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Dynamical Zeta Functions and Dynamical Determinants for Hyperbolic Maps

– A Functional Approach –

Springer

Preface

On July 9, 2001, Springer invited me to contribute a monograph on dynamical systems. I immediately accepted, offering to write a book on dynamical zeta functions and dynamical determinants: Dynamical determinants are functions defined from weighted periodic orbit data of a *differentiable* dynamical system. The zeroes of these functions describe a large part of the spectrum of an associated transfer operator. In other words they play the role of Fredholm determinants for the (usually not compact) transfer operator. This spectrum contains key information on the statistical properties of the dynamics. Dynamical zeta functions[1] can often be written as an alternated product of dynamical determinants, and their poles hence describe a usually smaller part of the spectrum of the transfer operator. Deep important results on dynamical zeta functions and dynamical determinants were obtained in the 80's and 90's by Ruelle and Pollicott, and the corresponding spectra are often called Ruelle–Pollicott resonances.

In 2001, in spite of the existence in preprint form of the pioneering work of Blank–Keller–Liverani [37], I could not foresee the major changes that would occur in the theory in the next decade. What Blank–Keller–Liverani did, for an Anosov diffeomorphism $T : M \to M$, was to construct a Banach space of *anisotropic* distributions on M on which the transfer operator

$$\mathscr{L}\varphi = \frac{\varphi \circ T^{-1}}{|\det DT \circ T^{-1}|}, \tag{0.1}$$

(defined initially for $\varphi \in L^\infty(M)$) had a spectral gap. This was the first time that a spectral gap was obtained for a transfer operator of a hyperbolic map without using symbolic dynamics. Since Markov partitions and the passage from an invertible to an expanding (one-sided) symbolic dynamics cause a great loss of information (starting from a C^r diffeomorphism, one is reduced to a one-sided Hölder shift), the work of Blank–Keller–Liverani was an important stepping-stone in the theory of

[1] The 1990 book by Parry and Pollicott [131] is the primary reference on dynamical zeta functions for hyperbolic dynamics, but it does not cover dynamical determinants, the symbolic dynamics approach used there being unsuitable for this purpose.

smooth chaotic dynamics. We call this symbolic dynamics-free spectral approach, using anisotropic spaces of distributions, the *functional approach.*

Between 2005 and 2008, seven papers[2] by various authors (Liverani, Tsujii, Gouëzel, myself) appeared ([15, 119, 120, 87, 88, 28, 31]), generalising and enhancing the results of Blank–Keller–Liverani, and connecting the spectral data to zeroes of dynamical determinants

$$d_{T^{-1},g}(z) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{T^m(x)=x} \frac{\prod_{k=0}^{m-1} g(T^k(x))}{|\det(\mathrm{Id} - DT^{-m}(x))|},$$

for hyperbolic diffeomorphisms T, and $g = |\det DT^{-1}|$, or more general weights.

There are two basic kinds of anisotropic Banach spaces (see [17]): Liverani and co-workers use a *geometric definition* [119, 120, 87, 88], based on cones in the tangent space. The approach of [28, 31] uses cones in the co-tangent space via the Fourier transform, in the spirit of Sobolev spaces, and it can be called *microlocal.* This microlocal approach has been adopted and developed by the semi-classical community, defining anisotropic Hilbert spaces for C^∞ dynamics. (We only mention here the discrete-time results of Faure–Roy [67], Faure–Roy–Sjöstrand [68], referring to Gouëzel [86] and Zworski [184, §4] and references therein for flows.)

Over the past few years, a clear picture of the functional approach has gradually emerged. The time seemed finally ripe for a book. So I was delighted when Ute Motz contacted me again in 2009 to nudge me gently into getting my act together. A substantial amount of time in the preparation of this book has then been spent over agonising decisions on what to include: I have decided to leave out completely both *continuous-time* dynamics and *piecewise smooth* dynamics. The goal of this book, intended for researchers and graduate students in dynamical systems, is thus to give a self-contained and (hopefully) reader-friendly account of the "microlocal" version [28, 31] of the functional approach, in the setting of finitely differentiable *hyperbolic diffeomorphisms.* In order to allow easier entry in the topic, we present the arguments first in the much easier toy model of differentiable (locally) *expanding maps.* One of the features of this book is that no knowledge[3] of microlocal analysis or pseudodifferential operator theory is requested. We hope that this will demystify the construction and the use of anisotropic spaces, as well as demonstrate that pedestrian (dyadic decomposition) techniques in Fourier space give very precise results. The book also aims to show the role played by the kneading determinant ideas of Milnor and Thurston [123] in this theory.

I would like to express here my deep thanks to David Ruelle, Gerhard Keller, Predrag Cvitanoviç, Mark Pollicott, Hans Henrik Rugh, David Fried, Carlangelo Liverani, Sébastien Gouëzel, Masato Tsujii, Frédéric Naud, Stéphane Nonnenmacher, Maciej Zworski, and Semyon Dyatlov (in chronological order) for many enlightening conversations over the years. I thank Dmitry Todorov, for questions on Sec-

[2] A previous important paper by Kitaev [112] was devoted to the dynamical determinant, with no spectral interpretation.

[3] See Section 1.4 for the very short list of black boxes that we use.

tions 2.2–2.4 of a preliminary draft and Alexander Adam who found several typos in preliminary versions. I am much indebted to Malo Jézéquel for a very sharp critical reading of the September 2016 version. I also thank the four reviewers of the September 2016 manuscript for constructive remarks.

I am also grateful to the audiences of several minicourses for their questions: In Rio de Janeiro (IMPA) in 2007, in Vienna in 2008 (Erwin Schrödinger Institute, semester on Hyperbolic Dynamics), in Porto in 2010, in Paris (UPMC) and Pisa in 2016. A short stay at the Newton Institute in the Fall of 2013 helped me go back to the book project after two difficult years. The most crucial phases of the writing were achieved during the Summers of 2010, 2011, 2014, 2015, 2016, and 2017 in the Centre for Mathematical Sciences, Lund University, which provided an office — and free access to the printer for several thousands of pages. I am also very grateful to the Mittag-Leffler Institute in Djursholm, where the book was finally completed in the Fall of 2017 during the Research Program on Fractal Geometry and Dynamics.

Last, but not least, I thank Tomas Persson, for reading and commenting on several drafts of this book, and much more.

Djursholm, December 3, 2017 *Viviane Baladi*

Contents

Index of notations

$[\alpha]$	The largest integer $\leq \alpha$, for $\alpha \in \mathbb{R}$.
$\mathbf{1}_X$	The indicator function of a set X.
b^{Op}	Pseudodifferential operator for the symbol $b : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, see (D.3).
$[\mathscr{B}_1, \mathscr{B}_2]_\theta$	Complex interpolation at $\theta \in [0,1]$ between Banach spaces $\mathscr{B}_1$ and $\mathscr{B}_2$. §2.3.1.
$\mathscr{B}^{t,s}$	Anisotropic Banach space with real regularity indices $s < 0 < t$, Definition 5.15. Noted $\mathscr{B}^{t,s}(V)$ or $\mathscr{B}^{t,s}(T,V)$ when associated with a hyperbolic map. §5.1.1.
C^r	For an arbitrary real number $r \geq 0$: The space of $C^{[r]}$ functions with a $C^{[r]-r}$-Hölder $[r]$-th derivative, with the corresponding norm. See also Remark 2.19.
$C^r(K)$	For a compact subset K of $\mathbb{R}^d$ or compact Riemann manifold, and an arbitrary real number $r \geq 0$: Those elements of C^r which are supported in K.
C^r_*	If $r > 0$ is not an integer, $C^r_* = C^r$. If $r \geq 0$ is an integer, the corresponding Zygmund space (2.80). See also Remark 2.19.
$C^r_0(\mathbb{R}^d)$	Compactly supported functions in $C^r(\mathbb{R}^d)$.
$C^{t,s}_*(\mathbb{R}^d)$	Anisotropic Banach space with real regularity indices $s < 0 < t$ from [28]. Noted $C^{t,s}_*(T,V)$ when associated with a hyperbolic map T. Definition 4.23.
$\mathbf{C}$	A cone in $\mathbb{R}^d$. Section 4.2.1. $\mathbf{C} \Subset \mathbf{C}'$ means $\overline{\mathbf{C}} \subset \text{interior}(\mathbf{C}') \cup \{0\}$.
$\chi^\pm_\mu(A,T)$	The smallest (χ^-), respectively largest (χ^+), Lyapunov exponent of the cocycle A over the dynamical system (T,μ). Also denoted $\chi^\pm_\mu(A)$.
$\mathscr{D}(z)$	The kneading operator associated with a dynamical system and a weight on a Banach space, (3.10) and (3.38) for expanding dynamics, and (6.8) for hyperbolic dynamics.
$d_{T,g}(z)$	(Fredholm–Ruelle) dynamical determinant of a map T and a weight g. Section 3.1 for expanding T. Section 6.1 for hyperbolic T.
$d^{(\ell)}_{T,g}(z)$	(Fredholm–Ruelle) dynamical determinant of a map T and a weight g, on ℓ-forms. §3.1.1 for expanding T. Section 6.4 for hyperbolic T.

$\det^\flat$	The flat determinant of a transfer operator. Section 3.3 for expanding maps. Section 6.1 in the hyperbolic case.
det	The determinant of a nuclear operator. Appendix A.
$\mathrm{Erg}(T)$	The ergodic T-invariant (Borel) probability measures. Appendix B.
$\mathbb{F}$	Continuous Fourier transform on $\mathbb{R}^d$. See (2.12).
$h_\mu(T)$	The Kolmogorov entropy of the T-invariant probability measure μ.
H_p^t	Isotropic Sobolev space with integrability index $1 < p < \infty$ and regularity index $t \in \mathbb{R}$, (2.11).
$H_p^{t,v}$	Anisotropic Triebel–Sobolev space with integrability index $1 < p < \infty$ and real regularity indices $v < -t < 0$. See (4.5). Used in [15, 20, 21].
$L_p(\mu)$	Space of functions φ with $\lvert\varphi\rvert^p$ integrable with respect to a a probability measure μ, for $1 \le p < \infty$. If $p = \infty$, space of μ-a.e. bounded functions. (As usual, we consider in fact equivalence classes, where two measurable functions are equivalent if they are equal μ-a.e.)
$L_p(X,\mu)$	Space of functions φ with $\lvert\varphi\rvert^p$ Lebesgue integrable, for $1 \le p < \infty$, where $X = \mathbb{R}^d$ or $X = M$ a compact manifold. If $p = \infty$, space of Lebesgue a.e. bounded functions. (As usual, we consider in fact equivalence classes.).
$\mathscr{L}_g$	Transfer operator for a map T and a weight g. See (2.5) in Section 2.1 for expanding T. See (4.2) in Section 4.1 for hyperbolic T.
$\mathscr{L}_g^{(\ell)}$	Transfer operator for a map T and a weight g, acting on ℓ-forms. Section 3.1.1 for expanding T. Section 6.4 for hyperbolic T.
$\lambda_{\overrightarrow{\omega}}^{(t,m)}$	The t-weighted hyperbolicity index of the inverse branches of an iterated expanding map T^m (2.55).
$\lambda^{(t,s,m)}(x)$	The (t,s)-weighted hyperbolicity index of an iterated hyperbolic map T^m, (4.11), uses the contracting and expanding indices $\lambda_x(T^m)$ and $\nu_x(T^m)$ from (4.9).
$P_{top}(\phi)$	The topological pressure of a weight function ϕ (for a map T). Appendix B.
$Q^{t,s,p}(g)$	Bound for the essential spectral radius of $\mathscr{L}_g$ on $\mathscr{B}_p^{t,s}$ in the hyperbolic case. Theorem 5.1.
$r_{ess}(L\vert_{\mathscr{B}})$	The essential spectral of a bounded linear operator L on a Banach space $\mathscr{B}$. Definition A.1.
ψ_n	For the isotropic Paley–Littlewood (dyadic) decomposition, (2.70).
$\psi_{\Theta,n,\pm}$	For the anisotropic Paley–Littlewood (dyadic) decomposition, (4.25).
$R_*^{t,p}(g)$	Bound for the essential spectral radius on $H_p^t(M)$ in the expanding case. Theorem 2.15.
$R^{t,s,p}(g)$	Bound for the essential spectral radius of $\mathscr{L}_g$ on $W_{p,*}^{t,s}$ or $W_{p,**}^{t,s}$ in the hyperbolic case. Theorem 4.6.
$\mathscr{S}$	The set of rapidly decreasing functions on $\mathbb{R}^d$. Appendix C.
$\mathscr{S}'$	The set of temperate distributions on $\mathbb{R}^d$. Appendix C.
S^m	The class $S^m = S^m_{1,0}$ of C^∞ symbols on $\mathbb{R}^d \times \mathbb{R}^d$. Definition D.1.
$\mathrm{sp}(\mathscr{M}\vert_{\mathscr{B}})$	The spectrum of a bounded linear operator $\mathscr{M}$ on a Banach space $\mathscr{B}$. Appendix A.1.
$\mathbb{S}^{d-1}$	The unit sphere in $\mathbb{R}^d$.

$\mathbb{T}^d$	The d-dimensional torus.
Θ	Cone systems $\Theta = (\mathbf{C}_+, \mathbf{C}_-, \varphi_+, \varphi_-)$ are formed of two cones and two smooth functions on $\mathbb{S}^{d-1}$, with $\varphi_+ + \varphi_- = 1$. Definition 4.10. Called polarizations in [28] and [31]. $\Theta < \Theta'$ is defined in (4.12), it implies $\mathbf{C}_+ \Subset \mathbf{C}'_+$ and $\mathbf{C}'_- \Subset \mathbf{C}_-$.
$\mathrm{tr}^\flat$	The flat trace of an operator. Section 3.3 for expanding maps. Section 6.1 for hyperbolic maps.
tr	The trace of a nuclear operator. Appendix A.
A^{tr}	The transposition of a finite-dimensional matrix A.
$W^{t,s}_{p,*}$	Anisotropic Banach space with integrability index $1 < p < \infty$ and real regularity indices $s < 0 < t$. Noted $W^{t,s,p}_{\dagger}$ and $W^{t,s,p}_{*}$ in [28]. Noted $W^{t,s}_{p,*}(T,V)$ when associated with a hyperbolic map T. Sections 4.2 and 4.3.
$W^{t,s}_{p,**}$	A variant of the anisotropic Banach space $W^{t,s}_{p,*}$. Section 4.2. Noted $W^{t,s,p}_{\dagger\dagger}$ in [28].
$\zeta_{T,g}(z)$	Dynamical (Ruelle) zeta function of a map T and a weight g. Section 3.1 for expanding T. Section A.3 for hyperbolic T.

Chapter 1
Introduction

Abstract Transfer operators associated with a dynamical system T and a weight g are important tools to understand the statistical properties of T, under appropriate smoothness and hyperbolicity conditions. Dynamical determinants are associated with the periodic orbits of T weighted by g. We define the spectral and determinantal resonances of the pair (T,g) and state the main results of the book, linking these resonances and establishing their properties. We briefly discuss the anisotropic spaces and the techniques used to prove these results. We illustrate these results by simple examples.

1.1 Statistical properties of chaotic differentiable dynamical systems.

The roots of the modern theory of chaotic dynamical systems reach back to Poincaré in the nineteen-hundreds. Chaos here means essentially sensitive dependence on initial conditions, illustrated by the famous metaphor of the Brazilian butterfly whose fluttering wings may induce a Texan tornado weeks, or maybe even years later. It turns out that chaotic dynamical systems, although by essence unpredictable, admit a very good statistical description, in the sense that it is often possible to describe the average *asymptotic* — that is, long-term — behaviour of *almost all initial conditions.* This first section is a very brief introduction to the topics of this book. To make this "introduction to the introduction" easier to read, we postpone precise statements and most of the references to the literature to later sections of the book.

In this book, we focus on dynamics taking place on a finite dimensional differentiable manifold M, assuming that M is compact. More precisely, we consider *discrete-time dynamical systems* represented by the iterates T^n of a transformation $T : M \to M$ with

$$T^n = \overbrace{T \circ \cdots \circ T}^{n \text{ times}}, \qquad n \in \mathbb{Z}^+,$$

viewing $T^n(x)$ as the *state at time* $n \in \mathbb{Z}^+$ of our system, starting from an *initial condition* $x \in M$. (If T is invertible, we can also consider negative n, setting $T^{-|n|} = (T^{-1})^{|n|}$.)

We shall make assumptions guaranteeing that T is chaotic, in the sense that arbitrarily close distinct initial points may become separated if one waits long enough. This already mentioned "sensitive dependence on initial conditions" often takes place at exponential speed, as in the elementary paradigm of the "angle-doubling" map $w \mapsto w^2$ on the circle $M = \{w \in \mathbb{C} \mid |w| = 1\}$, where the distance between $T^n(x)$ and $T^n(y)$ is 2^n times the distance between x and y if they are close enough (depending on n).

Our aim is to describe the long-time behaviour of generic initial conditions. Since generic is understood in a measure-theoretical sense, this task of *statistically describing* the asymptotics of "most" initial data is not rendered completely hopeless by the sensitive dependence property. Since M is a compact Riemann manifold, we have a natural reference probability measure on M: the Lebesgue measure dx. We therefore seek to understand invariant probability measures μ (i.e., $\mu(T^{-1}(E)) = \mu(E)$ for every Borel set) which are somehow related to the a priori measure dx.

Recall that [178] the fundamental *Birkhoff theorem* says that if μ is a T-invariant Borel probability measure such that (T,μ) is *ergodic* (i.e., "indecomposable" in the sense that $E = T^{-1}E$ only if $\mu(E) = 0$ or 1), then, for each "test" function (also called "observable") $\varphi \in L^1(d\mu)$ and for μ-almost all $x \in M$, the "time average converges to the space average:"

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(T^k(x)) = \int \varphi \, d\mu \,. \tag{1.1}$$

Of course, μ can be supported on a set of zero Lebesgue measure (the simplest example being a periodic orbit $\{T^k(x) \mid 0 \le k \le N-1\}$, with $T^N(x) = x$ for some finite $N \ge 1$). In this case "μ-almost everywhere" is not very meaningful... This motivates the following definition: Suppressing the observable, let δ_y denote the Dirac mass at y. The *ergodic basin* of a T-invariant probability measure μ is defined to be

$$\{x \in M \mid \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^k(x)} = \mu\} \tag{1.2}$$

(where convergence is understood in the weak sense). An ergodic T-invariant probability measure μ having an ergodic basin of *nonzero Lebesgue measure* is called a *natural* or *physical measure.* The simplest situation is when μ is both ergodic and absolutely continuous with respect to Lebesgue. This case is discussed at length in Chapter 2, when studying the (noninvertible) locally expanding maps of Part I. In the case of (invertible) hyperbolic maps of Part II, physical measures exist, but they are generically singular with respect to Lebesgue. In the settings considered in this book, physical measures coincide with the *SRB measures,* after the names of Sinai, Ruelle, and Bowen. We refer to Section 7.1 for a discussion of SRB measures in the hyperbolic case.

We shall discuss in this book dynamical systems possessing a single, or finitely many, physical measure(s) the (union of the) ergodic basin(s) of which have full Lebesgue measure in M. For an ergodic physical measure, one would like to quantify the spatial nature of the convergence (1.1). For a real-valued observable, assuming that $\int \varphi \, d\mu = 0$ (replacing if necessary φ by $\varphi - \int \varphi \, d\mu$), one says that the *central limit theorem* (CLT) holds for (T, μ) and φ if the "random variable"

$$x \mapsto \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ T^n(x)$$

(where x is distributed with law μ) converges in law to a normal distribution $\mathscr{N}(0, \sigma)$ for some $0 < \sigma < \infty$. In probability theory the central limit theorem holds, e.g., for independent identically distributed random variables [62], but also under weaker "independence" assumptions. Of course, a deterministic dynamical system does not behave like an i.i.d. random variable! Nevertheless "weak independence" assumptions involving the *correlation functions* or *rates of mixing* are sufficient for the CLT, as we explain next.

A statistical property which implies ergodicity is *mixing,* a kind of asymptotic independence: (T, μ) is mixing if for every pair of observables $\varphi, \psi \in L^2(d\mu)$

$$\lim_{n \to \infty} \int (\varphi \circ T^n) \, \psi \, d\mu = \int \varphi \, d\mu \int \psi \, d\mu \, . \tag{1.3}$$

If the observables are characteristic functions χ_A and χ_B of Borel sets $A, B \subset M$, this means

$$\lim_{n \to \infty} \mu(T^{-n} A \cap B) = \mu(A)\mu(B) \, , \tag{1.4}$$

which is indeed the classical notion of (asymptotic) independence. Often, (1.3) is equivalent to a more natural (and often more convenient) formulation of mixing (using dx for Lebesgue measure again):

$$\lim_{n \to \infty} \int (\varphi \circ T^n) \, \psi \, \mathrm{d}x = \int \varphi \, d\mu \int \psi \, \mathrm{d}x \, . \tag{1.5}$$

The left-hand side is now a well-defined expression even if μ is unknown. In particular, taking $\psi \equiv 1$ we find that (1.5) corresponds to *convergence to equilibrium:*

$$\lim_{n \to \infty} \int (\varphi \circ T^n) \, \mathrm{d}x = \int \varphi \, d\mu \, . \tag{1.6}$$

The expression

$$C_{\varphi,\psi}(n) = \int (\varphi \circ T^n) \, \psi \, d\mu - \int \varphi \, d\mu \int \psi \, d\mu \, , \tag{1.7}$$

viewed as a function of $n \in \mathbb{Z}_+$ is called the *correlation function* of (T, μ) for the observables φ and ψ. The correlation function measures the loss of memory or asymptotic independence. By a slight abuse of language, the difference

$$C^{(e)}_{\varphi,\psi}(n) = \int (\varphi \circ T^n)\, \psi \, \mathrm{d}x - \int \varphi \, d\mu \int \psi \, \mathrm{d}x \tag{1.8}$$

may also be called (empirical) correlation function.

Mixing alone is in general not enough to imply the CLT: We need to study the *speed* at which the convergence (1.3) (or (1.5)) takes place. This is one of the motivations to study the rate of decay of correlations functions. For this, it is in general necessary to restrict the test functions to a functional subspace $\mathscr{B}_f$ of $L^2(d\mu)$. In many cases, $C^{(e)}_{\varphi,\psi}(n) \sim C_{\varphi,\psi}(n)$ when $n \to \infty$ if the observables belong to such an appropriate function space. The rule of thumb is that if it is possible to find a space $\mathscr{B}_f$ of observables such that $|C_{\varphi,\psi}(n)| \le C(n)$ for all $n \in \mathbb{Z}_+$ and all $\varphi, \psi \in \mathscr{B}_f$, then the central limit theorem holds for observables in $\mathscr{B}_f$ *if the speed of mixing* $C(n)$ is summable, i.e., if $\sum_n C(n) < \infty$. See [116] for precise statements, their proofs, and references (in particular to previous work of Gordin, as well as an important martingale approximation theorem by Kipnis and Varadhan). The dynamical systems studied in this book enjoy exponential decay of correlations and thus summable speed of mixing.

If the correlation function $C_{\varphi,\psi}(n)$ decays exponentially with n for φ and ψ, it is natural to consider its (a priori formal) Fourier transform

$$\widehat{C}_{\varphi,\psi}(\eta) = \sum_{n \in \mathbb{Z}} e^{in\eta} C_{\varphi,\psi}(n) \tag{1.9}$$

(setting $C_{\varphi,\psi}(n) = C_{\psi,\varphi}(-n)$ for negative n if T is not invertible). Exponential decay (with a uniform rate for φ and ψ in some function class) would imply that $\widehat{C}_{\varphi,\psi}(\eta)$ is analytic in a strip $\{|\Im \eta| < L'\}$, with $L' > 0$ independent of φ and ψ in this function class. In some cases $\widehat{C}_{\varphi,\psi}(\eta)$ admits an extension meromorphic in a larger strip $\{|\Im(\eta)| < L\}$ where the possible location of its poles (the *correlation resonances*) only depend on the dynamical system (T, μ) (and not on the observables). These resonances are often described by the set of poles of a dynamical zeta function, or the set of the inverse of the zeroes of a dynamical Ruelle–Fredholm determinant $d(z)$ (*determinantal resonances*), as we shall explain in this book. The main tool to prove this relationship is a linear operator $\mathscr{L}$ associated to the dynamics, called the (Ruelle) transfer operator, whose dual preserves Lebesgue measure, i.e.

$$\int \mathscr{L}(\varphi)\, \mathrm{d}x = \int \varphi \, \mathrm{d}x \,.$$

In nontrivial cases, the operator $\mathscr{L}$ is not compact on any Banach or Hilbert space $\mathscr{B}$ (of functions or distributions on M) which is large enough to contain all smooth functions. However, $\mathscr{L}$ can often be proved to be quasicompact on suitably chosen $\mathscr{B}$, in the following sense: The operator $\mathscr{L}$ is bounded on $\mathscr{B}$, with spectral radius

$\rho(\mathscr{L},\mathscr{B}) = 1$, and there exists a real number $r_{ess} = r_{ess}(\mathscr{L},\mathscr{B})$, strictly smaller than 1, such that the intersection of the spectrum of $\mathscr{L}$ on $\mathscr{B}$ with the annulus $\{z \in \mathbb{C} \mid r_{ess} < |z| \le 1\}$ consists of isolated eigenvalues of finite multiplicities. We call these eigenvalues the *spectral resonances.* In the cases studied in this book, the fixed point(s) of $\mathscr{L}$ correspond to the physical measures while the spectral resonances can be related to the above mentioned correlation resonances. It will turn out that for noninvertible expanding maps one can simply choose $\mathscr{B} = \mathscr{B}_f$ to be a space of functions with a suitable modulus of continuity, while in the case of invertible hyperbolic diffeomorphisms, one must work with spaces $\mathscr{B}$ of *anisotropic distributions* (containing strictly $\mathscr{B}_f$). One of the purposes of this book is to give a detailed and readable account of the construction of these anisotropic Banach spaces.

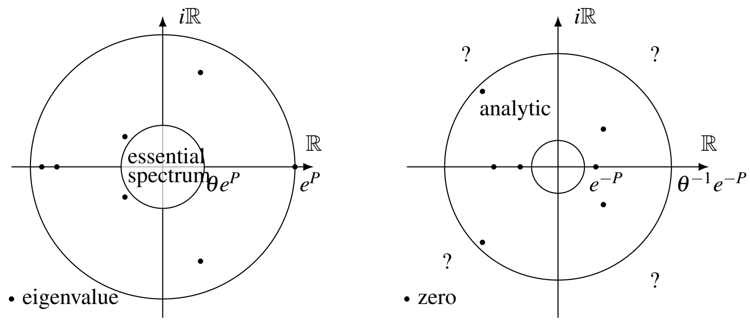

Fig. 1.1 Typical spectrum of the transfer operator. Typical extension and zeroes of the dynamical determinant.

One of the questions which immediately come to mind is: Do the resonances depend on the choices of $\mathscr{B}_f$ and $\mathscr{B}$? One of the answers given by this book is that, although the transfer operator $\mathscr{L}$ is not even compact on $\mathscr{B}$, in many cases a formal trace can be associated to $\mathscr{L}$ and its iterates $\mathscr{L}^n$. These traces are obtained by summing suitable weights (depending only on the dynamics T) over the fixed points of T^n. In particular, they do not depend on the choice of any space of functions of distributions. A *(Ruelle–Fredholm) determinant* $d(z)$ can then be associated to $\mathscr{L}$ by using these traces (just like for a finite matrix $\mathscr{Q}$, where $\det(\mathrm{Id} - z\mathscr{Q}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{tr}\,(\mathscr{Q}^m)$). Another main purpose of this book is to give a presentation of the proof of the fact that that $d(z)$ admits an analytic extension to some disc where the inverses of its zeroes (the *determinantal resonances*) are in bijection with the eigenvalues of $\mathscr{L}$. (See Figure 1.1.) Since $d(z)$ only depends on T, this gives a negative answer to the question asked in the first sentence of this paragraph. (We shall also express the determinant $d(z)$ as an alternated product of zeta functions, the poles of which are related to the resonances.)

The discussion above implies that, in good cases, the SRB measure μ is associated to an isolated eigenvalue of finite multiplicity of a bounded operator. It is natural to try and exploit perturbation theory of discrete eigenvectors to study how the SRB measure μ_T of a dynamical system T depends on T. This can be done, and it leads to *linear response* theorems and linear response formulas, which are also presented in this book.

We finish this introductory section by mentioning that suitably weighted transfer operators $\mathscr{L}_g$ can be used to study other interesting T-invariant probability measures, such as the measure of maximal entropy or other equilibrium (Gibbs) states. We refer in particular to Chapter 7 which contains hitherto unpublished proofs (of previously known results). Note that the corresponding determinants $d_g(z)$ (and thus the resonances) depend then on both T and g.

1.2 Transfer operators. Dynamical determinants. Resonances

In this book we consider two kind of dynamical systems on a C^∞ compact connected Riemann manifold M: Locally expanding C^r maps $T : M \to M$ and hyperbolic C^r diffeomorphisms $\tilde{T} : \tilde{V} \to \tilde{T}(\tilde{V})$, where[1] $r > 1$, and $\tilde{V} \subset M$ is an (open) isolating neighbourhood for a compact invariant hyperbolic set Λ for T. (We refer to Chapters 2 and 4, respectively, for formal definitions.) The main setting of interest is the hyperbolic one, we view expanding systems as a toy model in which the ideas of the constructions are easier to present.

Let $\mathrm{d}x$ denote Lebesgue measure on M, and let $L_p(M)$, for $1 \le p \le \infty$, denote the classical spaces of functions on M with $\|\varphi\|_{L_p} = (\int_M |\varphi|^p \,\mathrm{d}x)^{1/p}$. The Koopman operators of T and $\tilde{T}$ are the pullback operators defined respectively by

$$T^*(\varphi) = \varphi \circ T \,, \quad \tilde{T}^*(\varphi) = h(\varphi \circ \tilde{T}) \,, \quad \varphi \in L_p(M) \,,$$

usually for $p = 2$, but we can take any $1 \le p \le \infty$, where the cutoff function h is smooth, supported in $\tilde{V}$ and $\equiv 1$ in Λ.

The (Ruelle) transfer operators are defined by

$$\mathscr{L}_g(\varphi)(x) = \sum_{y:T(y)=x} g(y)\varphi(y) \,, \quad \varphi \in L_\infty(M) \,,$$

for a C^α function $g : M \to \mathbb{C}$, with $\alpha \in (0, r]$ in the locally expanding case, and by

$$\tilde{\mathscr{L}}_{\tilde{g}}(\varphi)(x) = (\tilde{g}\varphi)(\tilde{T}^{-1}(x)) \,, \quad \varphi \in L_\infty(M) \,,$$

for a C^{r-1} function $\tilde{g} : M \to \mathbb{C}$ supported in $\tilde{V}$, in the hyperbolic case. For the sake of comparison with the results of [28, 31], we shall mostly use in fact the diffeomorphism $T = \tilde{T}^{-1} : V \to T(V)$, with $V = \tilde{T}(\tilde{V})$, and the weight $g = \tilde{g} \circ \tilde{T}^{-1}$ on V, with $g \equiv 0$ outside of V, writing

$$\mathscr{L}_g(\varphi)(x) = g(x)\varphi(T(x)) \,.$$

[1] We exclude the easier case where T is analytic.

In both cases, the data will thus be a pair denoted (T,g).

The central object of this book is the transfer operator $\mathscr{L}_g$ and its spectrum (the Koopman operator will only appear in Chapter 7) associated with a pair (T,g). The operator $\mathscr{L}_g$ is bounded on $L_\infty(M)$, or on any space $L_p(M)$ for $1 \le p \le \infty$, but its spectrum there is not very interesting. We view as "interesting" a spectrum which is as close as possible to the spectrum of a compact operator: More precisely, for a given pair (T,g), we seek Banach spaces $\mathscr{B}$ of distributions on M (i.e. $\mathscr{B} \subset (C^\infty(M))^*$) on which the transfer operator is bounded with essential[2] spectral radius r_{ess} as small as possible. If the essential spectral radius is strictly smaller than the spectral radius, then part of the spectrum of the operator is discrete (where by discrete spectrum we mean isolated eigenvalues of finite multiplicity). In order to exclude trivial cases, we add the constraint that $\mathscr{B}$ should contain all C^{r-1} functions on M. In view of Lemma A.3, it is advantageous to require that C^{r-1} functions are dense in $\mathscr{B}$. This motivates the following definition:

Definition 1.1 (Spectral (Ruelle) resonances). Define $r_{inf} := \inf_{\mathscr{B}} r_{ess}(\mathscr{L}|_{\mathscr{B}}) \ge 0$, where the infimum is over all Banach spaces $\mathscr{B}$ so that[3]

$$\mathscr{L}_g : \mathscr{B} \to \mathscr{B} \text{ is bounded, and } C^{r-1}(M) \subset \mathscr{B} \subset (C^\infty(M))^* \, , \; \overline{C^{r-1}(M)} = \mathscr{B} \, . \quad (1.10)$$

(The closure in the last identity is for the norm of $\mathscr{B}$, i.e. we require $C^{r-1}(M)$ to be dense in $\mathscr{B}$.) The spectral resonances of the pair (T,g) are the complex numbers γ with $|\gamma| > r_{inf}$ so that there exists a Banach space $\mathscr{B}$ satisfying (1.10) with $r_{ess}(\mathscr{L}|_{\mathscr{B}}) < |\gamma|$, and γ is in the spectrum of $\mathscr{L}_g$ on $\mathscr{B}$. (It is thus an isolated eigenvalue of finite multiplicity of $\mathscr{L}_g$ on $\mathscr{B}$.)

In the expanding setting, the Banach spaces $\mathscr{B}$ giving small essential spectral radius will be spaces of functions. This is because the transfer operator is associated with the inverse branches, which are contractions, and "contractions improve regularity," in the following sense: The C^r norm of a function $\varphi \circ F$ composed with a contraction F is strictly smaller than the C^r norm of φ "modulo a compact perturbation." This will be explained in detail in Chapter 2, where we recall the work of Ruelle [141] and Gundlach–Latushkin [91] for the Banach space associated with the C^t norm before proving new bounds on Sobolev spaces $H^t_p(M)$. (See also the previous book [14].) In the hyperbolic case, coexistence of attracting and contracting directions will force us to work with (anisotropic) distributions. The construction will be carried out in Chapters 4–6. Indeed, the construction of anisotropic spaces is the core technical part of this book. In Chapter 4 we shall present the scales $W^{t,s}_{p,*}(T,V)$ and $W^{t,s}_{p,**}(T,V)$ from [28] and in Chapter 5 the scale $\mathscr{B}^{t,s}(T,V)$ from [31]. (Note that the scales we construct satisfy $\mathscr{B} \subset (C^{r-1}(M))^* \subset (C^\infty(M))^*$.)

[2] The infimum over those positive real numbers ρ so that the spectrum of the operator outside of the disc of radius ρ consists of isolated eigenvalues of finite multiplicity. Appendix A.1.

[3] All inclusions are continuous.

Remark 1.2 (Parabolic case). The above definition of spectral resonances works well in the expanding and hyperbolic[4] settings of this book, but the results of Rugh [152] show that the definition must be changed in the so-called parabolic case, i.e. where $DT^m(x)$ may have an eigenvalue equal to one for some periodic orbits. (As in the famous Pomeau–Manneville map.)

We now move to the dynamical determinants associated with a pair (T,g) as above.

For a finite matrix (see (3.1)), or more generally a trace class or nuclear operator $\mathscr{Q}$ (Appendix A), we can define a Fredhom determinant as follows

$$\det(\mathrm{Id} - z\mathscr{Q}) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{tr}\,(\mathscr{Q}^m)\,, \tag{1.11}$$

and[5] we have $\det(\mathrm{Id} - z\mathscr{Q}) = \prod(1 - z\gamma_j)$, where the γ_j are the finitely many or countably many eigenvalues of $\mathscr{Q}$.

The results in Chapters 2, 4, and 5 show that, when the differentiability r is large, then we can find a Banach space $\mathscr{B}$ for which the essential spectral radius of the transfer operator associated with (T,g) is small. In the analytic case, the transfer operators can be shown ([137, 73, 74, 150, 151, 160, 161]) to be compact, even nuclear, on suitable Banach or Hilbert spaces, containing all analytic functions. (In this case, as mentioned in the previous paragraph, traces and determinant are well defined.) This is not possible in finite differentiability. However, following Ruelle, one can introduce dynamical Fredholm–Ruelle determinants $d_{T,g}(z)$, where the role of the trace of $\mathscr{L}_g^m$ is played by a weighted sum over the periodic orbits of T, as we explain next.

If T is locally expanding then we introduce a formal[6] trace, the "flat trace" in Chapter 3, and show that it coincides with

$$\mathrm{tr}^\flat \mathscr{L}_g^m = \sum_{x:T^m(x)=x} \frac{\prod_{k=0}^{m-1} g(T^k(x))}{|\det(\mathrm{Id} - DT^{-m}(x))|}\,,$$

where we set $DT^{-m}(x) = ((DT^m)(x))^{-1}$ if $T^m(x) = (x)$.

In Chapter 6, for hyperbolic $\tilde{T} = T^{-1}$, the "flat trace" satisfies

$$\mathrm{tr}^\flat \mathscr{L}_g^m = \sum_{x:T^m(x)=x} \frac{\prod_{k=0}^{m-1} g(T^k(x))}{|\det(\mathrm{Id} - DT^m(x))|} = \sum_{x:\tilde{T}^m(x)=x} \frac{\prod_{k=0}^{m-1} \tilde{g}(\tilde{T}^k(x))}{|\det(\mathrm{Id} - D\tilde{T}^{-m}(x))|}\,.$$

[4] For nonuniformly hyperbolic dynamics, for example Collet–Eckmann logistic maps, it would be interesting to make Definition 1.1 compatible with the results of Keller–Nowicki [110] e.g. See Problem 2.43.

[5] In the nuclear case, this last claim holds if $\mathscr{Q}$ is $2/3$-nuclear.

[6] See also Remark 3.1 there.

We then define the Fredholm–Ruelle dynamical determinant[7] by

$$d_{T,g}(z) = \exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \mathrm{tr}^\flat \mathcal{L}_g^m . \tag{1.12}$$

Since the cardinality of $\{x \mid T^m(x) = x\}$ grows at most exponentially fast, the hyperbolicity assumption implies that the formal determinants $d_{T,g}(z)$ have a nonzero radius of convergence R_0, which in general is finite. In the disc of radius R_0 the function $d_{T,g}(z)$ cannot vanish. Our goal is to extend this function holomorphically to a larger disc where its zeroes will coincide with inverse eigenvalues of $\mathcal{L}_g$ on suitable spaces. (This task was first carried out by Ruelle [142] in the expanding case.) With this in mind, we formulate the next definition:

Definition 1.3 (Determinantal (Ruelle–Pollicott) resonances). Define $R_{sup} \le \infty$ by

$$R_{sup} := \sup\{R > 0 \mid d_{T,g}(z) \text{ admits a holomorphic extension to } \{|z| \le R\}\} .$$

The determinantal resonances of the pair (T,g) are the complex numbers w of modulus $> 1/R_{sup}$ so that $z = 1/w$ is a zero of $d_{T,g}(z)$ in the disc of radius R_{sup}.

Remark 1.4 (Parabolic case). Again, the above definition works well in the expanding and hyperbolic settings of this book, but the results of Rugh [152] show that it must be changed in the parabolic case. In particular the parabolic points must be removed from the definition of the determinant:

$$\exp - \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{\substack{x: T^m(x)=x \\ \det(\mathrm{Id} - DT^{-m}(x)) \neq 0}} \frac{\prod_{k=0}^{m-1} g(T^k(x))}{|\det(\mathrm{Id} - DT^{-m}(x))|} .$$

See also [26].

Finally, since the title of the book mention zeta functions, they should appear in the introduction, lest an unhappy reader asks for his or her money back: The dynamical zeta function of a pair (T,g) is

$$\zeta_{T,g}(z) = \exp \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{x: T^m(x)=x} \prod_{k=0}^{m-1} g(T^k(x)) .$$

We shall see that in the present differentiable discrete-time settings the dynamical zeta function contain less spectral information than the dynamical determinant: The formulas (3.8) and (6.38) below show how to express $\zeta_{T,g}(z)$ as an alternated product of dynamical determinants. The possibility of cancellations, in particular, implies that knowing the zeroes of each individual dynamical determinant in the product is

[7] In the analytic setting, it turns out that the flat traces and thus the dynamical determinant coincide with the traces and determinant of the nuclear operators involved.

more precise than knowing the poles and zeroes of the zeta function. Nevertheless, the dynamical zeta function is a natural object, especially for hyperbolic flows (the Selberg zeta function e.g. is a dynamical zeta function [144]) and deserves to be studied. (Note also that if the dynamics is not differentiable, then we cannot even define the dynamical determinant!)

1.3 Main results. Examples.

In a nutshell, the main results discussed in this book give bounds $r(T,g) < \infty$ so that the spectral and determinantal resonances coincide outside of the disc of radius $r(T,g)$. This is obtained by finding a Banach space $\mathscr{B}$ so that the essential spectral radius of $\mathscr{L}_g$ on $\mathscr{B}$ is not larger than $r(T,g)$ (Theorems 2.15, 4.6, and 5.1), and then exploiting the information gathered along the way (in particular, Lasota–Yorke or enhanced Lasota–Yorke estimates) to view the dynamical determinant as a formal Fredholm determinant for the (non compact) operator $\mathscr{L}_g$ (Theorems 3.5 and 6.2). For pedagogical reasons, this program is carried out first in the expanding case (in Chapters 2 and 3) and then in the hyperbolic case (in Chapters 4–6).

Finally, in Chapter 7, we give an ergodic interpretation of the peripheral eigenvectors of $\mathscr{L}_g$ on the anisotropic Banach spaces for hyperbolic T, as follows.

Let us discuss first the simplest case of so-called SRB measures: If $(\tilde{V},\Lambda)$ is attracting and transitive for $\tilde{T} = T^{-1}$ (for example if T is Anosov and transitive on $V = M$), we have[8] for every continuous function φ and almost all $x \in \tilde{V}$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ \tilde{T}^k(x) = \int \varphi \, \mathrm{d}\mu \, ,$$

where μ is an ergodic T-invariant probability measure. In general [4] this measure is singular with respect to Lebesgue. The measure μ is called a physical, or SRB measure for $\tilde{T}$. The survey [182] by Lai-Sang Young gives an excellent general introduction to SRB measures. We shall[9] see in §7.1.3 that SRB measures for hyperbolic maps can be obtained from the eigenvectors of the transfer operator associated with the pair $(T, |\det DT|)$. (The dual of this transfer operator has Lebesgue measure as a fixed point.)

Next, let $L_{inf} \geq 0$ be the largest real number $L \geq 0$ so that for all $\varphi, \psi \in C^{r-1}(M)$ the Fourier transform of the correlation function

$$C_{\varphi,\psi}(k) = \int (\varphi \circ \tilde{T}^k) \psi \, \mathrm{d}\mu - \int \varphi \, \mathrm{d}\mu \int \psi \, \mathrm{d}\mu$$

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[8] For expanding T, we have $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi \circ T^k(x) = \int \varphi \, \mathrm{d}\mu$ for every continuous function φ and almost all $x \in M$, where μ is an ergodic T-invariant probability measure. In addition μ is absolutely continuous with respect to Lebesgue. [14]

[9] The corresponding result for expanding maps is classical, see e.g. [14] and references therein.

admits a meromorphic extension to the strip $\{|\Im\eta| \le L\}$. The results of §7.1.3 give a positive lower bound on L_{inf} and show that the location[10] of the poles of the correlation function in the strip are in bijection with the spectral (or determinantal) resonances of $(T, |\det DT|)$ there. In fact, the proof of Theorem 7.11 implies that, if ψ is supported in $\tilde{V}$, then the same properties hold for the empirical correlation defined by integrating with respect to Lebesgue measure

$$\int (\varphi \circ \tilde{T}^k)\psi \, \mathrm{d}x - \int \varphi \, \mathrm{d}\mu \int \psi \, \mathrm{d}x \,. \tag{1.13}$$

Consider now a general hyperbolic map $\tilde{T}$ (lifting the assumption that $(\tilde{V}, \Lambda)$ is an attractor for $\tilde{T}$). Positive weights g then give rise to other interesting invariant ergodic probability measures μ_g, for example the measure of maximal entropy. They are called equilibrium or Gibbs measures and are also discussed in Chapter 7. The results there show in particular that the Fourier transform of the correlations

$$\int (\varphi \circ \tilde{T}^k)\psi \, \mathrm{d}\mu_g - \int \varphi \, \mathrm{d}\mu_g \int \psi \, \mathrm{d}\mu_g$$

have meromorphic extensions in domains where their poles are in bijection with the spectral resonances. (There is no obvious candidate for the empirical correlation (1.13) in this more general setting, since we do not have an a priori reference measure.)

Transfer operators associated with nonpositive weights also carry relevant physical or statistical information. For example the weight $g = \exp(i\sigma\varphi)$ allows to prove the central limit theorem. Other limit theorems and the almost sure invariance principle can be obtained by spectral methods. See e.g. [85] and references therein.

We move to another important application of anisotropic Banach spaces, the proof of stability of the resonances under small deterministic and random perturbations. Indeed, a key feature of these spaces is that, although they depend on the stable and unstable directions of T, the dependence is weak enough (at a technical level, one works with invariant cones in the tangent or cotangent space) that the same space will work for all diffeomorphisms T_ε close enough to T in the C^r topology. This means that the abstract spectral perturbation theory of Keller–Liverani [109] can be adapted in this setting. This was first done by Gouëzel–Liverani [87]. We describe this theory in Appendix A.3, and we use it to prove spectral stability of the discrete eigenvalues and eigenvectors of $\mathscr{L}_g$ (in §2.5 for expanding maps and §5.3 for hyperbolic maps). In the case of attractors and $g = |\det DT|$, this gives a new proof of Ruelle's [146, 147, 102] linear response for Anosov diffeomorphisms. (Other relevant physical data can be studied by this approach, see Remark 5.27.)

A final application of anisotropic spaces is a very short proof by Tsujii of Anosov's theorem of ergodicity of volume-preserving Anosov flows on compact

[10] The residues of the poles depend on φ and ψ. In particular, they can vanish for some (non-generic) φ and ψ.

connected manifolds. This previously unpublished argument is the content of Section 7.2.

We next present a few basic examples.

The simplest chaotic dynamical systems is the angle-doubling map $T(x) = 2x$ modulo 1 on the circle $M = S^1$. Normalised Lebesgue measure μ is an invariant ergodic probability measure for T. For the weight $g = 1/2 = 1/|\det DT|$, it is easy to compute the dynamical determinant and the zeta function:

$$d_{T,g}(z) = 1 - z\,, \quad \zeta_{T,g}(z) = \frac{1-z/2}{1-z}\,. \tag{1.14}$$

(To prove the above claims, use the Taylor expansion of $\log(1-w)$ and that there are exactly $2^m - 1 = 2^m|\det(1 - DT^{-m})|$ points in the set $\{x \mid T^m(x) = x\}$.) By the results recalled in Chapter 2, for any $t > 1$, the essential spectral radius of $\mathcal{L}_g$ on C^t is equal to 2^{-t}, and the only eigenvalue of $\mathcal{L}_g$ of modulus $> 2^{-t}$ is $\gamma = 1$, which is simple (corresponding to the fixed constant function). This reflects the fact that, in this linear case, Lebesgue measure is exponentially mixing with rate 2^{-t} for C^t test functions, while it mixes superexponentially for analytic test functions. In this simple example, there are no resonances.

Keller and Rugh [111] constructed C^∞ (in fact, real analytic) examples of expanding circle maps T with a spectral resonance γ (for $g = 1/|T'|$) so that the modulus of γ is larger than the inverse of the weakest expansion. See also Remarks 2.4 and 3.4.

We would like to mention that considering manifolds with boundaries can drastically change the resonances. For example, the doubling map on $I = [0,1]$ defined by $T(x) = 2x$ modulo 1 leaves Lebesgue measure invariant. Since $\{x \mid T^m(x) = x\}$ now has 2^m elements, one finds, for the weight $g = 1/|T'| = 1/2$,

$$d_{T,g}(z) = \prod_{k=0}^{\infty}(1 - 2^k z)\,, \quad \zeta_{T,g}(z) = \frac{1}{1-z}\,.$$

One can prove that the transfer operator $\mathcal{L}_g$ acting on C^t has an eigenvalue at $\gamma = 2^{-k}$ for every integer $0 \le k \le t$, with corresponding eigenvector a Bernoulli polynomial. (This was probably first observed by Gaspard [79].) Only the eigenvector for $k = 0$ (the constant function) is periodic, however.

We move next to hyperbolic examples. The simplest hyperbolic diffeomorphism is the Anosov diffeomorphism T obtained by considering the linear automorphism $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ on the two-torus M. (This is Arnold's celebrated cat map.) The matrix has one real eigenvalue, λ, inside the unit circle and the other, λ^{-1}, outside the unit circle. Let $g = |\det DT| \equiv 1$, and note that Lebesgue measure is preserved and ergodic (it is the SRB measure, which in this nongeneric linear case is non singular with respect to Lebesgue). It is not difficult to check (see e.g. [105, Prop 1.8.1] and its proof) that the number of fixed points of T^m is equal to $|\det(\mathrm{Id} - A^m)|$, so that

$$d_{T,g}(z) = 1 - z\,.$$

Therefore, just like for the linear angle-doubling map, there are no determinantal resonances except for $z = 1$. The results of Chapter 5–7 imply that for any $t > 0$ there is a Banach space containing $C^t(M)$ on which the spectrum of the transfer operator is the union of a simple eigenvalue at $\gamma = 1$ (corresponding to the fixed constant function) with a subset of the closed disc of radius λ^t. In particular, the correlations decay superexponentially for analytic observables (in this linear case, this can be checked directly by using Fourier coefficients, see [14]). The first examples of Anosov diffeomorphisms with nontrivial resonances for $g = |\det DT|$ are surprisingly recent and are due to Slipantschuk et al. [161] (based on [160]) and Adam [1] (based on an idea of Naud [127]). See also Remark 6.5.

Axiom A diffeomorphisms (Definition 4.3) are classical examples of non Anosov hyperbolic diffeomorphisms. The simplest case is the time-one map of a Morse–Smale gradient flow, where Λ is a set of finitely many hyperbolic periodic orbits. Then it is easy to write out the rational functions $d_{T,g}(z)$ and $\zeta_{T,g}(z)$, for example for $g = |\det DT|$. However, since T is not transitive on Λ, our methods do not allow to construct an anisotropic Banach space on which $\mathcal{L}_g$ has good spectral properties if g is not supported in an isolating neighbourhood of one of the periodic orbits. See §4.1.3 and the reference there to the recent work of Dang and Rivière [56].

1.4 Main techniques

We go over the basic ingredients of the proofs of the main theorems: The first step is to bound the essential spectral radius of the transfer operator $\mathcal{L}_g$. We use the — by now [14] standard — tool given by Hennion's theorem. To prove the needed Lasota–Yorke estimates, we decompose for every $m \geq 1$

$$\mathcal{L}_g^m = (\mathcal{L}_g^m)_b + (\mathcal{L}_g^m)_c\,,$$

where $(\mathcal{L}_g^m)_c$ is compact (and thus can be disregarded), while $(\mathcal{L}_g^m)_b$ is bounded, with a norm which can be estimated precisely.

The main computations for $\mathcal{L}_g^m$ and its decomposition are carried out locally, in charts. This can be done with the help of finite smooth partitions of unity on the manifold (the partitions, in particular their cardinality, depend on m), and appropriate "fragmentation," respectively, "reconstitution" lemmas, allowing us to decompose and then regroup the sums over the partition of unity. The number of terms in the final sum depends on the m-th iterate and grows exponentially — it is controlled by a variational expression, obtained using thermodynamic techniques from [31] explained in Appendix B.

This "Lego™" approach is also carried out in the Fourier parameter, where Paley–Littlewood (dyadic) decompositions are used. This allows to reduce the key estimates to ordinary integration by parts. These estimates are carried out for the operator $\mathscr{L}_g^m$ in charts, instead[11] of an ancillary countable matrix of operators as in [31]. We find this direct approach more intuitive (the price to pay, a few additional computations, see e.g. Step 2 in the proof of Proposition 6.9, seems worthwhile).

The second step is to study the dynamical determinant $d_{T,g}(z)$. There, we use the "kneading operator" ideas inspired from the work of Minor and Thurson [123, 27]. This program is carried out through an enhanced Lasota–Yorke decomposition, where $\mathscr{Q}_c := (\mathscr{L}_g^m)_c$ is not only compact, but has the property that $\mathscr{Q} = \mathscr{Q}_c^N$ is nuclear, for some integer $N \geq 1$ depending on the dimension d, the regularity $r > 1$, and the smoothness parameters t and s used to define the Banach norms. We call such a decomposition[12] "nuclear power decomposition," and we refer to §3.2.1 for an outline of the argument. The main tool to show nuclearity is the powerful theory of approximation numbers, as developed in particular by Pietsch. (Appendix A.4 lists needed facts about nuclear operators and approximation numbers.)

The proof of the results on the dynamical determinants hold in any dimension $d \geq 1$ and any regularity $r > 1$. When the regularity is small with respect to the dimension, two elementary but painful computations are required to prove Lemma 3.22 in §3.3.4 and Lemma 6.11 in §6.3, respectively.

The book is essentially self-contained, building up from integration by parts ((2.66) and (2.68)) and the continuous Fourier transform. In particular, we define the Paley–Littlewood decompositions from scratch. In the expanding case, if r, α, and t are integers, then the results in Chapters 2 and 3 for the Hilbert space $\mathscr{B} = H_2^t(M)$ require almost no "black box." For noninteger parameters or $p \neq 2$, only a couple of basic results are used in black box mode (the Marcinkiewicz multiplier theorems Theorem 2.9 and 2.31, and the pseudolocal property Lemma C.1). In the hyperbolic case, few "black boxes" are used for the spaces $W^{t,s}_{p,*/**}(T,V)$ of Chapter 4 and even less for the (optimal) Banach space $\mathscr{B}^{t,s}(T,V)$ of Chapter 5. (Appendix D requires more "black boxes," but it is not used anywhere in the book and is only included for the sake of comparison for those readers already familiar with pseudodifferential techniques.)

The functional approach advocated in this book bypasses the symbolic dynamics methods using Markov partitions which were first exploited ([139, 39, 131]) to study equilibrium states and dynamical zeta functions. However, we should not throw out the baby with the bath water: Some of the techniques developed by Ruelle, Bowen, and others between the late 1960's and the early 1980's, namely thermodynamic formalism (see Appendix B), and the properties of expansive homeomorphisms enjoying specification (see e.g. Theorem 6.6 in Chapter 6) turn out to be very useful, and we shall use them without any inhibitions.

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[11] As this book was going to press, Jézéquel [100] introduced an "intermediate" finite-dimensional ancillary matrix description which has its advantages, for example the proof of Proposition 3.15 becomes simpler.

[12] Such decompositions do not seem available for the "geometric" anisotropic spaces introduced by Liverani et al. [37, 87, 88, 59, 19]. They are implicitly performed however for the microlocal spaces of Faure–Roy–Sjöstrand [68].

Note: Except if otherwise mentioned, the problems in this book are open problems, but some are not difficult.

Comments

The main reference on dynamical zeta functions and the symbolic dynamics approach of Sinai, Bowen, and Ruelle [39] to transfer operators of differentiable[13] hyperbolic dynamics is the book [131] by Parry and Pollicott. Ruelle's surveys [144] and [145, §1] include zeta functions of flows, in particular the Selberg zeta function, and connections with number theory. The slightly more recent surveys [13, 148] also discuss dynamical determinants, but not anisotropic spaces. At the end of each chapter, a section of Comments gives pointers to the literature. We refer to them for references (Ruelle, Pollicott, Fried, Rugh etc.) using symbolic dynamics.

The functional approach to transfer operators, avoiding symbolic dynamics, can be traced back to the pioneering paper of Blank–Keller–Liverani [37] described in the Preface. Soon after this, we introduced in [15] the (microlocal) use of classical anisotropic Triebel–Sobolev spaces for transfer operators of hyperbolic maps, via the continuous Fourier transform in charts. It required a strong assumption of smoothness of the dynamical foliation, which was successfully discarded shortly thereafter: Liverani's "geometric" philosophy [37] of working with distributions defined by integrating along pieces of manifolds close to the dynamical foliations was honed to perfection in Gouëzel and Liverani' papers [87, 88]. More or less simultaneously, Tsujii and I published two articles [28, 31], inspired by Tsujii's previous work [9] with Avila and Gouëzel. In [28, 31] the "microlocal" anisotropic approach was carried out for the first time without any assumption of regularity on the dynamical foliations. (The survey [29] gives a simplified presentation of part of the results of [28], together with the expanding map toy model.) In [31], we obtained a new proof of Kitaev's [112] result on the domain of holomorphic[14] extension of the dynamical determinant $d_{T,g}(z)$ and a spectral interpretation of its zeroes (which was missing from [112]). We refer to the preface and the Comments section of Chapters 4 and 5 for more references (in particular to the "semi-classical" literature). Finally, we refer to the survey part of [17] for a comparison of the "geometric" [87, 88] and "microlocal" [28, 31] approaches and a discussion of (other) geometric and microlocal anisotropic spaces from [59, 20, 21, 25, 60, 19] adapted to piecewise smooth hyperbolic systems.

Cvitanović was one of the first [5, 6, 53] to recognize the importance of Ruelle's work on resonances in the study of physical and statistical properties of dynamical systems. His contributions to the physics (nonrigourous) literature contain a wealth of examples. Liverani's survey [118] provides an excellent mathematical discussion of the functional approach and the relevance of the spectral information of transfer operator. Gouëzel [86] gives a superb account of the semi-classical approach of Faure and Tsujii and connections with the spectrum of the Laplacian in constant

[13] The analytic setting will not be discussed in detail in this book.

[14] As explained in the preface, Liverani and Tsujii [119, 120] had previously obtained suboptimal results on $d_{T,g}(z)$.

curvature. Zworski's recent survey [184] gives a wide overview of the semi-classical approach and connections with scattering theory (see also the book [65]). We end by mentioning a recent reader-friendly survey by Galatolo [77].

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