text{max} = \left[ \log_2 \left( \sup_{x \in \text{supp } G} \sup_{|\xi| = 1} \left| t D_x T \cdot \xi \right| \right) \right] + 6 \]
and
\[ h^-_{\text{min}} = \left[ \log_2 \left( \inf_{x \in \text{supp } G} \inf_{|\xi| = 1 \in \xi \notin \text{C}_+} \left| t D_x T \cdot \xi \right| \right) \right] - 6. \]

If \( (n, \sigma), (l, \tau) \in \Gamma \) we write \( (l, \tau) \not\to (n, \sigma) \) if one of the following conditions holds (and \( (l, \tau) \not\to (n, \sigma) \) otherwise)
\begin{itemize}
  \item \( (\tau, \sigma) = (+, +) \) and \( n < l - h^+_\text{max} \);
  \item \( (\tau, \sigma) = (-, -) \) and \( l + h^-_{\text{min}} < n \);
  \item \( (\tau, \sigma) = (+, -) \) and \( n > h^-_{\text{min}} \) or \( l > -h^+_\text{max} \).
\end{itemize}

Now pick \( \hat{C}_+ \subseteq \{0\} \cup \text{int } C_+ \) a closed cone such that
\[ \forall x \in b \text{ supp } G : t D_x T \left( \mathbb{R}^d \setminus C_+ \right) \subseteq \{0\} \cup \text{int } C'_+. \]

Let \( \tilde{\varphi}_- : S^{d-1} \to [0,1] \) be a Gevrey function such that \( \tilde{\varphi}_- (\xi) = 0 \) if \( \xi \in \hat{C}_+ \cap S^{d-1} \), and \( \tilde{\varphi}_- (\xi) = 0 \) if \( \xi \notin C_+ \cap S^{d-1} \). Set \( \tilde{\varphi}_- = 1 \) and \( \tilde{\psi}_l = \chi_{l+1} - \chi_{l-2} \) for \( l \geq 1 \). Then define \( \tilde{\psi}_{\Theta, n, \sigma} \) for \( (n, \sigma) \in \Gamma \) as \( \psi_{\Theta, n, \sigma} \), but replacing \( \varphi_-, \varphi_- \) and \( \psi_l \) respectively by \( \tilde{\varphi}_-, \tilde{\varphi}_- \) and \( \tilde{\psi}_l \) if \( n \geq 1 \). If \( n = 0 \), set \( \tilde{\psi}_{\Theta, n, \sigma} = \chi_1 \).

These definitions ensure that \( \tilde{\psi}_{\Theta, n, \sigma} \) is equal to 1 on the support of \( \psi_{\Theta, n, \sigma} \). If \( \hat{C}_+ \) is chosen close enough to \( C_+ \) then there exists \( C (T, G) > 0 \) such that for all \( (l, \tau), (n, \sigma) \in \Gamma \), if \( (l, \tau) \not\to (n, \sigma) \) and max \( n, l \geq C (T, G) \) then
\[ d \left( \text{supp } \psi_{\Theta, n, \sigma}, t D_x T \left( \text{supp } \tilde{\psi}_{\Theta, l, \tau} \right) \right) \geq 2^{\text{max}(n,l)-C(T,G)}. \]

Now set for \( (n, \sigma), (l, \tau) \in \Gamma \) and \( \varphi \in S' \)
\[ S_{n, \tau}^{l, \sigma} : = \psi_{\Theta, n, \sigma} (D) \circ L \circ \tilde{\psi}_{\Theta, l, \tau} \varphi. \]

We shall see in Lemma 2.8 that \( S_{n, \tau}^{l, \sigma} \) defines a bounded operator from \( B_{l+3} \to B_{n+1} \). and then define an operator \( M : B_{n}^{\sigma} \to B_{n}^{\sigma} \) which acts as the matrix with entries \( S_{n, \tau}^{l, \sigma} \). The operator \( M \) will be well-defined thanks to (27) and Lemma 2.14, and will play the role of a transfer operator, just like in [3].

Define \( b : \mathbb{R}^d \to \mathbb{R} \) by
\[ b (x) = 1 \text{ if } |x| \leq 1, b (x) = \frac{1}{|x|^{d+1}} \text{ otherwise } . \]

For \( m \geq 0 \) and \( x \in \mathbb{R}^d \), set \( b_m (x) = 2^{dm} b (2^m x) \) and recall that there exists \( C > 0 \) such that for all \( n, m \in \mathbb{N} \) and \( x \in \mathbb{R}^d \) we have (see footnote 29 of chapter 2 of [1])
\[ \lambda_{n} \leq b_m (x) \leq C b_{\min(n,l)} (x). \]

The following lemma is a refinement of [3, Lemma 4.13] when \( T \) and \( G \) are Gevrey.

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8 We do not need to consider only regular cone-hyperbolic diffeomorphism as in [3] thanks to our choice of norm, but recall that this would not give optimal results if we were studying the spectral radius of the transfer operator in finite differentiability (see footnote 7).
Lemma 2.5. There are constants $C > 0$, $R > 0$ and $c > 1$ such that for all $s \in \mathbb{N}^*$, for all $(l, \tau), (n, \sigma) \in \Gamma$ such that $(l, \tau) \not\rightarrow (n, \sigma)$ and $\max(n, l) \geq C (T, G)$, for all bounded $u \in C^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we have

$$|\xi_{n, \sigma}^l u (x)| \leq CR^s s^{e_2 - s \max(n, l)} \int_{\mathbb{R}^d} b_{\min(n, l)} (x - y) |u (T (y))| \, dy.$$ 

As in appendix C of [3], to prove Lemma 2.5, we only need to show

$$|V_{n, \sigma}^l (x, y)| \leq CR^s s^{e_2 - s \max(n, l)} b_{\min(n, l)} (x - y),$$

where

$$V_{n, \sigma}^l (x, y) = \frac{1}{(2\pi)^{2d}} \int_{\mathbb{R}^d} e^{i (x - w) \xi + i (T (w) - T (y)) \eta} G (w) \psi_{\theta', n, \sigma} (\xi) \psi_{\Theta, l, \tau} (\eta) \, dw \, dw$$

is defined in such a way that

$$S_{n, \sigma}^l u (x) = \int_{\mathbb{R}^d} V_{n, \sigma}^l (x, y) u (T (y)) |\det (D_y T)| \, dy.$$ 

For all $w \in \mathbb{R}^d$, define the quadratic form $\Phi (w)$ on $\mathbb{R}^d \times \mathbb{R}^d$ by $\Phi (w) (\xi, \eta) = |DT (w)^* \cdot \eta - \xi|^2$. Thus if $(l, \tau) \not\rightarrow (n, \sigma)$, $\xi \in \supp (\psi_{\theta', n, \sigma})$ and $\eta \in \supp (\tilde{\psi}_{\Theta, l, \tau})$, with $\max(n, l) \geq C (T, G)$, we have, thanks to (21),

$$\Phi (w) (\xi, \eta) \geq C_1 \left(2^{\max(l, n)} \right)^2 \geq C_2 \left(2^{\max(n, l)} \right)^2 \max(\|\xi\|, \|\eta\|) \geq C_3 \left(\|\xi\|, \|\eta\|\right)^2,$$

for some constants $C_1, C_2, C_3 > 0$ that only depend on $T$ and $G$. Define also for $w \in \mathbb{R}^d$ and $k = 1, \ldots, d$ the linear form $l_k (w)$ on $\mathbb{R}^d \times \mathbb{R}^d$ by $l_k (w) = i (\partial_k T (w) \eta - w_k \eta_k)$. Thus the maps $w \mapsto \Phi (w)$ and $w \mapsto l_k (w)$ for $k = 1, \ldots, d$ are Grevzy.

We state and prove two lemmas before showing Lemma 2.5.

Lemma 2.6. For every non-zero natural integer $s$, the kernel $V_{n, \sigma}^l$ may be written as

$$V_{n, \sigma}^l (x, y) = (2\pi)^{-2d} \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i (x - w) \xi + i (T (w) - T (y)) \eta} F_s (\xi, \eta) \psi_{\theta', n, \sigma} (\xi) \tilde{\psi}_{\Theta, l, \tau} (\eta) \, dw \, dw,$$

where the function $F_s$ is the sum of at most $(4d)^s$ terms of the form

$$(\xi, \eta, w) \mapsto \pm \frac{\partial^a G (w)}{\Phi (w) (\xi, \eta)} \partial^{b_1} l_{k_1} (w) (\xi, \eta) \ldots \partial^{b_l} l_{k_l} (w) (\xi, \eta) \partial^l \Phi (w) (\xi, \eta) \ldots \partial^l \Phi (w) (\xi, \eta),$$

with $l \leq s$ and $|a| + |b_1| + \ldots + |b_l| + |\gamma_1| + \ldots + |\gamma_l| = s$.

Proof. Set $F_0 (\xi, \eta, w) = G (w)$ and then, performing integration by parts$^9$ in $w$ as in appendix C of [3], we may set

$$F_{s+1} (\xi, \eta, w) = \sum_{k=1}^d \partial_k \left( \frac{l_k (w) F_s (\ldots, w)}{\Phi (w)} \right).$$

Thus, each term of the generation $s$ gives rise to at most $d (2 s + 1) \leq 4 d (s + 1)$ terms of the generation $s + 1$. \hfill \Box

Notice that there is a constant $\delta > 0$ such that if $\max(n, l) \geq C (T, G)$ then $\max(\|\xi\|, \|\eta\|) \geq \delta$ if $\xi \in \supp (\psi_{\theta', n, \sigma})$ and $\eta \in \supp (\tilde{\psi}_{\Theta, l, \tau})$. Consequently, there is a constant $C_4$ such that if $h$ is a quadratic form (resp. a linear form) on $\mathbb{R}^d \times \mathbb{R}^d$ and $\alpha, \beta$ are multi-indices in $\mathbb{N}^d$, and $\xi \in \supp (\psi_{\theta', n, \sigma})$ while $\eta \in \supp (\tilde{\psi}_{\Theta, l, \tau})$, then we have

$$|\partial^\alpha \partial^\beta h (\xi, \eta)| \leq C_4 \|h\| \max(\|\xi\|, \|\eta\|)^{2 - |\alpha| - |\beta|},$$

(resp. $\leq C_4 \|h\| \max(\|\xi\|, \|\eta\|)^{1 - |\alpha| - |\beta|}$).

---

$^9$That is applying the formula

$$\int_{\mathbb{R}^d} e^{i f (w) g (w)} \, dw = i \int_{\mathbb{R}^d} e^{i f (w)} \left( \sum_{k=1}^d \frac{\partial_k (w) g (w)}{\sum_{k=1}^d (\partial_k (w))^2} \right) \, dw.$$
Lemma 2.7. There are constants $C$ and $c$ such that, for all multi-indices $\alpha$ and $\beta$ in $\mathbb{N}^d$, there is a constant $C_{\alpha,\beta}$ such that, for each natural integer $s$, for all $(l, \tau)$ and $(n, \sigma)$ in $\Gamma$ such that $\max(n, l) \geq C(T,G)$ and $(l, \tau) \rightarrow (n, \sigma)$, for all $\xi \in \text{supp } \left( \psi_{\Theta, n, \sigma} \right)$ and $\eta \in \text{supp } \left( \psi_{\Theta, l, \tau} \right)$, we have

$$\left| \partial^\alpha \partial^\beta F_s (\xi, \eta, w) \right| \leq C_s C_{\alpha,\beta} (s + 1)^{\left| \alpha \right| + \left| \beta \right|} s^{c s + 2 - n \left| \alpha \right| - \left| \beta \right|} \max(n, l).$$

Proof. Pick a term $H$ of the form given by (23). Then $\partial^\alpha \partial^\beta H$ may be written as the sum of at most $(2(2s + \left| \alpha \right| + \left| \beta \right|))^{\left| \alpha \right| + \left| \beta \right|}$ terms of the form:

$$\left( \xi, \eta, w \right) \mapsto \pm \frac{\partial^\alpha G (w)}{\left( \Phi (w) (\xi, \eta) \right)^{s + 1 + t}} \partial^\alpha \partial^\beta l_k (w) (\xi, \eta) \cdot \partial^\alpha \partial^\beta l_k (w) (\xi, \eta) \cdot \partial^\gamma \Phi (\xi, \eta) \cdot \partial^\gamma \Phi (\xi, \eta)$$

with the same notation as in (23), and, in addition,

$$\alpha = \alpha_1 + \cdots + \alpha_s + \alpha_1 + \cdots + \alpha_{t+s}$$

$$\beta = \beta_1 + \cdots + \beta_s + \beta_1 + \cdots + \beta_{t+s}$$

$$\gamma_{t+1} = \cdots = \gamma_{t+t} = 0$$

and $|t| \leq \left| \alpha \right| + \left| \beta \right|$. Now, if $w \in \text{supp } G$, $\xi \in \text{supp } \left( \psi_{\Theta, n, \sigma} \right)$, and $\eta \in \text{supp } \left( \psi_{\Theta, l, \tau} \right)$, the modulus of the term given by (25) is smaller than

$$C_s C_{\alpha,\beta} (s + 1)^{\left| \alpha \right| + \left| \beta \right|} \frac{\partial^\alpha G \left( \xi, \eta, w \right)}{\left( \Phi (w) (\xi, \eta) \right)^{s + 1 + t}} \left\| \partial^\alpha \partial^\beta l_k \right\|_\infty \left\| \partial^\gamma \Phi \right\|_\infty \left\| \partial^\gamma \Phi \right\|_\infty \left. \left( \operatorname{supp} G \right)^{c s + 2 - n \left| \alpha \right| - \left| \beta \right|} \max(n, l) \right|.$$
Now we can define the operator \( M_b : B^s_{\Gamma} \rightarrow B^s_{\Gamma} \) by

\[
M_b \left( (u, \tau)_{(l, \tau) \in \Gamma} \right) = \left( \sum_{(l, \tau) \in \Gamma} S_{n, \sigma}^{l, \tau} u \right)_{(n, \sigma) \in \Gamma}.
\]

Lemma 2.8 provides the following estimate, which is satisfied as soon as \( s \geq 1 \) (\( C \) being independent of \( s \)), and will play the role of Lemma 4.17 of [3] in (30):

\[
\|M_b\|_{\mathcal{L}(B^s_{\Gamma}, B^{s'}_{\Gamma})} \leq C 2^{\max(s h^+_{\max} - s h^-_{\min})}.
\]

Recall that \( h^+_{\max} \) and \( h^-_{\min} \) have been defined respectively in (19) and (20).

Now we are going to study the remainder term

\[
M_c : (u, \tau)_{(l, \tau) \in \Gamma} \mapsto \left( \sum_{(l, \tau) \in \Gamma} S_{n, \sigma}^{l, \tau} u \right)_{(n, \sigma) \in \Gamma}
\]

which will turn out in Lemma 2.14 to be nuclear (see [11, V.1] for a definition). Thus the operator

\[ M = M_b + M_c \]

which acts as the matrix with entries \( S_{n, \sigma}^{l, \tau} \), shall be well-defined. First recall the definition of the approximation numbers.

**Definition 2.10.** Let \( E, F \) be Banach spaces and \( A : E \rightarrow F \) be a bounded operator. For all \( k \in \mathbb{N} \) let

\[ a_k (A) = \inf \left\{ \| A - B \|_{\mathcal{L}(E,F)} : \text{rank } B < k \right\}. \]

We will need the following proposition, the fact that the constant \( C \) does not depend of the choices of the Banach spaces, while not stated in [18, 2.3.11], follows easily from the proof given there.

**Proposition 2.11.** There is a constant \( C > 0 \) such that for all Banach spaces \( E, F \) and every operator \( A : E \rightarrow F \), if \( \sum_{k \in \mathbb{N}} a_k (A) < +\infty \), then \( A \) is nuclear and

\[
\|A\|_{\text{nuc}} \leq C \sum_{k \in \mathbb{N}} a_k (A),
\]

where \( \|A\|_{\text{nuc}} \) denotes the nuclear operator norm of \( A \) (see [11, V.1] for a definition).

The following lemma is a refinement of [3, 4.21].

**Lemma 2.12.** Let \( K \) be a compact subset of \( \mathbb{R}^d \) containing the support of \( G \). There are constants \( C, R, c \) and \( A \) such that, for any natural integer \( s \), for each non-zero natural integer \( N \), and for all \( (n, \sigma) \in \Gamma \) with \( n < N \), there is an operator \( F_{n, \sigma, N} : C^\infty (K) \rightarrow B^s_{n+2} \) of rank at most \( A N \) such that, for any \( u \in C^\infty (K) \), we have

\[
\| \beta \cdot (F_{n, \sigma, N} u - \psi_{\Theta, n, \sigma} (D) u) \|_{L^1} \leq CR^s s^{c s} 2^{-s N} \| u \|_{L^1}.
\]

In order to prove Lemma 2.12, notice that, since \( \chi \) has been chosen Gevrey, there are constants \( C, R \) and \( c \) such that, for all multi-indices \( \alpha \) and all \( (n, \sigma) \in \Gamma \), we have

\[
\| \partial^\alpha \psi_{\Theta, n, \sigma} \|_{\infty} \leq CR^{(|\alpha| |\alpha|^{2n} 2^{-n |\alpha|}}.
\]

Performing integration by parts as in the proof of Lemma 2.5, we easily prove the following lemma.

**Lemma 2.13.** There are constants \( C, R, c \) and \( A \) such that for all integers \( s \), all \( (n, \sigma) \in \Gamma \), \( u \in C^\infty (K) \) and \( x \in \mathbb{R}^d \) such that \( d(x, K) \geq 1 \) we have

\[
|\psi_{\Theta, n, \sigma} (D) u (x)| \leq \frac{CR^s s^{c s} x^{2^{-n s} \| u \|_{L^1}}}{d(x, K)}.
\]

**Proof of Lemma 2.12.** We may suppose that \( K \subseteq [-1, 1]^d \). Then choose a Gevrey function \( \varphi : \mathbb{R}^d \rightarrow [0, 1] \) such that \( \varphi (x) = 1 \) if \( x \in [-1, 1]^d \) and \( \varphi (x) = 0 \) if \( x \notin [-2, 2]^d \). Define \( \varphi_N \) by \( \varphi_N (x) = \varphi (4^{-N} x) \) and define \( H (u) = \varphi_N \psi_{\Theta, n, \sigma} (D) u \). From Lemma 2.13 and the fact that \( \varphi_N (x) = 1 \) if \( |x| \leq 4^N \), it comes that

\[
\| \beta (H (u) - \psi_{\Theta, n, \sigma} (D) u) \|_{L^1} \leq CR^s s^{c s} 2^{-s N} \| u \|_{L^1}.
\]
for some constants $\tilde{C}, \tilde{R}$ and $\tilde{s}$. Then the function $H(u)$ is supported in $[-4^{N+1}, 4^{N+1}]^d$ and may be seen as a function on the torus $\mathbb{R}^d / (4^{N+1})^d$. Thus we can define its Fourier coefficients
\[ c_\alpha (u) = \frac{\sqrt{2} \times 4^{-d} 2^{N+1}}{2} \int_{\mathbb{R}^d} e^{-i\alpha x} H(u)(x) \, dx, \]
for $\alpha \in 4^{-N-1} \pi \mathbb{Z}^d$. Then set
\[ F(u)(x) = \varphi_{N+1}(x) \sum_{|\alpha| \leq 2^{N+5}} c_\alpha (u) e^{i\alpha x}. \]
Since $H(u) - F(u)$ is supported in $[-2 \times 4^{N+1}, 2 \times 4^{N+1}]$ we have
\[ \|\beta (H(u) - F(u))\|_{L^1} \leq C_{10} 4^N \|\beta (H(u) - F(u))\|_{L^\infty} \leq C_{11} 4^{(2d+1)N} \sum_{|\alpha| > 2^{N+5}} |c_\alpha (u)|, \]
for some constants $C_{10}$ and $C_{11}$. Then notice that
\[ c_\alpha (u) = \frac{\sqrt{2} \times 4^{-d} 2^{N+1}}{2} \mathcal{F}_{\varphi_{N+1}} (\psi_{\theta_n, \sigma} F u)(\alpha) \]
\[ = \frac{\sqrt{2} \times 4^{-d} 2^{N+1}}{2} 4^d \int_{\mathbb{R}^d} \mathcal{F}_{\varphi} (4^N (\alpha - \xi)) \psi_{\theta_n, \sigma} (\xi) F u (\xi) \, d\xi. \]
But if $\xi \in \text{supp} (\psi_{\theta_n, \sigma})$ we have $|\alpha - \xi| \geq \frac{|\alpha|}{2}$, consequently
\[ |c_\alpha (u)| \leq C_{13} 4^d 2^{Nd} \sup_{|x| \geq \frac{|\alpha|}{2}} |\mathcal{F}_{\varphi} (4^N x)| \|u\|_{L^1} \]
\[ \leq C_{13} 4^{dN} \sup_{|x| \geq \frac{|\alpha|}{2}} |\mathcal{F}_{\varphi} (x)| \|u\|_{L^1}. \]
Since $\varphi$ is Gevrey, we get
\[ \sup_{|x| \geq \frac{4^N |\alpha|}{2}} |\mathcal{F}_{\varphi} (x)| \leq \tilde{C} \tilde{R}^s \tilde{s}^4 \pi^{-N s} |\alpha|^{-s} \]
which implies (with different constants)
\[ |c_\alpha (u)| \leq \tilde{C} \tilde{R}^s \tilde{s}^4 \pi^{-N s} |\alpha|^{-s} 4^{dN} \|u\|_{L^1}, \]
thus (we may suppose $s \geq d + 1$)
\[ \|\beta (H(u) - F(u))\|_{L^1} \leq \tilde{C} \tilde{R}^s \tilde{s}^4 \pi^{-N s} 4^{(3d+1)N} \left( \sum_{|\alpha| \leq 2^{N+5}} |\alpha|^{-s} \right) \|u\|_{L^1} \]
\[ \leq \tilde{C} \tilde{R}^s \tilde{s}^4 \pi^{-N s} 4^{(3d+1)N + (d+1)(N+1)} \pi^{-d+1} \left( \sum_{|\alpha| \in \mathbb{Z}^d \setminus \{0\}} |\alpha|^{-d+1} \right) \|u\|_{L^1}, \]
which gives (with different constants)
\[ \|\beta (\psi_{\theta_n, \sigma} (D) u - F(u))\|_{L^1} \leq \tilde{C} \tilde{R}^s \tilde{s}^4 \pi^{-2sN} \|u\|_{L^1}. \]
Finally, set $F_{n,\sigma, N} = \chi_{n+1} (D) F$. We get the announced estimates by noticing that there is a constant $C$, that may depend on $K$, such that if $u \in C^\infty (K)$ then $\|\beta \chi_{n+1} (D) u\|_{L^1} \leq C \|\beta u\|_{L^1}$. And the rank of $F_{n,\sigma, N}$ is smaller than
\[ \# \{ \alpha \in 4^{-N-1} \pi \mathbb{Z}^d : |\alpha| \leq 2^{N+5} \} \leq C 2^{dN} \times 4^{N+1}. \]
\[ \square \]

The following lemma is a quantitative version of [3, Lemma 4.20].

**Lemma 2.14.** There are constants $C$, $R$ and $c$ such that for each non-zero integer $s$ the operator $M_c : E^{d,s}_1 \to E^{d,s}_1$ is nuclear with nuclear norm smaller than $CR^s s^c$. 

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Proof. As in the proof of Lemma 4.20 in [3], use Lemma 2.12 to construct, for $N$ sufficiently large, an operator $P_N$ of rank at most $CN^2A^N$ with $\|P_N - M_c\| \leq L(\|r^{s,\alpha}_L\|, \|s^{t,\beta}_L\|) \leq CR^s s^{q} 2^{-sN}. Then deduce

\begin{equation}
(28) \quad a^{(s)}_{CN^2A^N}(M_c) \leq CR^s s^{q} 2^{-sN}
\end{equation}

for $N$ sufficiently large (the exponent $s$ stresses the fact that the approximation numbers depend on the space $M_c$ is acting on). It is easy to get bounds on the operator norm of $M_c$ showing that the (decreasing) sequence of approximation numbers of $M_c$ has $l_1$ norm smaller than some $CR^s s^{q} 2^{-sN}$ when $s$ tends to $\infty$ (use $P_N$ and (28)).

Thus, by Proposition 2.11, we have the result for sufficiently large $s$ and Lemma 4.20 in [3] provides enough information to get it for every non-zero integer $s$ (anyhow, we shall only use it for large integers). \qed

2.2. Global space and transfer operator. Let $T : M \to M$ be a Gevrey diffeomorphism with hyperbolic basic set $\Lambda$ and let $g : M \to \mathbb{C}$ be a Gevrey function. Thanks to the hyperbolicity of $T$ on $\Lambda$, we may find, as in [3], a finite covering of $\Lambda$ by open subsets $(V_\omega)_{\omega \in \Omega}$ of $\mathbb{R}^d$ and polarizations $(\Theta_\omega, (C_{\omega, +}, (\omega, +, (\varphi_{\omega, +}, (\varphi_{\omega, -}) \in \Omega$ such that

1. the subset $\bigcup_{\omega \in \Omega} V_\omega$ of $M$ is an isolating neighbourhood for $\Lambda$;
2. for all $\omega \in \Omega$ the set $\kappa_\omega(V_\omega)$ is bounded in $\mathbb{R}^d$;
3. for all $\omega \in \Omega$ and $x \in V_\omega \cap \Lambda$ the cone $C_{\omega, +}(C_{\omega, +})$ contains the normal subspace of $E_{\omega}^\perp$ and the cone $C_{\omega, -}(C_{\omega, -})$ contains the normal subspace of $E_{\omega}^\perp$;
4. for all $\omega, \omega' \in \Omega$ if $V_\omega \cap V_{\omega'} \neq \emptyset$ then the map $T_{\omega, \omega'} = \kappa_\omega \circ T \circ \kappa_{\omega'} : \kappa_{\omega'}(V_\omega) \to \kappa_\omega(V_\omega)$ extends to a Gevrey bilipschitz cone-hyperbolic diffeomorphism $T_{\omega, \omega'} : \mathbb{R}^d \to \mathbb{R}^d$.

Now define the Banach spaces

\begin{equation}
B_{\omega}^s = \oplus_{\omega \in \Omega} B_{\omega} \quad \text{and} \quad \hat{B}_{\omega}^s = \oplus_{\omega \in \Omega} \hat{B}_{\omega}^s
\end{equation}

whose norms are

\begin{equation}
\|(u_{\omega})_{\omega \in \Omega}\|_{B_{\omega}^s} = \max_{\omega \in \Omega} \|u_{\omega}\|_{B_{\omega}^s} \quad \text{and} \quad \|(u_{\omega})_{\omega \in \Omega}\|_{\hat{B}_{\omega}^s} = \max_{\omega \in \Omega} \|u_{\omega}\|_{\hat{B}_{\omega}^s}.
\end{equation}

We are going to define an ad hoc the operator which will play the role of transfer operator. Let $V$ be an isolating neighbourhood for $\Lambda$ whose closure is contained in $\bigcup_{\omega \in \Omega} V_\omega$. Without loss of generality, we may suppose that $g$ is supported in $V$ (the dynamical determinant only depends on the values of $g$ on $\Lambda$). Choose a Gevrey partition of unity $(\varphi_{\omega})_{\omega \in \Omega}$ for $V$ subordinated to $(V_\omega)_{\omega \in \Omega}$: that is, for every $\omega \in \Omega$, the Gevrey function $\varphi_{\omega} : M \to [0, 1]$ is supported in $V_\omega$ and, for all $x \in V$, we have $\sum_{\omega \in \Omega} \varphi_{\omega}(x) = 1$. Choose also for every $\omega \in \Omega$ a Gevrey function $h_{\omega} : [0, 1] \to [0, 1]$ such that $h_{\omega}(x) = 1$ for all $x \in \kappa_{\omega}(\supp(\varphi_{\omega}))$ and $h_{\omega}(x) = 0$ for all $x \notin \kappa_{\omega}(V_\omega)$.

If $m$ is a non-zero integer, choose a Gevrey partition of unity $(\varphi_{\omega})_{\omega \in \Omega^{m+1}}$ of $\cap_{\omega \in \Omega} T^{-i}V$ subordinated to $(V_\omega)_{\omega \in \Omega^{m+1}}$, where for $\omega = (\omega_0, \ldots, \omega_m) \in \Omega^{m+1}$ we set

\begin{equation} \quad V_{\omega} = \bigcap_{i=0}^{m} T^{-i}V_{\omega_i}. \end{equation}

Then, if $\omega, \omega' \in \Omega^{m+1} \times \Omega \times \Omega$ define the operator $L_{\omega, \omega'} : C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)$ by

\begin{equation} \quad L_{\omega, \omega'}u = (\varphi_{\omega'} \varphi_{\omega}g^{(m)}) \circ \kappa_{\omega'}^{-1}(h_{\omega} \circ T_{\omega'}) \circ (u \circ T_{\omega'}) \end{equation}

where $T_{\omega'} = T_{\omega_{m-1} \cdots T_{\omega_0}}$ if $\omega = (\omega_0, \ldots, \omega_m)$. Now, apply the construction of §2.1 to $L = L_{\omega, \omega'}$ to get a bounded operator $M_{\omega, \omega'} = M_{\omega, \omega'} + M_{\omega', \omega} : B_{\omega} \to B_{\omega}$ be the operator acting as the matrix $(M_{\omega, \omega'})_{\omega, \omega' \in \Omega}$. Finally set

\begin{equation} \quad (M^m_b) = (M^m)_{b}, \quad \text{where} \quad (M^m)_b = \sum_{\omega \in \Omega} M_{b, \omega}. \end{equation}
As the notation suggests, $\mathcal{M}^m$ is the $m$th power of the operator$^{10}$ obtained in the case $m = 1$, this comes from the link between $\mathcal{M}$ and a concrete transfer operator (see [3]).

From estimate (27), Lemma 2.14, and the hyperbolicity of $T$ on $\Lambda$, we get that there exist constants $C > 0$ and $0 < \lambda < 1$, that do not depend on $m$, and constants $C(m), R(m) > 0$, and $\epsilon(m) > 1$, that may depend on $m$, such that for all integer $s$ we have, from (27) and Lemma 2.14,

\[(30) \quad \| (\mathcal{M}^m)_B \|_{L(B_2^Z, B_1)} \leq C(m) \lambda^{ms} C^s \text{ and } \| (\mathcal{M}^m)_c \|_{\mathcal{L}_{nuc}(B_2^Z, B_1)} \leq C(m) R(m) \epsilon^m C^s. \]

### 2.3. Flat traces and flat determinants.

In [3], a "flat trace" $\text{tr}^b$ is defined for some bounded operators from $B_2^Z$ to $\hat{B}_2^Z$. We will use the following properties of the flat trace:

- the flat trace is defined on a subspace of $\mathcal{L}(B_2^Z, \hat{B}_2^Z)$ and is linear;
- if $A$ can be written as $A = \sum_{i \in \mathbb{N}} l_i \otimes e_i$ with $l_i \in (B_2^Z)^\prime$, $e_i \in \hat{B}_2^Z$ and $\sum_{i \in \mathbb{N}} ||l_i|| \|e_i\| < +\infty$ then the flat trace of $A$ is defined and

\[(31) \quad \text{tr}^b (A) = \sum_{i \in \mathbb{N}} l_i (e_i), \]

in particular we have $|\text{tr}^b (A)| \leq \|A\|_{\mathcal{L}_{nuc}(B_2^Z, \hat{B}_2^Z)}$;
- for all $m \in \mathbb{N}^*$, the flat trace of $\mathcal{M}^m$ is defined and

\[(32) \quad \text{tr}^b (\mathcal{M}^m) = \sum_{z \in \Lambda} \frac{g^{(m)}(x)}{|\text{det} (I - D_z T^m)|}; \]

- there is an integer $L$ such that

\[(33) \quad \forall m_1, \ldots, m_n \geq L : \text{tr}^b \left( \prod_{i=1}^n (\mathcal{M}^m)_b \right) = 0. \]

Now let $A(z) = \sum_{k \geq 1} A_k z^k$ be a formal power series of bounded operators from $B_2^Z$ to $\hat{B}_2^Z$ such that $A(0) = 0$. If the flat trace of $A_k$ is defined for all $k$ then define the formal power series $\text{tr}^b (A(z)) = \sum_{k \geq 1} \text{tr}^b (A_k) z^k$. Now, if we can define in this way $\text{tr}^b (A(z)^n)$ for all $n$, then we define a new power formal series

\[
\text{det}^b (I - A(z)) = \exp \left( -\sum_{n=1}^{+\infty} \frac{1}{n!} \text{tr}^b (A(z)^n) \right).
\]

This is well defined since $A(0) = 0$. Specifying to the case $A(z) = z^{\mathcal{M}^m}$, we have for all $m \geq 1$

\[(34) \quad \text{det}^b (I - z^{\mathcal{M}^m}) = d_{\mathcal{M}^m}^b (g^{(m)})(z) \]

recalling (1) and (32) (in particular the formal power series in the left-hand side has an infinite radius of convergence). Note that, by (31), if $A$ is a nuclear operator then $\text{det}^b (I - A) = 0$ and converges to the Fredholm determinant of $i \circ \hat{A}$, where $i$ is the inclusion of $\hat{B}_2^Z$ in $B_2^Z$ and $\hat{A}$ is any kernel of $A$.

Now, elementary combinatorics shows that, if $A(z)$ and $B(z)$ are such that $A(0) = B(0) = 0$, and, for all $C_1(z, \ldots, C_n(z) \in \{A(z) B(z)\}$, the flat trace $\text{tr}^b (C_1(z) \ldots C_n(z))$ is defined and invariant under circular permutations of the $C_i(z)$, then we have the equality between formal power series

\[(35) \quad \text{det}^b (I - (A(z) + B(z) - A(z) B(z))) = \text{det}^b (I - A(z)) \text{det}^b (I - B(z)). \]

If $L$ is an integer and $\mathcal{N}$ is an operator such that the flat trace of $\mathcal{N}^m$ is defined for all $m \geq 1$, we can define a formal power series $A(z)$ by

\[
\exp \left( \sum_{k=1}^{L} \frac{1}{k} \mathcal{N}^k z^k \right) = I - A(z).
\]

Then for all $n \geq 1$, the coefficients of $A(z)^n$ are linear combination of powers of $\mathcal{N}$, and thus $\text{tr}^b (A(z)^n)$ is defined. We assert then that we have, as formal power series,

\[(36) \quad \text{det}^b (I - A(z)) = \exp \left( \sum_{k=1}^{L} \frac{1}{k} \text{tr}^b (\mathcal{N}^k) z^k \right)\]

$^{10}$More precisely, we have $\mathcal{M}^m = \mathcal{M} \circ (i \circ \mathcal{M})^{m-1}$ where $i$ denotes the inclusion of $\hat{B}_2^Z$ into $B_2^Z$. 


which may be loosely stated as

\[(\alpha L)^{s}\exp\left(\sum_{k=1}^{L} 1/k N^k z^k\right) = \exp\left(\sum_{k=1}^{L} 1/k \text{tr}^b(N^k) z^k\right).\]

To prove \((36)\), one may reduce to the case \(L = 1\) by using \((35)\), and then the proof becomes straightforward.

2.4. Proof of Theorem 2.2. First of all, we must refine the decomposition \((29)\) in order to make the dependence on \(m\) of the constants appearing in \((30)\) explicit. To do so, we choose an integer \(L\) large enough so that \(\alpha L^2 C < 1\) (where \(\alpha\) and \(C\) are the constants appearing in \((30)\) and \((33)\) holds. Then if \(m \geq L\) write \(m = qL + r\) with \(L \leq r < L\) and then define

\[(\mathcal{M}^m)_b = (\mathcal{M}^L)_b(\mathcal{M}^r)_b\]

and

\[(\mathcal{M}^m)_c = (\mathcal{M}^L)_c(\mathcal{M}^r)_c + \sum_{k=0}^{q-1}(\mathcal{M}^L)_b(\mathcal{M}^L)_c\mathcal{M}^{m-k-1}L\]

If \(1 \leq m < L\) just keep \((\mathcal{M}^m)_b = (\mathcal{M}^L)_b\) and \((\mathcal{M}^m)_c = (\mathcal{M}^L)_c\). Thus for all \(m \geq 1\) we have

\[\mathcal{M}^m = (\mathcal{M}^m)_b + (\mathcal{M}^m)_c\]

and, for some new values of the constants \(C > 0, c > 1, 0 < \lambda < 1\), we have for all integers \(m \geq 1\) and \(s \geq 1\)

\[\|\mathcal{M}^m\|\mathcal{L}(B^s_z,B^z_z) \leq C^{m+s}(\lambda^m)\]

\[\text{and} \quad \|\mathcal{M}^m\|\mathcal{L}(B^s_z,B^z_z) \leq (C^s s^c)\]

Notice also that \((33)\) still holds for our new decomposition.

Now choose a new \(L\) such that \(\gamma = \lambda^L C < 1\) (with our new values of \(\lambda\) and \(C\)) and \((33)\) holds. Then we define the sequence \((b_{k,L})_{k \geq L}\) by

\[(1 - z) \exp\left(\sum_{n=1}^{L-1} \frac{1}{n!}z^n\right) - 1 = - \sum_{k \geq L} b_{k,L}z^k\]

for every complex number \(z\). From basic complex analysis, it is easy to prove that there is a constant \(\alpha\) such that for every positive real number \(r\) we have

\[\sum_{k \geq L} |b_{k,L}|r^k \leq \alpha \exp(\alpha r^L).\]

Up to enlarging \(\alpha\), we also have

\[\sum_{k \geq L} |b_{k,L}|r^k \leq \alpha r^L\]

if in addition \(r \leq 1\). Then define (both series converge thanks to \((39)\), respectively in operator and nuclear operator norms, for any choice of \(s\))

\[\mathcal{M}_b(L)(z) = \sum_{k \geq L} b_{k,L}z^k (\mathcal{M}^k)_b \quad \text{and} \quad \mathcal{M}_c(L)(z) = \sum_{k \geq L} b_{k,L}z^k (\mathcal{M}^k)_c\]

and notice that

\[I - \mathcal{M}_b(L)(z) - \mathcal{M}_c(L)(z) = (I - z\mathcal{M}) \exp\left(\sum_{k=1}^{L-1} \frac{1}{k} \mathcal{M}^k\right)\]

and that for all \(n \geq 1\) we have the equality between formal power series \(\text{tr}^b(\mathcal{M}_{b,L}(z)^n) = 0\), as defined in §2.3, and so \(\det^b(I - \mathcal{M}_{b,L}(z)) = 1\). Now set \(c_1 = \max\left(2, C, (2\alpha)^2\right)\) and \(c_2 = \max\left(-\frac{2}{\ln(\lambda)}, -\frac{2L}{\ln(\gamma)}\right)\). Then pick \(|z| \geq c_1\) and choose \(s = [c_2 \ln|z|]\). With \((40)\) we find

\[\|\mathcal{M}_{b,L}(z)\|\mathcal{L}(B^s_z,B^z_z) \leq \frac{1}{2}\]

and from \((41)\) we can write

\[(I - z\mathcal{M}) \exp\left(\sum_{k=1}^{L-1} \frac{1}{k} \mathcal{M}^k\right) = (I - \mathcal{M}_{b,L}(z))(I - (I - \mathcal{M}_{b,L}(z))^{-1} \mathcal{M}_{c,L}(z)),\]
from which we get, with\(^{11}\) \(\det \left( I - \mathcal{M}_{b,L} (z) \right) = 1, (34), (35),\) and (37)

\[
d_{T,g} (z) = \exp \left( - \sum_{k=1}^{L-1} \frac{1}{k} \text{tr}^{2} (\mathcal{M}^{m}) z^{k} \right) \det \left( I - (I - \mathcal{M}_{b,L} (z))^{-1} \mathcal{M}_{c,L} (z) \right)
\]

\[
= \exp \left( - \sum_{k=1}^{L-1} \frac{1}{k} \text{tr}^{2} (\mathcal{M}^{m}) z^{k} \right) \det \left( I - (I - \mathcal{M}_{b,L} (z))^{-1} \mathcal{M}_{c,L} (z) \right).
\]

(43)

Now, the second factor of (43) is the Fredholm determinant of a nuclear operator whose nuclear operator norm may be bounded thanks to\(^{12}\) (38) and (42), thus [11, Theorem 1.3 p.95] yields

\[
|d_{T,g} (z)| \leq \exp \left( \sum_{k=1}^{L-1} \frac{|z|^{k}}{k} \text{tr}^{2} (\mathcal{M}^{k}) \right) \left( 1 + 2 \| \mathcal{M}_{c,L} (z) \|_{\mathcal{L}_{\text{nuc}} (B_{2}^{s},B_{2}^{s})} \right) \exp \left( 8 \| \mathcal{M}_{c,L} (z) \|_{\mathcal{L}_{\text{nuc}} (B_{2}^{s},B_{2}^{s})}^{2} \right)
\]

recall that \(s = s(|z|) = \lceil c_{2} s \ln |z| \rceil \). Finally, with (38) and (39), this gives

\[
|d_{T,g} (z)| = O \left( \exp \left( c \exp \left( c |z|^{\ln (\ln |z|)} \right) \right) \right)
\]

(44)

for some constant \(c\).

If \(L = 1\) then we may replace (39) by the much better estimates

\[
\sum_{k \geq L} |b_{k,L} | r^{k} \leq r,
\]

and thus the estimates (44) becomes

\[
|d_{T,g} (z)| = O \left( \exp \left( c |z|^{\ln (\ln |z|)} \right) \right).
\]

Note that \(L\) may be chosen equal to 1 if \(T\) and \(g\) are respectively replaced by \(T^{m}\) and \(g^{(m)}\) for \(m \geq L\), which ends the proof of the theorem with \(L = m_{0}\).

\[\square\]

3. Hyperbolic dynamics with explicit dynamical determinants

In this section, we realise a wide class of entire functions as dynamical determinants. In particular, we shall materialise all the possibilities considered in Theorem 1.4 as well as the counter-examples of Proposition 1.8. We shall also construct dynamical determinants, associated with finitely differentiable weights, which cannot be holomorphically continued to the whole complex plane. Unfortunately, it seems that our approach is far too crude to prove that our bounds from Theorem 2.2 for Gevrey maps are sharp. We shall materialise all the possibilities considered in Theorem 1.4 as well as the counter-examples of Proposition 1.8. The strategy is the following: we first construct a subshift of finite type and a weight for which the zeta function cannot be continued meromorphically to the whole complex plane (see [2] and [17, Example 1 p.165]).

The strategy is the following: we first construct a subshift of finite type and a weight for which the zeta function is explicit, then we use Whitney’s extension theorem [22] as in [6] to get a hyperbolic dynamics on a manifold with the same dynamical zeta function, and finally we show that in this particular case the dynamical determinant may be obtained from the dynamical zeta function.

3.1. Symbolic dynamics with explicit weighted zeta functions. Denote by \((\Sigma,\sigma)\) the full (two-sided) shift on two symbols that is

\[
\Sigma = \{0,1\}^\mathbb{Z} \text{ and } \sigma : (x_{i})_{i \in \mathbb{Z}} \mapsto (x_{i+1})_{i \in \mathbb{Z}}.
\]

For all \(\theta \in [0,1]\) define a distance on \(\Sigma\) by \(d_{\theta} (x,y) = \theta^{k}\) where \(k = \inf i \in \mathbb{N} : x_{i} \neq y_{i} \text{ or } x_{-i} \neq y_{-i}\) (with the convention \(\theta^{\infty} = 0\)).

Recall that if \(G : \Sigma \to \mathbb{C}\) is a function, the weighted zeta function associated to \((\sigma,G)\) is the formal power series defined by

\[
\zeta_{\sigma,G} (z) = \exp \left( \sum_{n=1}^{+\infty} \frac{1}{n} \left( \sum_{\sigma^{n} = x = x_{0}}^{n-1} G (\sigma^{k} x, z)^{n} \right) \right).
\]

Notice that \(\zeta_{\sigma,1}\) is the well-known Artin-Mazur zeta function, and that the radius of convergence of \(\zeta_{\sigma,G}\) is non-zero as soon as \(\sigma\) is bounded. We are going to construct weights \(G\) for which \(\zeta_{\sigma,G}\) is given by (45).

\[\text{The condition of invariance under circular permutation of the flat trace is obviously verified here since either all the operators are equal or there is at least one nuclear operator among them and so the flat trace coincide with the trace of nuclear operators.}\]

\[\text{We also use the fact that the inclusion of } B_{2}^{s} \text{ into } B_{2}^{s} \text{ has operator norm smaller than 1.}\]
Proposition 3.1. Let \( h \) be a holomorphic function defined on a neighbourhood of 0 and whose expansions in power series at zero is \( h(z) = \sum_{k=0}^{+\infty} \alpha_k z^k \). Denote by \( \rho \) its convergence radius, and assume that for all \( k \in \mathbb{N} \) we have \( \alpha_k \neq -1 \). Then there is a function \( G : \Sigma \to \mathbb{C} \) such that

\[
(45) \quad \zeta_{\sigma, G}(z)^{-1} = 1 - 2z - z(1 - z)h(z).
\]

Moreover for all \( \theta \in \left[ \frac{1}{2}, 1 \right] \), the function \( G \) is Lipschitz for the distance \( d_\theta \) and if \( \alpha_k \in [-1, +\infty[ \) for all \( k \in \mathbb{N} \) then \( G \) is strictly positive.

Proof. Set \( \beta_m = \frac{1+\alpha_m}{1+\alpha_{m-1}} \) if \( m \geq 1 \) and \( \beta_0 = 1 + \alpha_0 \) and define \( G : \Sigma \to \mathbb{C} \) by

\[
G(x) = \begin{cases} 
\beta_m & \text{if } x_0 = \cdots = x_{m-1} = 0 \text{ and } x_m = 1 \\
1 & \text{if } x_0 = \cdots = x_i = \cdots = 0,
\end{cases}
\]

where \( x = (x_i)_{i \in \mathbb{Z}} \). An easy computation shows that \( G \) is Lipschitz for the distance \( d_\theta \) provided that \( \theta \in \left[ \frac{1}{2}, 1 \right] \). For all \( N > 0 \) define a \( N+1 \times N+1 \) matrix \( P_N \) by

\[
\begin{cases} 
(P_N)_{0,i} = \beta_i & \text{if } 0 \leq i \leq N-1 \\
(P_N)_{0,N} = 1 \\
(P_N)_{i+1,1} = \beta_i & \text{if } 0 \leq i \leq N-1 \\
(P_N)_{N,N} = 1 \\
\end{cases}
\]

that is,

\[
P_N = \begin{bmatrix}
\beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{N-1} & 1 \\
\beta_0 & 0 & \cdots & \cdots & 0 \\
0 & \beta_1 & 0 & \cdots & \cdots & 0 \\
\cdots & \cdots & 0 & \beta_2 & \cdots & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \beta_{N-1} & 1
\end{bmatrix}.
\]

Then an elementary graph-theoretic argument provides that, for all integers \( k \geq 1 \) and all \( N > k \), we have

\[
(46) \quad \sum_{\sigma^i x = x}^{k-1} \prod_{i=0}^{k-1} G(\sigma^i x) = \text{tr}(P_N^k).
\]

Using an argument of dominated convergence (it is easy to show that \( |\text{tr}(P_N^k)| \leq 2^k \|G\|_\infty^k \) by reducing to the positive case), one may then show that, for positive small enough \( z \)

\[
\zeta_{\sigma, G}(z)^{-1} = \lim_{N \to +\infty} \det(I - zP_N).
\]

A computation provides

\[
\det(I - zP_N) = (1 - z) \left( 1 - \sum_{k=0}^{N-1} \left( \prod_{i=0}^{k} \beta_i \right) z^{k+1} \right) + z^{N+2} \prod_{i=0}^{N-1} \beta_i
\]

\[
= (1 - z) \left( 1 - \sum_{k=0}^{N-1} (1 + \alpha_k) z^{k+1} \right) + z^{N+2} (1 + \alpha_{N-1})
\]

and thus

\[
\zeta_{\sigma, G}(z)^{-1} = (1 - z) \left( 1 - \sum_{k=0}^{+\infty} (1 + \alpha_k) z^{k+1} \right) = 1 - 2z - z(1 - z)h(z).
\]

\( \Box \)

Remark 3.2. We could get a more general expression for (45), for instance by allowing more than two symbols. However, we shall not need this here.
3.2. Smooth hyperbolic dynamics with explicit dynamical determinant.

**Proposition 3.3.** There is a $C^\infty$ diffeomorphism $T$ of the sphere $S^4$ and a hyperbolic basic set $\Lambda$ for $T$ such that if $h$ is as in Proposition 3.1 with in addition that $\rho > 1$, then there is a function $g : S^4 \to \mathbb{C}$ such that

$$\zeta_{T,g} (z)^{-1} = 1 - 2z - z(1-z)h(z)$$

and

$$d_{T,g} (z) = \prod_{k=0}^{\infty} \left( \frac{z}{4^k+2} \right)^{(k+1)(k+2)(k+3)}.$$

Moreover $g$ is $C^r$ for all integers $r$ strictly smaller than $\frac{\ln \rho}{\ln 4}$, and, if $\alpha_k \in ]-1, +\infty[ $ for all integers $k$, then $g$ is strictly positive on $\Lambda$.

**Proof.** Let $G : \Sigma \to \mathbb{C}$ be the function given by Proposition 3.1. We next recall a construction due to Bowen [6], in order to check that it has some extra properties that suit us.

Let $(e_i)_{0 \leq i \leq 3}$ be the standard basis in $\mathbb{R}^4$. Set $R(k) = 0$ if $k \geq 0$ and $R(k) = 1$ if $k < 0$. Then, for $x = (x_i)_{i \in \mathbb{Z}} \in \Sigma$, define

$$I(x) = \sum_{k \in \mathbb{Z}} 4^{-|k|} e_{2x_k + R(k)}.$$

Then one easily checks that for $x, y \in \Sigma$ we have

$$\frac{5}{6}d_\Sigma (x, y) \leq d(I(x), I(y)) \leq \frac{8}{3}d_\Sigma (x, y),$$

where $d$ is the euclidean distance on $\mathbb{R}^4$. Thus $I$ induces a homeomorphism on its image $\Lambda$, which is a compact subset of $\mathbb{R}^4$. Define $V_i = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : 1 \leq x_2 \leq \frac{1}{2}, 0 \leq x_k \leq \frac{1}{2} \text{ for } k \neq 2i \}$ and $F_i = I(\{x \in \Sigma : x_0 = i\})$ for $i = 0, 1$. It is easy to check that $F_i$ is contained in $V_i$.

Define

$$L = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}.$$

For $x \in V_i$, set $G_i x = Lx - 4e_{2i} + \frac{1}{4}e_{2i+1}$ (for $i = 0, 1$). Then define $G$ on $V_0 \cup V_1$ by $G|_{V_i} = G_i$. One easily checks that $G \circ I = I \circ \sigma$. Viewing $\mathbb{R}^4$ as embedded in $S^4$, one may extends $G$ to a diffeomorphism $T$ of $S^4$, that coincides with $G_i$ on a neighbourhood $U_i$ of $V_i$ (see for instance [16]). Setting $U = U_1 \cup U_2$ one has $\bigcap_{k \in \mathbb{Z}} T^k(U) = \Lambda$. Thus $\Lambda$ is a basic set for $T$ with isolating neighbourhood $U$.

Now define $\tilde{g}$ on $F$ by $\tilde{g} = G \circ I^{-1}$. Let $r$ be an integer strictly smaller than $\frac{\ln \rho}{\ln 4}$. Choose $\theta \in ]\frac{1}{r}, 4^{-r}[$. Next, recalling (49) and that $G$ is Lipschitz for the distance $d_\beta$, there exists a constant $C$ such that for all $x, y \in \Lambda$ we have

$$m(x, y) = \frac{\ln (d_\beta^{-1}(I^{-1}(x), I^{-1}(y)))}{\ln 4} \geq \frac{\ln (\frac{5}{6} d(x, y))}{\ln 4},$$

and thus

$$\frac{\ln (d_\beta^{-1}(I^{-1}(x), I^{-1}(y)))}{\ln 4} \geq \frac{\ln (\frac{5}{6} d(x, y))}{\ln 4}.$$
Notice that for all \( n \in \mathbb{N}^* \) we have
\[
\frac{1}{|\det (I - L^n)|} = \frac{1}{16^n} \sum_{k=0}^{+\infty} (-1)^k \left( \frac{-4}{k} \right) \frac{1}{4^{nk}},
\]
and recall that \((-1)^k \left( \frac{-4}{k} \right) = \frac{(k+1)(k+2)(k+3)}{6}\) is an integer. Fubini’s theorem gives
\[
d_{T,g}(z) = \exp \left( \sum_{n=1}^{+\infty} \frac{1}{n} \sum_{x \in \Lambda} \frac{g^{(n)}(x)}{|\det (I - D_x T^n)|} z^n \right)
= \exp \left( \sum_{n=1}^{+\infty} \frac{1}{n \det (I - L^n)} \sum_{x \in \Lambda} g^{(n)}(x) z^n \right)
= \exp \left( \sum_{n=1}^{+\infty} \sum_{k=0}^{+\infty} (-1)^k \left( \frac{-4}{k} \right) \sum_{x \in \Lambda} g^{(n)}(x) \left( \frac{z}{4^{k+2}} \right)^n \right)
= \prod_{k=0}^{+\infty} \left( \zeta_{T,g} \left( z + \frac{1}{4^{k+2}} \right)^{-1} - \frac{1}{4^{k+2}} \right) \left( \frac{k+1)(k+2)(k+3)}{6} \right) .
\]

As an immediate consequence of Proposition 3.3, Lemma 1.11, Lemma 1.12 and Lemma 1.13, we get the three following corollaries.

**Corollary 3.4.** The counter-examples of Proposition 1.8 may be produced as dynamical determinants. Namely:

a) There are a \( C^\infty \) diffeomorphism \( T : S^4 \to \mathbb{R}^4 \), a hyperbolic basic set \( \Lambda \) for \( T \), and a \( C^\infty \) function \( g : S^4 \to \mathbb{R} \), strictly positive on \( \Lambda \), such that for any ordering \((\lambda_m)_{m \geq 0}\) of the resonances of \((T,g)\) (see Definition 1.1) we have for all \( n \geq 1 \) the trace formula
\[
(50) \quad \text{tr}^b \left( \mathcal{L}_g^n \right) = \sum_{T^n x = x} \frac{g^{(n)}(x)}{|\det (I - D_x T^n)|} = \sum_{m \geq 0} \lambda_m^n
\]
but the convergence of the right hand side is never absolute.

b) There are a \( C^\infty \) diffeomorphism \( T : S^4 \to \mathbb{R}^4 \), a hyperbolic basic set \( \Lambda \) for \( T \), a \( C^\infty \) function \( g : S^4 \to \mathbb{R} \), strictly positive on \( \Lambda \), an ordering \((\lambda_m)_{m \geq 0}\) of the resonances of \((T,g)\), and a permutation \( \sigma \) of \( \mathbb{N} \) such that \((\lambda_{\sigma(m)})_{m \geq 0}\) is an ordering of the resonances of \((T,g)\) and, for all \( n \geq 1 \), the trace formula (50) holds but the series \( \sum_{m \geq 0} \lambda_{\sigma(m)}^n \) does not converge.

**Corollary 3.5.** The dynamical determinant of a \( C^\infty \) diffeomorphism with \( C^\infty \) weight on a hyperbolic basic set may be of any (finite or infinite) genus\(^{14}\).

Recall that Theorem 1.4 gives a characterization of the genus of the dynamical determinant in terms of global trace formulae. Moreover from Corollary 3.5 and [12, Corollary 1 p.17, second part of the book], we deduce that there are dynamical determinants that are not Fredholm determinants of any nuclear operators (and so there is no "good" Banach space on which the associated transfer operators are nuclear).

**Corollary 3.6.** Let \( E \) be a subset of \( \mathbb{N}^* \). Then there are There are a \( C^\infty \) diffeomorphism \( T : S^4 \to \mathbb{R}^4 \), a hyperbolic basic set \( \Lambda \) for \( T \), and a \( C^\infty \) function \( g : S^4 \to \mathbb{R} \), strictly positive on \( \Lambda \), such that, for any ordering \((\lambda_m)_{m \geq 0}\) of the resonances of \((T,g)\), and for all \( n \in \mathbb{N}^* \), the series
\[
\sum_{m \geq 0} \lambda_m^n
\]
converges absolutely and its sum is \( \text{tr}^b \left( \mathcal{L}_g^n \right) \) if and only if \( n \in E \).

\(^{14}\)And even of any non-integral order according to footnote 4.
Roughly speaking, Corollary 3.6 asserts that global trace formula (4) may hold on any fixed subset of $\mathbb{N}^*$.

Remark 3.7. Writing $h(z) = \sum_{k=0}^{+\infty} a_k z^k$ if there are constants $C, c > 0$ and $\theta_0 \in [0, 1]$ such that for all $\theta \in [0, \theta_0]$ and $k \in \mathbb{N}$ we have
\[
|a_k| \leq C \theta^k - c \ln(|\theta|)
\]
then we can show that $g$ may be chosen Gevrey. However, the control of the growth of $d_{T,g}$ imposed by (51) is far too strong to prove that the bounds of Theorem 2.2 are sharp, and so we shall not provide a proof.

Remark 3.8. The weight $g$ produced by Corollaries 3.4, 3.5 or 3.6 being strictly positive on $\Lambda$, it is associated to some physically meaningful measure $\mu_g$ (the one appearing in (2), see [1, Chapter 7] for details). For example if $g = 1$ or $g = \|\det (DT)\|_g$ ($= 16$ in our case), $\mu_g$ is respectively the physical measure or the measure of maximal entropy for $T_A$ (for the $T$ we constructed these measures coincide).

It may be noticed that the weights produced by Corollaries 3.4, 3.5 and 3.6 may be taken arbitrarily close to 1 in the $C^\infty$ topology on a neighbourhood of $\Lambda$. The proof of this relies on the fact that, according to Lemmas 1.11 and 1.13, $h$ may be taken arbitrarily close to 0 in the topology of the uniform convergence on all compact subsets of $\mathbb{C}$ (but, to actually prove it, an investigation of a proof of Whitney’s extension theorem is needed).

Remark 3.9. Proposition 3.3 realises a lot of entire functions as inverses of dynamical zeta functions, thus we could have stated many variations on Corollaries 3.4, 3.5 and 3.6. For instance, one may construct weight $g$ for which the trace formula (50) always holds but the convergence is absolute only when $n$ is bigger than some fixed integer (replace $\frac{1}{\ln \ln n}$ in the expression of $a_n$ in the proof of Proposition 1.8 by $\frac{1}{\ln m}$ for some $a > 0$ and then state analogues of Lemma 1.11 and Lemma 1.12).

Remark 3.10. If in Proposition 3.3 we take $h(z) = h_{a,\rho}(z) = a \ln \left( 1 + \frac{z}{\rho} \right)$, where $\rho > 1$ and $a > 0$ is small, then we get weights $g = g_{a,\rho}$ strictly positive on $\Lambda$. From formulae (47) and (48), we know that the radius of convergence of $d_{T,g_{a,\rho}}$ is exactly $\rho_{eff} = 16\rho$. Let $r > 2$ be an integer, and choose $\rho$ such that $r < \frac{\ln \rho}{\ln 16}$, then if $[3, 1.5]$ predicted a radius of convergence greater than $\rho_{pred} = \exp \left( -P_{top} (\log g_{a,\rho} - \log 16) \right)$ for $d_{T,g_{a,\rho}}$. However since $g$ is strictly positive, [1, Theorem 6.2] and [1, Theorem 7.5] imply that $\exp \left( -P_{top} (\log g_{a,\rho} - \log 16) \right)$ is the smallest zero of $d_{T,g_{a,\rho}}$, which can be made arbitrary close to $\frac{1}{\rho}$ by taking $a$ close enough to 0. On the other hand, we may chose $\rho$ arbitrary close to $4^r$. Thus, for all $\epsilon > 0$, there is a choice of $a$ and $\rho$ such that
\[
\frac{\rho_{eff}}{\rho_{pred}} \leq 2048 + \epsilon.
\]
This means that [3, 1.5] described accurately the way the radius of convergence of the dynamical determinant grows when the regularity of the weight grows (up to a bounded multiplicative constant that could be made smaller than 2048 by playing on the parameter of the construction of Proposition 3.3).

References


\*The dynamical determinant $d_{T,g_{a,\rho}}$ cannot even be continued meromorphically outside the disc of center 0 and radius $16\rho$.\[5]


