Hamiltonian methods for the water wave problem

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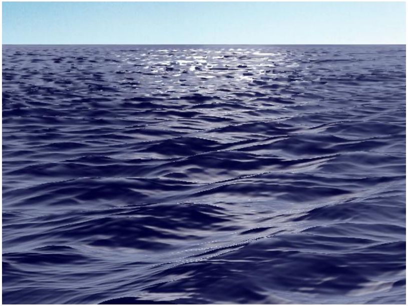


Contents

1. Lecture 1. The water waves equations, Hamiltonian formulation. Results based on Hamiltonian and reversible structure. Long time existence results.
2. Lecture 2. Hamiltonian Birkhoff normal form : finite dimensional systems and semilinear PDEs
3. Lecture 3 and talk at workshop. Hamiltonian Birkhoff normal form for quasi-linear PDEs . Paradifferential calculus, paradifferential normal form and the symplectic corrector

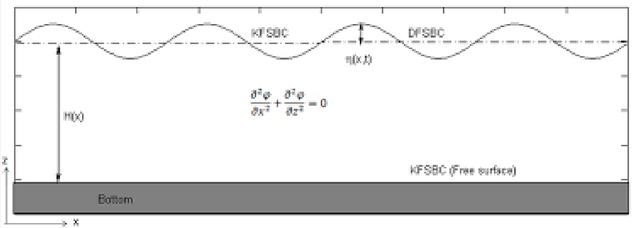
based on paper

Hamiltonian Birkhoff normal form for gravity-capillary water waves with constant vorticity: almost global existence,

M. Berti, A. Maspero and F. Murgante, arxiv 2022

Outline

Time evolution of space periodic water waves in Trieste gulf:

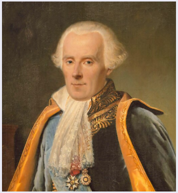


In section it is described by a bidimensional fluid, periodic in x

Incompressible Euler equations, 1757. Mémoires de l'Académie des Sciences de Berlin, "Principes généraux du mouvement des fluides"

$$\partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = -\nabla P \,, \quad \mathrm{div} \vec{u} = 0$$

Laplace: 1776. Suite des récherches sur plusieurs points du systéme du monde. Acad. R. Sci. Inst. France. Lagrange: 1781, 1786. Mémoire sur la théorie du mouvement des fluides. Nouv. Mém. Acad. Berlin.



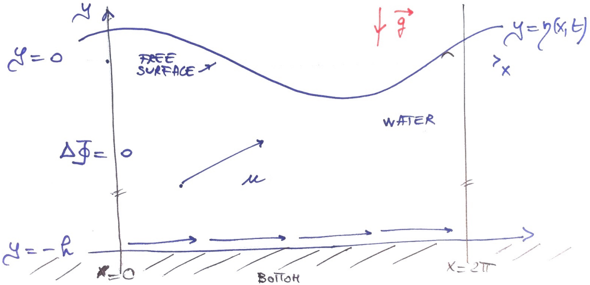
Water Waves : Euler equations for an incompressible fluid with constant vorticity γ in $\mathcal{D}_\eta(t) = \{-h < y < \eta(t,x)\}$ under gravity and capillarity

Equation of motions for $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ in $-h < y < \eta(t,x)$

$$\begin{cases} \partial_t \vec{u} + \vec{u} \cdot \nabla \vec{u} = -\nabla P - g e_y \\ \operatorname{div} \vec{u} = 0 \\ \operatorname{rot} \vec{u} = v_x - u_y = \gamma \end{cases}$$

Boundary conditions:

$$\begin{cases} \eta_t = v - u\eta_x & \text{at } y = \eta(t,x) \\ P + \kappa \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right) = P_0 & \text{at } y = \eta(t,x) \\ v = 0 & \text{at } y = -h \end{cases}$$



g = gravity, κ = surface tension coefficient, γ = vorticity

P = pressure of fluid, P_0 = atmospheric pressure, Curvature $= \partial_x \left(\frac{\eta_x}{\sqrt{1+\eta_x^2}} \right)$

Unknowns:

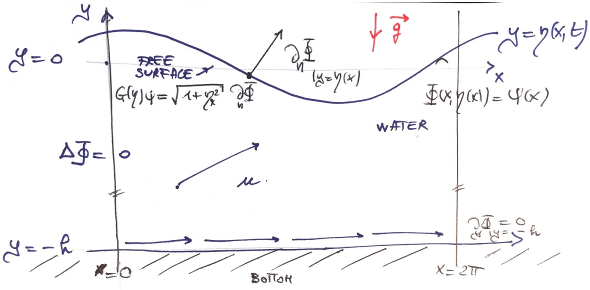
free surface $y = \eta(t,x)$ and the velocity field $\vec{u}(t,x,y)$

Hodge decomposition: $\vec{u}$ is the sum of a *Couette flow* and of an irrotational flow

$$\vec{u}(t,x,y) = \underbrace{\begin{pmatrix} -\gamma y \\ 0 \end{pmatrix}}_{\textit{vorticity } \gamma} + \underbrace{\nabla \Phi}_{\textit{irrotational}} \,, \qquad \Phi(t,x,y) = \text{ velocity potential}$$

$\vec{u}(t,x,y)$ is completely determined by $\eta(t,x)$ and $\psi(t,x) = \Phi(t,x,\eta(t,x))$

$$\begin{cases} \Delta \Phi = 0 & \text{in } -h < y < \eta(t,x) \\ \Phi = \psi & \text{at } y = \eta(t,x) \\ \partial_y \Phi = 0 & \text{at } y = -h \end{cases}$$



Reformulate the equations in terms of (η, ψ)

Zakharov-Craig-Sulem-Constantin-Wahlén formulation of WW with vorticity

$$\begin{cases} \eta_t = G(\eta)\psi + \gamma\eta\eta_x \\ \psi_t = -g\eta - \frac{\psi_x^2}{2} + \frac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} + \kappa\left(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\right)_x + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi \end{cases}$$

Dirichlet–Neumann operator

$$G(\eta)\psi(x) := \sqrt{1+\eta_x^2}\,\partial_n\Phi|_{y=\eta(x)} = (\Phi_y - \eta_x\Phi_x)|_{y=\eta(x)}$$

1. $G(\eta)$ is linear in ψ, non-local,
2. self-adjoint with respect to $L^2(\mathbb{T}_x)$
3. $G(\eta) \geq 0$, $G(\eta)[1] = 0$
4. $\eta \mapsto G(\eta)$ nonlinear, smooth,
5. $G(\eta)$ is pseudo-differential, $G(\eta) = D\tanh(hD) + OPS^{-\infty}$

Calderon, Craig, Lannes, Metivier, Alazard, Burq, Zuily, Delort...

Hamiltonian formulation

$$\partial_t \begin{pmatrix} \eta \\ \psi \end{pmatrix} = J_\gamma \nabla H(\eta, \psi) \,, \quad J_\gamma := \begin{pmatrix} 0 & Id \\ -Id & \gamma \partial_x^{-1} \end{pmatrix}$$

Hamiltonian

$$H(\eta,\psi) = \underbrace{\frac{1}{2}\int_{\mathbb{T}} \psi\, G(\eta)\psi\, dx}_{kinetic\ \ energy} + \underbrace{\frac{1}{2}\int_{\mathbb{T}} g\eta^2\, dx}_{potential\ \ energy} + \underbrace{\kappa \int_{\mathbb{T}} \sqrt{1+\eta_x^2}\, dx}_{capillary\ \ energy} + \underbrace{\frac{\gamma}{2}\int_{\mathbb{T}} (-\psi_x \eta^2 + \frac{\gamma}{3}\eta^3)\, dx}_{vorticity\ \ energy}$$

Wahlen coordinates (η, ζ) are Darboux coordinates:

$$\zeta := \psi - \frac{\gamma}{2}\partial_x^{-1}\eta$$

$$\partial_t \begin{pmatrix} \eta \\ \zeta \end{pmatrix} = J \nabla H(\eta, \zeta) \,, \quad J := \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$$

Symplectic 2-form

$$\Omega_\gamma \left(\begin{pmatrix} \eta_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} \eta_2 \\ \psi_2 \end{pmatrix} \right) = \left(E_\gamma \begin{pmatrix} \eta_1 \\ \psi_1 \end{pmatrix}, \begin{pmatrix} \eta_2 \\ \psi_2 \end{pmatrix} \right)_{L^2}, \quad \underbrace{E_\gamma := \begin{pmatrix} \gamma \partial_x^{-1} & -Id \\ Id & 0 \end{pmatrix}}_{symplectic\ \ tensor}$$

Hamiltonian vector field

$$dH(\eta,\psi)[\cdot] = \Omega_\gamma(X_H(\eta,\psi),\cdot) \quad \Longleftrightarrow$$

$$X_H(\eta,\psi) = J_\gamma \underbrace{\nabla H(\eta,\psi)}_{L^2-gradient}, \quad \underbrace{J_\gamma := E_\gamma^{-1} = \begin{pmatrix} 0 & Id \\ -Id & \gamma \partial_x^{-1} \end{pmatrix}}_{Poisson\ \ tensor}$$

Pull-back 2-form under a linear transformation B

$$B^* \Omega_\gamma \left(\begin{pmatrix} \eta_1 \\ \zeta_1 \end{pmatrix}, \begin{pmatrix} \eta_2 \\ \zeta_2 \end{pmatrix} \right) = \Omega_\gamma \left(B \begin{pmatrix} \eta_1 \\ \zeta_1 \end{pmatrix}, B \begin{pmatrix} \eta_2 \\ \zeta_2 \end{pmatrix} \right) = \Big(\underbrace{B^\top E_\gamma B}_{new\ \ symplectic\ \ tensor} \begin{pmatrix} \eta_1 \\ \zeta_1 \end{pmatrix}, \begin{pmatrix} \eta_2 \\ \zeta_2 \end{pmatrix} \Big)_{L^2}$$

Whalén transformation $B : (\eta, \zeta) \mapsto (\eta, \psi)$

$$B := \begin{pmatrix} Id & 0 \\ \frac{\gamma}{2}\partial_x^{-1} & Id \end{pmatrix}, \quad B^\top := \begin{pmatrix} Id & -\frac{\gamma}{2}\partial_x^{-1} \\ 0 & Id \end{pmatrix}$$

$$B^T E_\gamma B = E_0 = \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$$

standard symplectic tensor

Translation invariance

$$H \circ \tau_\varsigma = H, \qquad \tau_\varsigma : (\eta, \zeta)(x) \mapsto (\eta, \zeta)(x + \varsigma)$$

$\Rightarrow$ by Noether theorem

Momentum

$$\int_{\mathbb{T}} \zeta_x(x)\eta(x)\, dx$$

EXERCISE 1: the transformations τ_ς are symplectic

$$\tau_\varsigma^* \Omega_0 = \Omega_0 \,, \qquad \Longleftrightarrow \qquad \tau_\varsigma^\top E_0 \tau_\varsigma = E_0$$

EXERCISE 2: the Hamiltonian vector field generated by the momentum is the generator of the translations, and thus has flow τ_ς

Reversibility

$$H \circ S = H, \qquad \text{Involution: } S : (\eta, \zeta)(x) \mapsto (\eta, -\zeta)(-x)\,, \quad S^2 = \mathrm{Id}$$

Reversible vector field $X_H = J \nabla H$

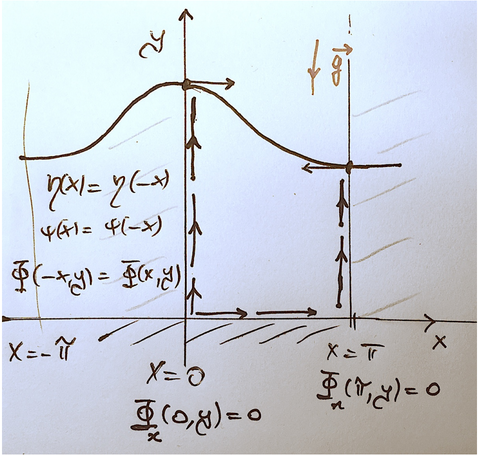
$$X_H \circ S = -S \circ X_H \quad \Longleftrightarrow \quad \Phi_H^t \circ S = S \circ \Phi_H^{-t}$$

Equivariance under the $\mathbb{Z}/(2\mathbb{Z})$-action of the group $\{\mathrm{Id}, S\}$

Recommended book: Moser-Zehnder: Lectures in Dynamical Systems

If $\gamma = 0$: Standing Waves : Invariant subspace: functions even in x

$$\eta(-x) = \eta(x)\,, \quad \psi(-x) = \psi(x)$$



Standing waves

Fluid confined between two walls

NOT for $\gamma \neq 0$

Invariant subspace: functions even in x

$$\eta(-x) = \eta(x)\,, \quad \psi(-x) = \psi(x)$$

Thus the velocity potential

$$\Phi(-x, y) = \Phi(x, y) \quad \Longrightarrow \quad \Phi_x(0, y) = 0$$

and, using also 2π periodicity,

$$-\Phi_x(\pi, y) = \Phi_x(-\pi, y) = \Phi_x(\pi, y) \quad \Longrightarrow \quad \Phi_x(\pi, y) = 0$$

$\Longrightarrow$ no flux of fluid outside the walls $\{x = 0\}$ and $\{x = \pi\}$.

Neumann boundary conditions at $x = 0$ and $x = \pi$

$$\eta_x(0) = \eta_x(\pi) = 0\,, \quad \psi_x(0) = \psi_x(\pi) = 0$$

Mass

$$\int_{\mathbb{T}} \eta(x)\, dx = \text{const.}$$

Phase space

$$\eta \in H^s_0(\mathbb{T}) := \Big\{\eta \in H^s(\mathbb{T}) \, : \, \int_{\mathbb{T}} \eta(x) dx = 0\Big\}$$

$$u \in H^s(\mathbb{T}) \;\Leftrightarrow\; u(x) = \sum_{k\in\mathbb{Z}} u_k e^{\mathrm{i}kx}\,, \quad \sum_{k\in\mathbb{Z}} |u_k|^2 \langle k \rangle^{2s} =: \|u\|^2_{H^s} < +\infty$$

The variable ζ is defined modulo constants: only the velocity field $\nabla_{x,y}\Phi$ has physical meaning:

$$\zeta \in \dot{H}^s(\mathbb{T}) = H^s(\mathbb{T})/\sim \qquad u(x) \sim v(x) \quad \Longleftrightarrow \quad u(x) - v(x) = c$$

Hamiltonian and reversible nature of water waves equation only recently **effectively** exploited

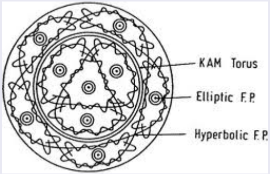
1. Existence of time quasi-periodic solutions. KAM for water waves
   Baldi, Berti, Feola, Franzoi, Giuliani, Haus, Maspero, Montalto, since 2015

   prior results of periodic solutions: Toland, Plotnikov, Iooss, Alazard, Baldi
2. Long time existence results. Birkhoff normal form for water waves
   Berti, Delort, Feola, Franzoi, Maspero, Murgante, Pusateri, since 2016
3. Benjamin-Feir instability of Stokes waves
   Berti, Maspero, Ventura, since 2022

Remarks:

- key role in dynamical systems of XX century;
- In $\mathbb{R}^d$ less relevant as dispersion prevails (but also here useful for local existence)

Expected scenario for nearly-integrable Hamiltonian systems close to an elliptic equilibrium



1. **KAM results**: These are solutions defined for all times

Definition: quasi-periodic solution with n frequencies

$u(t,x) = U(\omega t, x)$ where $U(\varphi, x) : \mathbb{T}^n \times \mathbb{T} \to \mathbb{R}$,
$\omega \in \mathbb{R}^n (= \textit{frequency vector})$ is irrational $\omega \cdot k \neq 0\,,\ \forall k \in \mathbb{Z}^n \setminus \{0\}$
$\Longrightarrow$ the linear flow $\{\omega t\}_{t \in \mathbb{R}}$ is DENSE on $\mathbb{T}^n$

selection of "initial conditions" giving rise to global solutions

2. **Long time existence: solution of size ϵ does it exists and remain in an $O(\varepsilon)$-ball for all $|t| \leq c\varepsilon^{-N}$.** For exponential times ?
3. **Arnold diffusion:** *What about a solution which does not start on a KAM torus for times $|t| > c\varepsilon^{-N}$?*

   Chaos? Growth of Sobolev norms?

In these lecture item 2 : long time existence results and Birkhoff normal form

Outline

Linearized system at $(\eta, \zeta) = (0, 0)$

$$\begin{cases} \partial_t \eta = G(0)\zeta + \frac{\gamma}{2} G(0) \partial_x^{-1} \eta, \\ \partial_t \zeta = -g\eta + \kappa \eta_{xx} + \frac{\gamma}{2} \partial_x^{-1} G(0)\zeta + \left(\frac{\gamma}{2}\right)^2 \partial_x^{-1} G(0) \partial_x^{-1} \eta \end{cases}$$

Dirichlet-Neumann operator at the flat surface $\eta = 0$ is

$$G(0) = D \tanh(hD) = |D| \tanh(h|D|)\,, \quad D = \frac{\partial_x}{\mathrm{i}}$$

Fourier multiplier notation: given $m : \mathbb{Z} \to \mathbb{C}$

$$m(D)h = \mathrm{Op}(m)h = \sum_{j \in \mathbb{Z}} m(j) h_j e^{\mathrm{i}jx}\,, \quad h(x) = \sum_{j \in \mathbb{Z}} h_j e^{\mathrm{i}jx}$$

Exercise: computation of the Dirichlet-Neumann operator $G(0)$

The solution of the elliptic problem:

$$\Delta \Phi = 0 \text{ in } \{-\mathtt{h} < y < 0\}, \ \Phi|_{y=0} = \psi, \ \partial_y \Phi = 0 \text{ at } y = -\mathtt{h}$$

where $\psi(x) = \sum_{j \in \mathbb{Z}} \psi_j e^{\mathrm{i} j x}$ is

$$\Phi(x,y) = \psi_0 + \sum_{j \neq 0} \frac{\psi_j}{\cosh(\mathtt{h} j)} \cosh(j(y+\mathtt{h})) e^{\mathrm{i} j x}$$

Thus

$$G(0)\psi := (\partial_y \Phi)(x,0) = \sum_{j \in \mathbb{Z}} j \tanh(\mathtt{h} j) \psi_j e^{\mathrm{i} j x} =: D \tanh(\mathtt{h} D) \psi$$

Complex variable

$$u = \frac{1}{\sqrt{2}}(M^{-1}(D)\eta + \mathrm{i}M(D)\zeta)\,, \quad M(D) := \Big(\frac{G(0)}{\kappa D^2 + g - \frac{\gamma^2}{4}\partial_x^{-1}G(0)\partial_x^{-1}}\Big)^{\frac{1}{4}}$$

Linear Water Waves

$$u_t = i\Omega(D)u$$

Dispersion relation

$$\Omega(\xi) = \sqrt{\Big(\kappa\xi^2 + g + \frac{\gamma^2}{4}\frac{\tanh(\mathtt{h}\xi)}{\xi}\Big)\xi\tanh(\mathtt{h}\xi)} \quad + \frac{\gamma}{2}\tanh(\mathtt{h}\xi)$$

Linear solutions: infinitely many harmonic oscillators

$$\dot{u}_j = \mathrm{i}\Omega_j(\kappa)u_j \qquad \textbf{all solutions}: \qquad u(t,x) = \sum_{j\in\mathbb{Z}\setminus\{0\}} u_j(0)\, e^{\mathrm{i}t\Omega_j(\kappa)}\, e^{\mathrm{i}jx}$$

are **periodic, quasi-periodic, almost periodic**

The Sobolev norm is constant

$$\|u(t,\cdot)\|_{H^s} = \|u(0,\cdot)\|_{H^s}$$

Linear frequencies of oscillations

$$\Omega(\xi) = \underbrace{\sqrt{\Big(\kappa\xi^2 + g + \frac{\gamma^2}{4}\frac{\tanh(\mathtt{h}\xi)}{\xi}\Big)\xi\tanh(\mathtt{h}\xi)}}_{even\ \ in\ \ \xi} \quad \underbrace{+\frac{\gamma}{2}\tanh(\mathtt{h}\xi)}_{odd\ \ in\ \ \xi}$$

1. For $\kappa > 0$ (superliner)

$$\Omega(\xi) \sim \sqrt{\kappa}|\xi|^{\frac{3}{2}} \quad \text{as} \quad |\xi| \to +\infty$$

2. $x \in \mathbb{T}$ and $u(x)$ zero average $\Rightarrow |\xi| \geq 1$
3. For $\gamma = 0$ the dispersion relation is EVEN $\Omega(\xi) = \Omega(-\xi)$

   on the subspace of even functions the frequencies $\Omega(j)$ are simple

Outline

Main question:

- for which time intervals $(-T_{\max}, T_{\max})$ solutions of the nonlinear water waves equations exist?

Major difficulties:

Quasi-linear PDEs

$$u_t = \mathrm{i}\Omega(D)u + N(u, \bar{u}), \quad \Omega(D) \sim |D|^{3/2}$$

$N =$ quadratic nonlinearity with derivatives of order $N(|D|^{3/2}u)$

Local existence. Hidden hyperbolic structure, with or without capillarity.
Nalimov, Yosihara, Craig,

S. Wu = initial data of arbitrary size in Sobolev spaces, 1999

Lindblad, Beyer-Gunther, Coutand-Shkroller, Shatah-Zeng, Lannes, Alazard-Burq-Zuily –Alinhac "good unknown"– Schweizer, Ifrim-Tataru, ...

For global existence huge difference between $x \in \mathbb{R}^d$ and $x \in \mathbb{T}^d$

Periodic boundary conditions $x \in \mathbb{T}$

NO dispersive effects of the linear PDE as for $x \in \mathbb{R}^2$, $x \in \mathbb{R}$ and data decaying at infinity:
Global well-posedness: S.Wu, Germain-Masmoudi-Shatah, Ionescu-Pusateri, Alazard-Delort, Ifrim-Tataru, Alazard-Burq-Zuily

Not available conserved quantities controlling high Sobolev norms

Theorem (M. Berti, A. Maspero, F. Murgante 2022)

For any value of the gravity $g > 0$, *depth* $\mathtt{h} \in (0, +\infty]$ *and vorticity* $\gamma \in \mathbb{R}$, *there is a zero measure set* $\mathcal{K} \subset (0, +\infty)$ *such that, for any surface tension coefficient* $\kappa \in (0, +\infty) \setminus \mathcal{K}$, *for any* N *in* $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, *there is* $s_0 > 0$ *and, for any* $s \geq s_0$, *there are* $\varepsilon_0 > 0, c > 0, C > 0$ *such that, for any* $0 < \varepsilon < \varepsilon_0$, *any initial datum*

$$(\eta_0, \psi_0) \in H_0^{s+\frac{1}{4}}(\mathbb{T}, \mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T}, \mathbb{R}) \quad \text{with} \quad \|\eta_0\|_{H_0^{s+\frac{1}{4}}} + \|\psi_0\|_{\dot{H}^{s-\frac{1}{4}}} < \varepsilon \,,$$

the gravity-capillary-vorticity water waves equations have a unique classical solution

$$(\eta, \psi) \in C^0\Big([-T_\varepsilon, T_\varepsilon], H_0^{s+\frac{1}{4}}(\mathbb{T}, \mathbb{R}) \times \dot{H}^{s-\frac{1}{4}}(\mathbb{T}, \mathbb{R})\Big) \quad \text{with} \quad T_\varepsilon \geq c\varepsilon^{-N-1} \,,$$

satisfying the initial condition $\eta|_{t=0} = \eta_0, \psi|_{t=0} = \psi_0$. *Moreover*

$$\sup_{t \in [-T_\varepsilon, T_\varepsilon]} \Big(\|\eta\|_{H_0^{s+\frac{1}{4}}} + \|\psi\|_{\dot{H}^{s-\frac{1}{4}}}\Big) \leq C\varepsilon$$

Remarks

1. This theorem extends Berti-Delort 2017 :
   (i) *non zero vorticity* γ;
   (ii) it is new also for $\gamma = 0$ since [BD] holds for initial data (η_0, ψ_0) *even* in x
2. Restriction on parameters to ensure the absence of *N-wave resonant interactions*:

$$\left|\Omega_{j_1}(\kappa) + \ldots + \Omega_{j_p}(\kappa) - \Omega_{j_{p+1}}(\kappa) - \ldots - \Omega_{j_N}(\kappa)\right| \gtrsim \max(|j_1|, \ldots, |j_N|)^{-\tau}$$

   among integers $j_1, \ldots, j_p, j_{p+1}, \ldots, j_N$ which are *not super-action preserving*, namely

$$\{|j_1|, \ldots, |j_p|\} \neq \{|j_{p+1}|, \ldots, |j_N|\}$$

   Tool: *sub-analytic* functions Delort-Szeftel '03
3. Key energy estimate for $\|(\eta, \psi)\|_{X^s} := \|\eta\|_{H_0^{s+\frac{1}{4}}} + \|\psi\|_{\dot{H}^{s-\frac{1}{4}}}$ as

$$\|(\eta, \psi)(t)\|_{X^s}{}^2 \lesssim_{s,N} \|(\eta, \psi)(0)\|_{X^s}^2 + \int_0^t \|(\eta, \psi)(\tau)\|_{X^s}^{N+3} \, d\tau$$

Highly non trivial facts: same X^s and $N + 3$

1. $T_\varepsilon \geq c\varepsilon^{-1}$, local existence theory, S. Wu., Lindblad, Beyer-Gunther, Coutand-Shkroller, Lannes, Shatah-Zeng, Alazard-Burq-Zuily, Ifrim-Tataru, ...
2. $T_\varepsilon \geq c\varepsilon^{-2}$, S. Wu, Ifrim-Tataru, in cases there are no 3-wave interactions: $e^{\mathrm{i}\Omega_j t} e^{\mathrm{i}kx}$

No integer solutions $j_1, j_2, j_3 \in \mathbb{Z} \setminus 0$ of

$$\begin{cases} \Omega_{j_1} \pm \Omega_{j_2} \pm \Omega_{j_3} = 0 \\ j_1 \pm j_2 \pm j_3 = 0 \end{cases}$$

Pure capillary, $\mathtt{h} = +\infty$. $\Omega_j = |j|^{\frac{3}{2}}$
Pure gravity, $\mathtt{h} = +\infty$. $\Omega_j = |j|^{\frac{1}{2}}$

3. Gravity-capillary irrotational even in x waves $T_\varepsilon \geq c\varepsilon^{-N}$, $\forall N$, Berti-Delort '17, we erase parameters (g, κ) to avoid multiple wave interactions

$$\begin{cases} \Omega_{j_1} \pm \ldots \pm \Omega_{j_{N+1}} = 0 \\ j_1 \pm \ldots \pm j_{N+1} = 0 \end{cases}$$

Theorem (Berti-Feola-Franzoi, '19)

For any value of g = gravity, κ = capillarity, $\mathtt{h}$ = depth, the solutions of gravity-capillary irrotational water waves exist for $T_\varepsilon \geq c\varepsilon^{-2}$

$$\Omega_j = \sqrt{j \tanh(\mathtt{h}j)(g + \kappa j^2)}$$

There are 3-waves resonances (Wilton-ripples)

$$\begin{cases} \Omega_{j_1} \pm \Omega_{j_2} \pm \Omega_{j_3} = 0 \\ j_1 \pm j_2 \pm j_3 = 0 \end{cases} \qquad j_1, j_2, j_3 \in \mathbb{Z} \setminus 0\,,$$

Key: Finitely many + Hamiltonian Birkhoff normal form

Theorem (Berti, Feola, Pusateri, '18) Conjecture of Zakharov-Dyachenko '94

The irrotational gravity water waves equations in deep water $\mathtt{h} = +\infty$ are an *integrable* system up to quartic terms $O(u^4)$ and $T_\varepsilon \geq c\varepsilon^{-3}$

NO PARAMETERS . Linear frequencies $\Omega(j) = g\sqrt{|j|}$

Recent extensions : S. Wu and Deng-Ionescu-Pusateri

Algebraic property to exclude "growth of Sobolev norms"

1. Hamiltonian
2. Reversibility

Dynamical systems heuristic explanation:

Water waves

$$u_t = \mathrm{i}\Omega(D)u + N_2(u,\bar{u})\,, \quad N_2(u,\bar{u}) = O(u^2)$$

Fourier and Action-Angle variables (θ, I)

$$u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx}\,, \quad u_j = \sqrt{I_j}e^{\mathrm{i}\theta_j}$$

$$\text{Sobolev norm } \|u\|_{H^s}^2 = \sum_{j\in\mathbb{Z}}(1+j^2)^s I_j$$

Small amplitude solutions

Rescaling $u \mapsto \varepsilon u$

$$u_t = \mathrm{i}\Omega(D)u + \varepsilon O(u^2)$$

in action-angle variables reads

$$\frac{d}{dt}I_j = \varepsilon f_j(\varepsilon,\theta,I)\,, \quad \frac{d}{dt}\theta_j = \Omega(j) + \varepsilon g_j(\varepsilon,\theta,I)$$

angles $\theta_j = \Omega(j)t$ “rotate fast”, actions $I_j(t)$ “slow” variables

"Averaging principle":

The effective dynamics of the actions is expected to be governed by

$$\frac{d}{dt} I_j = \varepsilon \langle f_j \rangle(\varepsilon, I)\,, \quad \langle f_j \rangle(\varepsilon, I) := \int_{\mathbb{T}^\infty} f_j(\varepsilon, \theta, I) d\theta$$

the average with respect to $\theta = (\theta_j)_{j \in \mathbb{Z}}$

If $\langle f_j \rangle(\varepsilon, I) \neq 0 \implies I_j(t)$ diverges ("secular terms" of Celestial mechanics)

Necessary condition for QP solutions and long time existence

$$\langle f_j \rangle(I) = 0$$

The condition $\langle f_j \rangle(I) = 0$ is implied by

Hamiltonian case: $f(\theta, I) = (\partial_\theta H)(\theta, I)$

$$\Longrightarrow \quad \int_{\mathbb{T}^\infty} (\partial_\theta H)(\theta, I) d\theta = 0$$

Reversible vector field (Moser)

$$\tfrac{d}{dt}\theta = g(I,\theta)\,, \ \tfrac{d}{dt} I = f(I,\theta)\,, \quad f(I,\theta)\,\mathrm{odd\,in}\,\theta, \ g(I,\theta)\,\mathrm{even\ in}\,\theta$$

$$\Longrightarrow \quad \int_{\mathbb{T}^\infty} f(\theta, I) d\theta = 0$$

Reversible vector field

$$X(\theta, I) = (g, f)(\theta, I)\,, \qquad X \circ S = -S \circ X\,, \quad S : (\theta, I) \mapsto (-\theta, I)$$

The water waves equations (written in complex variables) are reversible with respect to the involution

$$S : u(x) \mapsto \bar{u}(x)$$

that on the subspace of even functions

$$u(-x) = u(x)\,, \quad u(x) = \textstyle\sum_{j \in \mathbb{Z}} u_j e^{\mathrm{i}jx} = \sum_{j \in \mathbb{Z}} \sqrt{I_j} e^{\mathrm{i}\theta_j} e^{\mathrm{i}jx}\,,$$

is

Moser reversibility

$$(\theta, I) \mapsto (-\theta, I)$$

Alinhac "good unknown" which has to be introduced to get energy estimates (local existence theory) preserves the reversible structure, not the Hamiltonian one

Why Need of preserving the Hamiltonian structure

Poincaré-Birkhoff normal form in case of simple eigenvalues

$$\dot u_j = \mathrm{i}\Omega_j u_j + \underbrace{a|u_k|^2 u_j}_{Poincare'-Birkhoff \ \ resonant}, \qquad \forall j \in \mathbb{Z}$$

1. Reversible structure : vector field $f(u) := ([f(u)]_j)$ with $[f(u)]_j := a|u_k|^2 u_j$

$$f \circ S = -S \circ f\,, \quad S : (u_j) \mapsto (\overline{u_j})$$

implies

$$\bar a = -a \quad \Longrightarrow \quad a = \mathrm{i}\beta\,, \beta \in \mathbb{R}\,,$$

2. $\Rightarrow$

$$\frac{d}{dt}|u_j|^2 = \frac{d}{dt}(u_j\overline{u_j}) = \dot u_j \overline{u_j} + u_j \overline{\dot u_j}$$
$$= \Big[\mathrm{i}(\Omega_j + \beta|u_k|^2)u_j\Big]\overline{u_j} + u_j\Big[-\mathrm{i}(\Omega_j + \beta|u_k|^2)\overline{u_j}\Big] = 0$$

$\Rightarrow |u_j|^2$ are prime integrals [Berti-Delort]

False in presence of multiple eigenvalues

Birkhoff-resonant: $\Omega_j = \Omega_{-j}$

$$
\begin{cases}
\dot{u}_{-j} = \mathrm{i}\Omega_{-j} u_{-j} + \ b u_k \overline{u_{-k}} u_j \\
\dot{u}_j = \mathrm{i}\Omega_j u_j + \qquad \underbrace{a \overline{u_k} u_{-k} u_{-j}}_{Poincare' - Birkhoff\ resonant}
\end{cases}
$$

- Reversible structure implies $a, b \in \mathrm{i}\mathbb{R}$;
  - does not imply boundedness of the solutions
- Hamiltonian: $a \overline{u_k} u_{-k} u_{-j} = \mathrm{i}\partial_{\bar{u}_j} H \,, \ b u_k \overline{u_{-k}} u_j = \mathrm{i}\partial_{\bar{u}_{-j}} H \Longrightarrow a = -\overline{b}$

$$
H = \frac{a}{\mathrm{i}} \overline{u_k} u_{-k} u_{-j} \overline{u_j} + \overline{\left(\frac{a}{\mathrm{i}}\right)} u_k \overline{u_{-k}}\, \overline{u_{-j}} u_j
$$

“super-action” $J = |u_j|^2 + |u_{-j}|^2$ are prime integrals

- All the paradifferential transformations performed to prove local existence –as the celebrated Alinhac good unknown– are *NOT* symplectic
- In the last 2 papers (Berti-Feola-Franzoi '19) e (Berti-Feola-Pusateri '18) an a-posteriori identification argument implies that the quadratic and cubic Poincaré-Birkhoff normal forms are nevertheless Hamiltonian.

This argument does NOT work for any N

MAJOR GOAL OF THESE LECTURES and talk at workshop

recover, in paradifferential calculus, the nonlinear Hamiltonian structure, at any degree of homogeneity N

Develop a **systematic** paradifferential approach to Hamiltonian Birkhoff normal form for quasi-linear Hamiltonian PDEs

A symplectic Alinach good unknown up to homogeneity N

The nonlinear Alinach good unknown map [Alazard-Metivier, Alazard-Burq-Zuily]

$$\mathcal{G}\begin{pmatrix}\eta\\ \psi\end{pmatrix} := \begin{pmatrix}\mathrm{Id} & 0\\ -T_{B(\eta,\psi)} & \mathrm{Id}\end{pmatrix}\begin{pmatrix}\eta\\ \psi\end{pmatrix} = \begin{pmatrix}\eta\\ -T_{B(\eta,\psi)}\eta + \psi\end{pmatrix}$$

is **not** symplectic. T_B paraproduct

$$T_B u := \sum_{|j-k|<\delta|j|} \hat{B}(k-j)\hat{u}(j)e^{\mathrm{i}kx}$$

for the function $B(\eta,\psi)(x) := \Phi_y(x,\eta(x))$. However

Theorem: Symplectic good unknown up to homogeneity N

Let $N \in \mathbb{N}$. There exists a pluri-homogeneous smoothing operator $R_{\leq N}(\cdot)$ in $\Sigma_1^N \widetilde{\mathcal{R}}_q^{-\varrho}$ for any $\varrho \geq 0$ such that

$$D_{\leq N}(\eta,\psi) := (\mathrm{Id} + R_{\leq N}(\cdot)) \circ \mathcal{G}(\eta,\psi)$$

is symplectic up to homogeneity N, namely

$$\left[d_{(\eta,\psi)} D_{\leq N}(\eta,\psi)\right]^{\top} E_0 \left[d_{(\eta,\psi)} D_{\leq N}(\eta,\psi)\right] = E_0 + O((\eta,\psi)^{N+1}), \quad E_0 := \begin{pmatrix}0 & -\mathrm{Id}\\ \mathrm{Id} & 0\end{pmatrix}$$

Linearly Symplectic map : the matrix of operators $\begin{pmatrix} \mathrm{Id} & 0 \\ -T_{B(\eta,\psi)} & \mathrm{Id} \end{pmatrix}$ is **linearly symplectic** namely

$$\begin{pmatrix} \mathrm{Id} & 0 \\ -T_{B(\eta,\psi)} & \mathrm{Id} \end{pmatrix}^{\top} E_0 \begin{pmatrix} \mathrm{Id} & 0 \\ -T_{B(\eta,\psi)} & \mathrm{Id} \end{pmatrix} = E_0$$

Symplectic map : the nonlinear map

$$\mathcal{G}\begin{pmatrix} \eta \\ \psi \end{pmatrix} = \begin{pmatrix} \mathrm{Id} & 0 \\ -T_{B(\eta,\psi)} & \mathrm{Id} \end{pmatrix} \begin{pmatrix} \eta \\ \psi \end{pmatrix}$$

is **not symplectic** :

$$\mathrm{d}_{(\eta,\psi)}\mathcal{G}(\eta,\psi)^{\top}\ E_0\ \mathrm{d}_{(\eta,\psi)}\mathcal{G}(\eta,\psi) \neq E_0$$

$$\mathrm{d}_{(\eta,\psi)}\mathcal{G}(\eta,\psi) = \begin{pmatrix} \mathrm{Id} & 0 \\ -T_{B(\eta,\psi)} & \mathrm{Id} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -\mathrm{d}_{(\eta,\psi)} T_{B(\eta,\psi)}[\cdot] & 0 \end{pmatrix} \begin{pmatrix} \eta \\ \psi \end{pmatrix}$$

6) Global existence?

Question: **Do these solutions exist for all times?**

We do not know. Maybe not

Craig-Workfolk: for $\kappa = 0$, $\mathtt{h} = +\infty$ the water-waves PDEs are not integrable at the fifth order Birkhoff normal form

(could be Chaotic but with well defined flow)

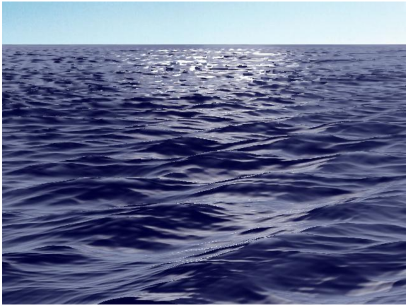
References: long time existence results

1. Berti M., Delort J.-M., *Almost Global Solutions of Capillary-gravity Water Waves Equations on the Circle*. UMI Lecture Notes 2018, ISBN 978-3-319-99486-4.
2. Berti M., Feola R., Franzoi L., *Quadratic life span of periodic gravity-capillary water waves*. Water Waves 3(1): 85-115, 2021.
3. Berti M., Feola R., Pusateri F., *Birkhoff Normal Form and Long Time Existence for Periodic Gravity Water Waves*. Comm. Pure Applied Math., 76, 7, Pages 1416-1494, 2023
4. M. Berti, A. Maspero, F. Murgante, *"Hamiltonian Birkhoff normal form for gravity-capillary water waves with constant vorticity: almost global existence"*, arxiv.org/abs/2212.12255, 2022.

Hamiltonian methods in water waves

Mittag-Leffler, Stockholm 7-8 September 2023

Massimiliano Berti, SISSA



1. Finite dimensional case
2. Semilinear PDEs
3. Quasi-linear PDEs

1. H. Hofer, E. Zehnder, *Symplectic invariants and Hamiltonian dynamics*, Birkhäuser, Chapter 1
2. B. Grebért, *Birkhoff normal form and Hamiltonian PDEs*, Lecture notes
3. D. Bambusi, *An introduction to Birkhoff normal form*, Lecture notes

Phase space $\mathbb{R}^{2n}$ with coordinates $(q,p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$

Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$

$$\dot q_j = \partial_{p_j} H\,, \quad \dot p_j = -\partial_{q_j} H\,, \quad j = 1, \ldots, n$$

Hamiltonian vector field

$$X_H = J \nabla_{(q,p)} H\,, \quad J = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$$

Symplectic form

$$dH[\cdot] = \Omega(X_H, \cdot), \quad \Omega = \sum_{j=1}^n dp_j \wedge dq_j\,, \quad \Omega(v_1, v_2) = (Ev_1, v_2)_{\mathbb{R}^{2n}},\ E = J^{-1} = -J$$

symplectic tensor E non-degenerate, i.e. E invertible, and antisymmetric, i.e. $E^T = -E$

Poisson bracket

$$\{F, G\} = \sum_{j=1}^n \partial_{p_j} F \partial_{q_j} G - \partial_{p_j} G \partial_{q_j} F = \Omega(X_F, X_G)$$

Hamiltonian

$$H(q, p) = \sum_{j=1}^{n} \omega_j \frac{p_j^2 + q_j^2}{2}$$

Hamilton's Equation of motion

$$\dot{q}_j = \omega_j p_j \, , \quad \dot{p}_j = -\omega_j q_j \, , \quad j = 1, \ldots, n$$

The "actions" $\frac{p_j^2 + q_j^2}{2}$ are prime-integrals. Orbits included in tori:

$$I_j(t) = \frac{p_j^2(t) + q_j^2(t)}{2} = \frac{p_j^2(0) + q_j^2(0)}{2}$$

Harmonic oscillator Hamiltonian

$$H = \sum_{j=1}^n \omega_j \frac{p_j^2+q_j^2}{2}$$

Action-angle variables

$$q_j = \sqrt{2I_j}\cos(\theta_j)\,, \quad p_j = -\sqrt{2I_j}\sin(\theta_j)$$

Symplectic form

$$\Omega = dI \wedge d\theta = \sum_{j=1}^n dI_j \wedge d\theta_j$$

Hamiltonian

$$H = \omega \cdot I = \sum_{j=1}^n \omega_j I_j, \quad \omega := (\omega_1, \dots, \omega_n) \text{ frequency vector}$$

Hamilton's equations

$$\dot\theta = \partial_I H\,, \quad \dot I = -\partial_\theta H\,,$$
$$\dot\theta_j = \omega_j\,,\ \dot I_j = 0\,, \quad \theta_j(t) = \theta_j(0) + \omega_j t\,,\ I_j(t) = I_j(0)$$

actions $(I_j)_{j=1,\dots,n}$ introduced as coordinates, angles $(\theta_j)_{j=1,\dots,n}$ rotate with frequencies ω_j

complex variables

$$u_j := \frac{p_j + \mathrm{i} q_j}{\sqrt{2}}, \qquad I_j = |u_j|^2 = u_j \overline{u_j}$$

Hamiltonian system

$$\dot{u}_j = \mathrm{i}\partial_{\bar{u}_j} H, \quad j = 1, \ldots, n$$

$$\partial_{\bar{u}_j} := \frac{1}{\sqrt{2}}(\partial_{q_j} + \mathrm{i}\partial_{p_j}), \quad \partial_{u_j} := \frac{1}{\sqrt{2}}(\partial_{q_j} - \mathrm{i}\partial_{p_j})$$

symplectic form

$$\Omega = \frac{1}{\mathrm{i}} \sum_{j=1}^{n} du_j \wedge d\bar{u}_j$$

Poisson bracket

$$\{F, G\} = \frac{1}{\mathrm{i}} \sum_{j=1}^{n} \left(\partial_{u_j} F \partial_{\bar{u}_j} G - \partial_{u_j} G \partial_{\bar{u}_j} F \right)$$

Harmonic oscillators

$$H_2 = \sum_{j=1}^{n} \omega_j u_j \overline{u_j}, \quad \dot{u}_j = \mathrm{i}\omega_j u_j, \quad u_j(t) = u_j(0) e^{\mathrm{i}\omega_j t}$$

motion = rotation in the complex plane of angle $\omega_j t$

Hamiltonian vector field X_H

$$u_t = X_H(u)$$

Φ is a Symplectic diffeomorphism $u = \Phi(v)$

$$\Phi^* \Omega = \Omega \,, \;\; i.e. \quad \Omega(d\Phi(v)\hat{v}_1, d\Phi(v)\hat{v}_2) = \Omega(\hat{v}_1, \hat{v}_2) \,, \;\; \forall \hat{v}_1, \hat{v}_2 \,,$$

$$(d\Phi(v))^\top E \, d\Phi(v) = E$$

New Hamiltonian system

$$v_t = X_K(v), \quad K = H \circ \Phi$$

$$H = H^{(2)} + \underbrace{H^{(3)} + H^{(4)} + \ldots}_{=:P}$$

where

$$H^{(2)} := \sum_{j=1}^{n} \omega_j |u_j|^2 \,, \qquad H^{(m)} = \sum_{\alpha,\beta \in \mathbb{N}^n, |\alpha|+|\beta|=m} c_{\alpha,\beta}\, u^\alpha \, \bar{u}^\beta$$

is a polynomial of order m

$$u^\alpha \, \bar{u}^\beta = u_1^{\alpha_1} \ldots u_n^{\alpha_n} \bar{u}_1^{\beta_1} \ldots \bar{u}_n^{\beta_n}, \quad \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \,, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n$$

Remark: if $\nabla H(0) = 0$ and $d^2 H(0)$ is positive definite, there exists a symplectic linear change of variables in which the Hamiltonian assumes this form (Weirstrass, see Hofer-Zehnder)

Question: *Is there a canonical change of variables in which the Hamiltonian assumes a simpler form? For example is it possible to remove the cubic terms? and the fourth order ones? etc*

Hamiltonian Birkhoff normal form theorem

Assume $H = H^{(2)} + P$ with P smooth and vanishing in a cubic way at the origin $u = 0$. For any $r \geq 3$, there exists a symplectic change of coordinates $(\Phi - \mathrm{Id})(u, \bar{u}) = O(|u|^2)$, defined in a small neighborhood of 0, such that

$$H \circ \Phi = H^{(2)} + Z + R$$

where Z is a polynomial of order r such that

$$\{H^{(2)}, Z\} = 0$$

and R vanishes in $(u, \bar{u})$ with order $r + 1$

$$\{H^{(2)}, u^\alpha \bar{u}^\beta\} = \mathrm{i}\omega \cdot (\alpha - \beta) u^\alpha \bar{u}^\beta \qquad \Longrightarrow \qquad Z = \sum_{\omega \cdot (\alpha - \beta) = 0} c_{\alpha,\beta} u^\alpha \bar{u}^\beta$$

Remark: advantage of complex coordinates : $\mathrm{Ad}_{H^{(2)}} := \{H^{(2)}, \ \}$ has eigenvectors $u^\alpha \bar{u}^\beta$ with eigenvalues $\mathrm{i}\omega \cdot (\alpha - \beta)$

Non resonant case

$\omega \cdot k \neq 0$ for any $0 < |k| \leq r \implies \omega \cdot (\alpha - \beta) = 0$ only if $\alpha = \beta \implies$

$$Z = \sum_{\alpha} c_\alpha u^\alpha \bar{u}^\alpha = \prod_{j=1}^{n} |u_j|^{2\alpha_j}$$

depends only on the actions $I_j := |u_j|^2$

Dynamical consequence:

$$\frac{d}{dt} I_j = \{I_j, H^{(2)} + Z\} = 0$$

$\implies I_j$ are prime-integrals of Z

$$\frac{d}{dt} I_j = \{I_j, H^{(2)} + Z + R\} = \{I_j, R\} = O(|I|^{\frac{r+1}{2}})$$

so

$$|I(t)| \leq |I(0)| + C \int_0^t |I(\tau)|^{\frac{r+1}{2}} d\tau$$

Claim

There exists $c > 0$ such that if $I(0) = \varepsilon^2$ then $|I(t)| \leq 2\varepsilon^2$ for any $0 < t < c\varepsilon^{-(r-1)}$

Boostrap argument:

$$E := \Big\{ t > 0 \ : \ |I(t)| \leq 2\varepsilon^2 \Big\}, \quad E \neq \emptyset, \quad T := \sup E > 0$$

Or $T = +\infty$ or $T < +\infty$. Claim $\exists \underline{c} > 0$ such that $T > \underline{c}\varepsilon^{-(r-1)}$. If not $\forall c > 0$ we have $T \leq c\varepsilon^{-(r-1)}$ so

$$|I(T)| \leq |I(0)| + C \int_0^T |I(\tau)|^{\frac{r+1}{2}} d\tau \leq \varepsilon^2 + TC(2\varepsilon^2)^{\frac{r+1}{2}} \leq \varepsilon^2 + c\varepsilon^{r-1} C(2\varepsilon^2)^{\frac{r+1}{2}} \leq \frac{3}{2}\varepsilon^2$$

for $c > 0$ small enough. Contradict that $T = \sup E$

1. If there are multiple frequencies :
   what about stability of the dynamics of the normal form ?
2. case $\Big(\omega_1 = \omega_2, (\omega_j)_{3 \leq j \leq n}\Big)$ non-resonant. Then

$$\omega_1(\alpha_1 - \beta_1 + \alpha_2 - \beta_2) + \omega_3(\alpha_3 - \beta_3) + \ldots + \omega_n(\alpha_n - \beta_n) = 0$$

if and only if $\alpha_1 - \beta_1 + \alpha_2 - \beta_2 = 0$, $\alpha_j = \beta_j$ for $j = 3, \ldots, n$

Super-action

$$J_1 := I_1 + I_2 = |u_1|^2 + |u_2|^2\,, \ \ I_j = |u_j|^2\,, \ j = 3, \ldots, n$$

$J_1, \ I_3, \ldots, I_n$ are prime integrals of normal form Z

$$\tfrac{d}{dt} J_1 = \{J_1, u^\alpha \bar{u}^\beta\} = \{|u_1|^2 + |u_2|^2, u^\alpha \bar{u}^\beta\} = \mathrm{i}(\alpha_1 - \beta_1 + \alpha_2 - \beta_2) u^\alpha \bar{u}^\beta = 0$$

Can be generalized

- Our case $\Omega_j = \Omega_{-j}$ for any $j \in \mathbb{Z}$, double frequencies
- Thus we require $(\Omega_{|j|})$ are non-resonant
- Restriction on parameters to ensure the absence of *N-wave resonant interactions*:

$$\left|\Omega_{j_1}(\kappa) + \ldots + \Omega_{j_p}(\kappa) - \Omega_{j_{p+1}}(\kappa) - \ldots - \Omega_{j_N}(\kappa)\right| \gtrsim \max(|j_1|, \ldots, |j_N|)^{-\tau}$$

among integers $j_1, \ldots, j_p, j_{p+1}, \ldots, j_N$ which are *not super-action preserving*, namely

$$\{|j_1|, \ldots, |j_p|\} \neq \{|j_{p+1}|, \ldots, |j_N|\}$$

- If N odd we eliminate all the monomials,
- If N even we keep only the super-action preserving monomials; for example if $N = 4$

$$|u_j|^2 |u_k|^2 , \quad |u_j|^2 u_k \overline{u_{-k}} , \quad u_j \overline{u_{-j}} u_k \overline{u_{-k}}$$

Lemma

The flow Φ_F^τ at time τ of a Hamiltonian vector field X_F

$$\Phi_F(\tau) : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad \Phi_F(\tau)[u_0] = u(\tau), \quad \Phi_F(0) = \mathrm{Id}$$

$$\frac{d}{d\tau} u(\tau) = X_F(u(\tau)), \quad u(0) = u_0,$$

is symplectic.

$\Longrightarrow$ it is sufficient to transform the Hamiltonian

$$H \circ \Phi_F(\tau)$$

Set $\mathrm{Ad}_F H := \{F, H\}$

Lemma (Lie expansion)

$$H \circ \Phi_F(1) = \sum_{\ell=0}^{L} \frac{1}{\ell!} \mathrm{Ad}_F^{\ell} H + \frac{1}{L!} \int_0^1 (1-\tau)^L \mathrm{Ad}_F^L H \circ \Phi_F(\tau)\, d\tau$$
$$= H + \{F, H\} + \frac{1}{2}\{F, \{F, H\}\} + \dots$$

Proof: Taylor expansion of $H \circ \Phi_F(\tau)$ at $\tau = 0$. We have

$$\frac{d}{d\tau} H \circ \Phi_F(\tau) = \{F, H\} \circ \Phi_F(\tau) = \mathrm{Ad}_F(H) \circ \Phi_F(\tau)\,.$$

Iterating

$$\frac{d^{\ell}}{d^{\ell}\tau} H \circ \Phi_F(\tau) = \mathrm{Ad}_F^{\ell} H \circ \Phi_F(\tau)$$

Aim: kill

$$H^{(3)} = \sum_{|\alpha|+|\beta|=3} c_{\alpha,\beta} u^\alpha \bar{u}^\beta \,, \qquad u^\alpha \bar{u}^\beta := \prod_{j=1}^n u_j^{\alpha_j} \prod_{j=1}^n \bar{u}_j^{\beta_j}$$

Take an auxiliary cubic Hamiltonian

$$F^{(3)} = \sum_{|\alpha|+|\beta|=3} f_{\alpha,\beta} u^\alpha \bar{u}^\beta$$

Transformed Hamiltonian under the flow of $X_{F^{(3)}}$

$$H + \{F^{(3)}, H\} + \tfrac{1}{2}\{F^{(3)}, \{F^{(3)}, H\}\} + \dots$$
$$= H^{(2)} + \underbrace{H^{(3)} + \{F^{(3)}, H^{(2)}\}}_{new\ \ cubic\ \ term} + quartic\ \ monomials$$

$$H^{(3)} + \{F^{(3)}, H^{(2)}\} = \sum_{\alpha,\beta} \big(H^{(3)}_{\alpha,\beta} + \mathrm{i}\omega \cdot (\alpha - \beta) F^{(3)}_{\alpha,\beta} \big) u^\alpha \bar{u}^\beta$$

if $\omega \cdot (\alpha - \beta) \neq 0$ then $F^{(3)}_{\alpha,\beta} := -\frac{H^{(3)}_{\alpha,\beta}}{\mathrm{i}\omega\cdot(\alpha-\beta)}$

Higher orders: by induction.

Remarks

- The subsequent transformations are closer and closer to identity and do not change the lower order normal form
- In the non-resonant case the normal form is unique.

  Important: thus whatever is the method and order of Birkhoff transformations the normal form is uniquely determined

ω non-resonant at any order, diophantine

$$|\omega \cdot k| \geq \tfrac{\gamma}{|k|^\tau}\,, \quad \forall k \in \mathbb{Z}^n \setminus \{0\}\,,$$

Compute the dependence of constants on r and optimize

Time of stability for any $r \in \mathbb{N}$

$$T_r = \tfrac{c_r}{\varepsilon^{r-1}},\ c_r = C(r!)^{-(\tau+1)} \quad \Longrightarrow$$

$$T_r = \frac{C}{\varepsilon^{r-1}(r!)^{\tau+1}} \overset{Stirling}{\approx} \frac{e^{r(\tau+1)}}{(r^{\tau+1}\varepsilon)^r}$$

$$r = \varepsilon^{-\frac{1}{\tau+1}} \quad \Longrightarrow \quad T_\varepsilon \leq e^{-\frac{C}{\varepsilon^\beta}}$$

All previous estimates depend on n and for PDEs $n = +\infty$

In finite dimension

$$H = \sum_{j=1}^{n} \omega_j u_j \overline{u_j} + H^{(3)} + H^{(4)} + \dots$$

we used non-resonance condition for $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$:

$$\omega \cdot k \neq 0\,, \ \forall 0 < |k| \leq r\,, \ k \in \mathbb{Z}^n \quad \Longrightarrow \quad \min_{0<|k|\leq r} |\omega \cdot k| > 0$$

For infinitely many frequencies $(\omega_j)_{j\in\mathbb{Z}} \in \mathbb{R}^{\mathbb{Z}}$, in general

$$\inf_{0<|k|\leq r, k\in\mathbb{Z}^\infty} |\omega \cdot k| = 0$$

Example $r = 3$. $Klein - Gordon : \omega_j = \sqrt{j^2 + m} = j + O(\frac{1}{j})$, $j > 0$

$$\omega_{j_1} - \omega_{j_2} - \omega_{j_3} = j_1 - j_2 - j_3 + O(\tfrac{1}{j_1}) + O(\tfrac{1}{j_2}) + O(\tfrac{1}{j_3})$$

- Hamiltonian **semilinear** PDEs

$$u_t + \mathrm{i}\Omega(D)u = f(u), \qquad f(u) \text{ no derivatives of } u$$

Bambusi, Grébert, Delort, Szeftel, '03, '06, '07

Examples:

Hamiltonian Wave equation

$$y_{tt} - y_{xx} + V(x)y = g(x,u), \quad x \in \mathbb{T}$$

Hamiltonian Schödinger

$$\mathrm{i}u_t = \partial_{xx}u + V(x)u + \partial_{\bar u}G(x,u,\bar u), \quad x \in \mathbb{T}$$

Phase space: Sobolev spaces

$$H^s(\mathbb{T}) := \Big\{ u(x) = \sum_{j\in\mathbb{Z}} u_j e^{\mathrm{i}jx} \ : \ \|u\|_s^2 := \sum_{j\in\mathbb{Z}} |u_j|^2 \langle j \rangle^{2s} < +\infty \Big\}$$

$$\langle j \rangle := \max\{1, |j|\}$$

Identify $u(x)$ with the sequence $(u_j)_{j\in\mathbb{Z}}$

Hamiltonian equation $\partial_t u = \mathrm{i}\nabla_{\bar u} H(u, \bar u)$

$$\dot u_j = (X_H)_j = \mathrm{i}\partial_{\bar u_j} H \,, \quad \forall j \in \mathbb{Z}$$

Example : cubic NLS

$$H(u, \bar u) = \int_{\mathbb{T}} |u_x|^2 dx + \tfrac{1}{2} \int_{\mathbb{T}} |u|^4 dx$$

$$\partial_t u + \mathrm{i} u_{xx} = \mathrm{i}|u|^2 u$$

$$\partial_t u_j + \mathrm{i}j^2 u_j = \mathrm{i} \sum_{j_1 - j_2 + j_3 = j} u_{j_1} \bar u_{j_2} u_{j_3}$$

Are the Birkhoff transformations well defined?

Is the auxiliary flow which generates the Birkhoff transformations

$$\partial_\tau u = X_F(u)$$

well defined in H^s?

Example: to remove the cubic Hamiltonian

$$H^{(3)} = \sum_{j_1, j_2, j_3 \in \mathbb{Z}} H_{j_1, j_2, j_3} u_{j_1} u_{j_2} \bar{u}_{j_3}$$

the auxiliary Hamiltonian is

$$F^{(3)} = \sum_{j_1, j_2, j_3 \in \mathbb{Z}} \frac{H_{j_1, j_2, j_3}}{\mathrm{i}(\Omega_{j_1} + \Omega_{j_2} - \Omega_{j_3})} u_{j_1} u_{j_2} \bar{u}_{j_3}$$

Questions

1. Which growth conditions for H_{j_1, j_2, j_3} in j_1, j_2, j_3?
2. Which lower bounds for $|\Omega_{j_1} + \Omega_{j_2} - \Omega_{j_3}|$?

Semi-linear Hamiltonians

$$H(u, \bar{u}) = \sum_{\substack{(j_1, \ldots, j_p) \in \mathbb{Z}^p, (\sigma_1, \ldots, \sigma_p) \in \{\pm\}^p \\ \sigma_1 j_1 + \ldots + \sigma_p j_p = 0}} H_{j_1, \ldots, j_p}^{\sigma_1, \ldots, \sigma_p} u_{j_1}^{\sigma_1} \ldots u_{j_p}^{\sigma_p}, \quad u^+ = u, u^- = \bar{u}$$

for some $\mu > 0$,

$$|H_{j_1, \ldots, j_p}^{\sigma_1, \ldots, \sigma_p}| \leq C \max_3 \{|j_1|, \ldots, |j_p|\}^\mu$$

$$\max_3\{n_1, \ldots, n_p\} := \text{third largest among integers } n_1, \ldots, n_p$$

Key properties

1. The Hamiltonian vector field X_H is bounded on H^s for any $s \geq s_0$
2. Stable class under solution of homological equation
3. contains

$$H(u, \bar{u}) = \int_{\mathbb{T}} |u|^4 dx = \sum_{j_1 - j_2 + j_3 - j_4 = 0} u_{j_1} \bar{u}_{j_2} u_{j_3} \bar{u}_{j_4}$$

Lemma

There exists $s_0 > 0$ such that the Hamiltonian vector field

$$X_H : H^s \to H^s \,, \quad \forall s \geq s_0$$

Example: Cubic Hamiltonian

$$H = \sum_{j_1+j_2-j=0} H^{+,+,-}_{j_1,j_2,j} u_{j_1} u_{j_2} \bar{u}_j$$

$$\dot{u}_j = [X_H]_j \,, \quad [X_H]_j = \mathrm{i} \sum_{j_1+j_2=j} H^{+,+,-}_{j_1,j_2,j} u_{j_1} u_{j_2} \,, \quad \forall j \in \mathbb{Z}$$

$$|H^{+,+,-}_{j_1,j_2,j}| \lesssim \max_3(|j_1|, |j_2|, |j|)^{\mu}$$

$$\max_3(|j_1|, |j_2|, |j|) = \min(|j_1|, |j_2|, |j|) \leq \min(|j_1|, |j_2|) = \max_2(|j_1|, |j_2|)$$

$$u * v = ((u * v)_j)_{j \in \mathbb{Z}}, \quad (u * v)_j := \sum_{j_1 + j_2 = j} u_{j_1} v_{j_2} = \sum_{j_1 \in \mathbb{Z}} u_{j_1} v_{j - j_1}$$

Young inequality

$$\| u * v \|_{\ell^2} \leq \| u \|_{\ell_1} \| v \|_{\ell^2}$$

PROOF.

$$\begin{aligned} \| u * v \|_{\ell^2}^2 &= \sum_j \Big| \sum_{j_1 \in \mathbb{Z}} u_{j_1} v_{j - j_1} \Big|^2 \leq \sum_{j \in \mathbb{Z}} \Big(\sum_{j_1 \in \mathbb{Z}} |u_{j_1}|^{\frac{1}{2}} |u_{j_1}|^{\frac{1}{2}} |v_{j - j_1}| \Big)^2 \\ &\leq \sum_j \Big(\sum_{j_1} |u_{j_1}| \Big) \sum_{j_1} |u_{j_1}| |v_{j - j_1}|^2 = \| u \|_{\ell^1} \sum_{j, j_1} |u_{j_1}| |v_{j - j_1}|^2 \\ &= \| u \|_{\ell^1} \sum_{j_1} |u_{j_1}| \sum_j |v_{j - j_1}|^2 = \| u \|_{\ell^1}^2 \| v \|_{\ell^2}^2 \end{aligned}$$

Exercise: ℓ^1 is an algebra

$$\|u * v\|_{\ell^1} \leq \|u\|_{\ell_1} \|v\|_{\ell^1}$$

Young inequality and algebra of ℓ^1 imply

Exercise: iterated Young inequality

$$\|u^{(1)} * \ldots u^{(n-1)} * u^{(n)}\|_{\ell^2} \leq \|u^{(1)}\|_{\ell^1} \ldots \|u^{(n-1)}\|_{\ell^1} \|u^{(n)}\|_{\ell^2}$$

Sobolev embedding: for $s > 1/2$

$$\|(|u_j|)\|_{\ell^1} = \sum_{j \in \mathbb{Z}} |u_j| \leq \Big(\sum_{j \in \mathbb{Z}} |u_j|^2 \langle j \rangle^{2s} \Big)^{\frac{1}{2}} \Big(\sum_{j \in \mathbb{Z}} \langle j \rangle^{-2s} \Big)^{\frac{1}{2}} \lesssim_s \|u\|_s$$

$$
\begin{aligned}
\|X_H(u)\|_s^2 &\lesssim \sum_{j\in\mathbb{Z}} \langle j\rangle^{2s} \Big(\sum_{j_1+j_2=j} |u_{j_1}||u_{j_2}| \max_3(\langle j_1\rangle, \langle j_2\rangle, \langle j\rangle)^{\mu} \Big)^2 \\
&\overset{\color{red}{s\geq 0}}{\lesssim} \sum_{j\in\mathbb{Z}} \Big(\sum_{j_1+j_2=j} \max(\langle j_1\rangle, \langle j_2\rangle)^s |u_{j_1}||u_{j_2}| \max_2(\langle j_1\rangle, \langle j_2\rangle)^{\mu} \Big)^2 \\
&\lesssim (I) + (II) \quad \text{where}
\end{aligned}
$$

$$
\begin{aligned}
(I) := \sum_j \Big(\sum_{j_1+j_2=j, \color{red}{|j_2|\leq|j_1|}} (\langle j_1\rangle^s |u_{j_1}|)(|u_{j_2}|\langle j_2\rangle^{\mu}) \Big)^2 &\lesssim \Big\| (\langle j_1\rangle^s |u_{j_1}|) * (|u_{j_2}|\langle j_2\rangle^{\mu}) \Big\|_{\ell^2}^2 \\
&\overset{Young}{\leq} \Big\| (\langle j_1\rangle^s |u_{j_1}|) \Big\|_{\ell^2}^2 \Big\| (|u_{j_2}|\langle j_2\rangle^{\mu}) \Big\|_{\ell^1}^2 \\
&\overset{Sobolev\ \ embedding}{\lesssim_s} \|u\|_s^2 \|u\|_{\mu+1}^2
\end{aligned}
$$

the contribution (II) is similar $\Longrightarrow$

$$
\|X_H(u)\|_s \lesssim_s \|u\|_{s_0} \|u\|_s \,, \quad \forall s \geq s_0 := \mu + 1
$$

$$F^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3} = \frac{H^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3}}{\mathrm{i}(\sigma_1\Omega_{j_1} + \sigma_2\Omega_{j_2} + \sigma_3\Omega_{j_3})}$$

Small divisors

$$|\sigma_1\Omega_{j_1} + \sigma_2\Omega_{j_2} + \sigma_3\Omega_{j_3}| \geq \frac{c}{\max_3\{|j_1|, |j_2|, |j_3|\}^\tau}$$

$$|F^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3}| \lesssim \underbrace{|H^{\sigma_1,\sigma_2,\sigma_3}_{j_1,j_2,j_3}|}_{\leq \max_3\{|j_1|,|j_2|,|j_3|\}^\mu} \max_3\{|j_1|, |j_2|, |j_3|\}^\tau \lesssim \max_3\{|j_1|, |j_2|, |j_3|\}^{\mu+\tau}$$

1. If the nonlinearity $f(u)$ contains derivatives then $f(u)$ is unbounded

Hamiltonians with m-derivatives

$$|H^{\sigma_1,\ldots,\sigma_p}_{j_1,\ldots,j_p}| \leq C \max_3\{|j_1|,\ldots,|j_p|\}^{\mu} \max\{|j_1|,\ldots,|j_p|\}^{m}$$

$$X_H : H^s \mapsto H^{s-m}, \quad m > 0$$

In general $\partial_\tau u = X_H(u, \bar{u})$ does not define a flow

2. What to do with only weak non-resonance conditions

$$|\Omega_{j_1} \pm \ldots \pm \Omega_{j_p}| \geq \frac{\gamma}{\max(|j_1|,\ldots,|j_p|)^{\tau}}$$

which are small in the biggest frequency! (loss of derivatives)

CHANGE OF PARADIGM

- Not reduce first the nonlinearity in sizes of u but in decreasing orders of operators:

$$\underbrace{u^2(x)}_{\varepsilon^2}\,\partial_{xx} u \text{ is much bigger that } \underbrace{u(x)}_{\varepsilon}\,\partial_x u \text{ acting on } e^{\mathrm{i}jx} \text{ for } j \gg 1$$

New procedure:

1. Paradifferential normal form:
   transform the water waves equations to a
   diagonal, constant coefficients in x paradifferential system
   up to **smoothing remainders**
   - Originated in KAM for quasi-linear PDEs, Berti, Baldi, Montalto. Reduction in order of linearized operator;
   - Nonlinear version: para-linearization of vector field; Berti-Delort,
2. Then implement "semilinear" normal form transformations which reduce the *size* of the nonlinear terms

Example: quasi-linear perturbation of KdV

$$u_t = u_{xxx} + u_{xxx}^3$$

Quasi-linear, duhamel iteration fails. Use Nash-Moser

Linearized equation

$$h_t = (1 + 3u_{xxx}^2(t,x))h_{xxx}$$

Strategy 1. Do at black-board. Reduce to constant coefficients

$$h_t = (1 + m_3)h_{xxx} + \text{lower order terms}$$

In this new coordinates it is constant coefficients. The transformations are composition operators: $x + \beta(t,x)$. Linearly symplectic version $(1 + \beta_x(x))u(x + \beta(x))$

Paralinearize

$$u_t = \mathrm{Op}^B(1 + 3u_{xxx}^2)(i\xi)^3)u + \underbrace{R(u)[u]}_{smoothing}$$

similarly reduce to constant coefficients. Paracomposition.

Outline

1. Paradifferential calculus

2. Birkhoff normal form for Hamiltonian Quasi-linear PDEs

Symbols $a \in \Sigma\Gamma_p^m$. m = order of symbol, p = size in $O(\|u\|^p)$

1. $a(u;x,\xi) = \sum_{q=p}^{N-1} a_q(u;x,\xi) + a_N(u;x,\xi)$ with $a_q \in \Gamma_q^m$ and $a_N = O(\|u\|^N)$
2. **Homogeneous symbol:**

$$a_q(u;x,\xi) = \sum_{(j_1,\ldots,j_q)\in\mathbb{Z}^q,(\sigma_1,\ldots,\sigma_q)\in\{\pm\}^q} a_{j_1,\ldots,j_q}^{\sigma_1,\ldots,\sigma_q}(\xi) u_{j_1}^{\sigma_1} \ldots u_{j_q}^{\sigma_q} e^{\mathrm{i}(\sigma_1 j_1+\ldots+\sigma_q j_q)x}$$

   for some $\mu \geq 0$, $\forall \beta \in \mathbb{N}$,

$$|\partial_\xi^\beta a_{j_1,\ldots,j_q}^{\sigma_1,\ldots,\sigma_q}(\xi)| \leq C|(j_1,\ldots,j_q)|^\mu \langle\xi\rangle^{m-\beta}$$

3. **Non-homogeneous symbol :** $\forall \alpha, \beta$ in $\mathbb{N}$, with $\alpha \leq s - s_0$

$$|\partial_x^\alpha \partial_\xi^\beta a(u;x,\xi)| \leq C\langle\xi\rangle^{m-\beta} \|u\|_{s_0}^{q-1} \|u\|_s$$

Exercise 1 : $a \in \Gamma_p^m \quad \Longrightarrow \quad \partial_x a \in \Gamma_p^m, \partial_\xi a \in \Gamma_p^{m-1}$

Exercise 2 : $u_x^2(x)\mathrm{i}\xi$ is a symbol in Γ_2^1, $\quad u_x^2(x)\mathrm{i}\xi = -\mathrm{i}\sum_{j_1,j_2} j_1 j_2 u_{j_1} u_{j_2} \xi e^{\mathrm{i}(j_1+j_2)x}$

Bony-Weyl quantization

$$\mathrm{Op}^{BW}(a(u;x,\xi)) = \mathrm{Op}^{W}(a_{\chi_q}(u;x,\xi))$$

where

$$a_{\chi_q}(u;x,\xi) := \sum_{\substack{(j_1,\ldots,j_q)\in\mathbb{Z}^q,(\sigma_1,\ldots,\sigma_q)\in\{\pm\}^q,\\ |j_1|,\ldots,|j_q|\le\delta\langle\xi\rangle}} a_{j_1,\ldots,j_q}^{\sigma_1,\ldots,\sigma_q}(\xi) u_{j_1}^{\sigma_1}\ldots u_{j_q}^{\sigma_q} e^{\mathrm{i}(\sigma_1 j_1+\ldots+\sigma_q j_q)x}$$

Weyl quantization

$$u(x) = \frac{1}{\sqrt{2\pi}}\sum_{j\in\mathbb{Z}} u_j e^{ijx}$$

$$\mathrm{Op}^{W}(a(x,\xi))u = \frac{1}{\sqrt{2\pi}}\sum_k \Big(\sum_j \hat{a}(k-j,\tfrac{k+j}{2})u_j\Big)e^{\mathrm{i}kx}$$

Advantage of Weyl $(\mathrm{Op}^{BW}(a))^* = (\mathrm{Op}^{BW}(\bar{a}))$

Standard quantization

$$\mathrm{Op}(a(x,\xi))u = \frac{1}{\sqrt{2\pi}}\sum_k \Big(\sum_j \hat{a}(k-j,j)u_j\Big)e^{\mathrm{i}kx}$$

Bony-Weyl paradifferential operators

$$\mathrm{Op}^{BW}(a(u;x,\xi))v = \sum_{j_1,\ldots,j_q,j,k} a_{j_1,\ldots,j_q}^{\sigma_1,\ldots,\sigma_q}\Big(\frac{j+k}{2}\Big) u_{j_1}^{\sigma_1}\ldots u_{j_q}^{\sigma_q} v_j e^{\mathrm{i}kx}$$

1. $|j_1|,\ldots,|j_q| \leq \delta|j|$, $\delta \ll 1$,
2. $k = \sigma_1 j_1 + \ldots + \sigma_q j_q + j$ (translation invariance)
3. $|j| \sim |k|$

Notation: $a_{j_1,\ldots,j_q}^{\sigma_1,\ldots,\sigma_q}(\xi) = a_{\vec{\jmath}}^{\vec{\sigma}}(\xi)$

Action on Sobolev spaces of a para-differential operator

Let $a \in \Gamma_q^m$. Then, $\exists s_0 > 1/2$, such that for any $s \in \mathbb{R}$,

$$\|\mathrm{Op}^{BW}(a(u;\cdot))v\|_{s-m} \leq C\|u\|_{s_0}^q\|v\|_s$$

$$\begin{aligned}
\|\mathrm{Op}^{BW}(a(u;\cdot))v\|_{s-m}^2 &\leq \sum_{k\in\mathbb{Z}} |k|^{2(s-m)} \Big(\sum_{j\sim k} \Big| a_{\vec{j}}^{\vec{\sigma}}\Big(\frac{j+k}{2}\Big)\Big| |u_{j_1}^{\sigma_1}| \ldots |u_{j_q}^{\sigma_q}||v_j| \Big)^2 \\
&\overset{|k|\sim|j|}{\lesssim} \sum_{k\in\mathbb{Z}} \Big(\sum_{j\sim k} |j|^{s-m}|j|^m |u_{j_1}^{\sigma_1}| \ldots |u_{j_q}^{\sigma_q}||v_j| \Big)^2 \\
&\lesssim \sum_{k\in\mathbb{Z}} \Big(\sum_{\sigma_1 j_1+\ldots+\sigma_q j_q = k} |u_{j_1}^{\sigma_1}| \ldots |u_{j_q}^{\sigma_q}||v_j||j|^s \Big)^2 \\
&= \Big\| (|u_j|) * \ldots * (|u_j|) * (|v_j||j|^s) \Big\|_{\ell^2}^2 \\
&\overset{Young\ +\ \ell^1\ is\ algebra}{\lesssim} \|(|u_j|)\|_{\ell^1}^q \| |v_j||j|^s \|_{\ell^2}^2 \\
&\overset{Sobolev\ \ embedding}{\lesssim} \|u\|_{s_0}^q \|v\|_s^2
\end{aligned}$$

Smoothing operators $\Sigma\mathcal{R}_p^{-\rho}$, $\rho > 0$

$$R(u)v = \sum_{q=p}^{N-1} R_q(u)v + R_N(u)v$$

1. **Homogeneous smoothing operators**

$$R_q(u)v = \sum_{(j_1,\ldots,j_q),j,(\sigma_1,\ldots,\sigma_q)} R^{\sigma_1,\ldots,\sigma_q}_{j_1,\ldots,j_q,j} u^{\sigma_1}_{j_1} \ldots u^{\sigma_q}_{j_q} v_j e^{\mathrm{i}(\sigma_1 j_1 + \ldots + \sigma_q j_q + j)x}$$

for some $\mu > 0$

$$|R^{\sigma_1,\ldots,\sigma_q}_{j_1,\ldots,j_q,j}| \lesssim \max_2(|j_1|,\ldots,|j_q|,|j|)^{\mu} \max(|j_1|,\ldots,|j_q|,|j|)^{-\rho}$$

2. **Non-homogeneous smoothing operators** $\mathcal{R}^{-\rho}$. $\exists \sigma > \mu : \forall u, v \in H^s$, $s + \rho > 0$,

$$\|R(u)[v]\|_{s+\rho} \lesssim_s \underbrace{\|u\|_\sigma^q \|v\|_s}_{\textit{if} \ \max(|j_1|,\ldots,|j_q|,|j|)=|j|} + \underbrace{\|u\|_\sigma^{q-1} \|u\|_s \|v\|_\sigma}_{\textit{if} \ \max(|j_1|,\ldots,|j_q|,|j|)=\max(|j_1|,\ldots,|j_q|)}$$

Two types of smoothing operators

1. $R(u) = \mathrm{Op}^{BW}(a(u;x,\xi))$ with a symbol $a(u;x,\xi) \in \Gamma_q^{-\rho}$

$$\left| a_{j_1,\dots,j_q}^{\sigma_1,\dots,\sigma_q}\Big(\frac{j+k}{2}\Big)\right| \leq C \quad \underbrace{|(j_1,\dots,j_q)|^{\mu}}_{=\max_2(|j_1|,\dots,|j_q|,|j|)^{\mu}} \qquad \underbrace{\langle j\rangle^{-\rho}}_{=\max(|j_1|,\dots,|j_q|,|j|)^{-\rho}}$$

Arise as remainders of composition operators: see next slide

2. $|R_{j_1,\dots,j_q,j}^{\sigma_1,\dots,\sigma_q}| \lesssim \max(|j_1|,\dots,|j_q|,|j|)^{\tau}$ with support condition

$$\max(|j_1|,\dots,|j_q|,|j|) \sim \max_2(|j_1|,\dots,|j_q|,|j|)$$

$$|R_{j_1,\dots,j_q,j}^{\sigma_1,\dots,\sigma_q}| \lesssim \max(|j_1|,\dots,|j_q|,|j|)^{\tau} \sim \max_2(|j_1|,\dots,|j_q|,|j|)^{\tau+\rho} \max(|j_1|,\dots,|j_q|,|j|)^{-\rho}$$

$\Longrightarrow$ that $R(u)$ is smoothing for any $\rho > 0$, with $\mu = \tau + \rho$, thus estimates for $\sigma \sim \rho$

Arise for example as remainders of Bony paradroducts : see later slide

Composition of paradifferential operators

Let $a \in \Sigma\Gamma_p^m$, $b \in \Sigma\Gamma_q^{m'}$. Then

$$\mathrm{Op}^{BW}(a) \circ \mathrm{Op}^{BW}(b) = \mathrm{Op}^{BW}((a\#b)_\rho) + R$$

where

$$(a\#b)_\rho = ab + \frac{1}{2\mathrm{i}}\{a,b\} + \dots \qquad \text{last term} \ \sim \partial_\xi^\rho a \, \partial_x^\rho b$$

and $R \in \Sigma\mathcal{R}_{p+q}^{-\rho+m+m'}$

Commutator

$$\left[\mathrm{Op}^{BW}(a), \mathrm{Op}^{BW}(b)\right] = \mathrm{Op}^{BW}\big(\tfrac{1}{\mathrm{i}}\{a,b\} + r_{-3}\big) + R$$

where the Poisson bracket

$$\{a,b\} := \partial_\xi a \partial_x b - \partial_x a \partial_\xi b$$

and $r_{-3} \in \Sigma\Gamma_{p+q}^{m+m'-3}$

This is the other main advantage of Weyl

Paraproduct

$$u^2 = \big(\mathrm{Op}^{BW}(2u)\big)u + R(u)u$$

and $R(u) \in \mathcal{R}_1^{-\rho}$ for any ρ in particular $\|R(u)u\|_{2s-\frac{1}{2}-} \lesssim \|u\|_s^2$

$$u^2 = \sum_n \sum_{n_1+n_2=n} u_{n_1} u_{n_2} e^{inx} = \underbrace{\sum_n \sum_{n_1+n_2=n, |n_1| \le \delta |n_2|} u_{n_1} u_{n_2} e^{inx}}_{=\mathrm{Op}^{BW}(u)u}$$

$$+ \underbrace{\sum_n \sum_{n_1+n_2=n, |n_2| \le \delta |n_1|} u_{n_2} u_{n_1} e^{inx}}_{=\mathrm{Op}^{BW}(u)u}$$

$$+ \underbrace{\sum_n \sum_{n_1+n_2=n, \delta |n_2| < |n_1| < \delta^{-1} |n_2|} u_{n_1} u_{n_2} e^{inx}}_{=:R(u)u}$$

Composition operator

$$u(x) \mapsto f(u)(x) := f(u(x))$$

Bony para-linearization

Let $f \in C^\infty$, $f(0) = f'(0) = 0$, and $u \in H^s$. Then

$$f(u) = \mathrm{Op}^{BW}(f'(u))u + R(u)u$$

where $R(u)u \in H^{2s-\frac{1}{2}-}$. Actually $R(u) \in \mathcal{R}^{-\rho}$: for all $s > \sigma$

$$\|R(u)v\|_{s+\rho} \lesssim_s \|u\|_s \|v\|_\sigma + \|u\|_\sigma \|v\|_s$$

Example with derivatives

$$
\begin{aligned}
u_x^2 = & \underbrace{\mathrm{Op}^{BW}(2u_x)[u_x]}_{=\mathrm{Op}^{BW}(2u_x)\mathrm{Op}^{BW}(\mathrm{i}\xi)[u]} + \underbrace{R(u)[u]}_{=-\sum_{\substack{j_1+j_2=j \\ |j_1|\sim|j_2|}} j_1 j_2 u_{j_1} u_{j_2} e^{\mathrm{i}jx}} \\
& = \mathrm{Op}^{BW}\Big(2u_x \mathrm{i}\xi + \underbrace{\frac{1}{2\mathrm{i}}\{2u_x, \mathrm{i}\xi\}}_{=u_{xx}}\Big)[u] + R(u)[u] \\
& = \mathrm{Op}^{BW}\big(\underbrace{a(u;x,\xi)}_{2u_x\mathrm{i}\xi + u_{xx} \in \Gamma_1^1} \big)[u] + \underbrace{R(u)[u]}_{\in \mathcal{R}^{-\rho}, \forall \rho>0}
\end{aligned}
$$

since

$$
\max\{|j_1|, |j_2|, |j|\} \sim \max_2\{|j_1|, |j_2|, |j|\}
$$

Indeed

$$
\max\{|j_1|, |j_2|, |j|\} \lesssim \max\{|j_1|, |j_2|\} \sim \max_2\{|j_1|, |j_2|\} \leq \max_2\{|j_1|, |j_2|, |j|\} \leq \max\{|j_1|, |j_2|, |j|\}
$$

- **Remark:** Arbitrariness in the cut-off : where to insert smoothing terms

Paralinearize an equation

$$\partial_t U = X(U), \quad U := \begin{pmatrix} u(x) \\ \bar{u}(x) \end{pmatrix}$$

means

$$\partial_t U = \underbrace{\mathrm{Op}^{BW}(A(U;x,\xi))U + R(U)[U]}_{=X(U)},$$

where $A(U;x,\xi)$ is a matrix of symbols and $R(U)$ are smoothing operators

Remark: The algebraic properties are preserved by paralinearization.

If X **is real-to-real**, i.e. X leaves invariant subspace of $U := \begin{pmatrix} u(x) \\ \bar{u}(x) \end{pmatrix}$, then

$$A(U;x,\xi) = \begin{pmatrix} a(U;x,\xi) & b(U;x,\xi) \\ \overline{b(U;x,-\xi)} & \overline{a(U;x,-\xi)} \end{pmatrix}$$

indeed

$$\overline{\mathrm{Op}^{BW}(a(x,\xi))} = \mathrm{Op}^{BW}(\overline{a(x,-\xi)})$$

Hamiltonian vector field

$$X(U) = J_c \nabla_{(u,\bar{u})} H(u,\bar{u}) = \begin{pmatrix} -\mathrm{i}\partial_{\bar{u}} H \\ \mathrm{i}\partial_u H \end{pmatrix}, \quad J_c := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

Linearly Hamiltonian structure of $X(U) = \mathrm{Op}^{BW}(A(U;x,\xi))U + R(U)[U]$

$$A(U;x,\xi) = J_c S(U), \quad S(U) := \begin{pmatrix} a(U;x,\xi) & b(U;x,\xi) \\ \overline{b(U;x,-\xi)} & \overline{a(U;x,\xi)} \end{pmatrix}, \quad \mathrm{Op}^{BW}(S) = (\mathrm{Op}^{BW}(S))^{\top}$$

$$\mathrm{Op}^{BW}(a) = \mathrm{Op}^{BW}(a)^{\top},\ a(U;x,\xi) = a(U;x,-\xi),\ \mathrm{Op}^{BW}(b) = \mathrm{Op}^{BW}(b)^{*},\ b(U;x,\xi) \in \mathbb{R}$$

transpose with respect to real scalar product:

$$\left\langle \begin{pmatrix} v_1^+ \\ v_1^- \end{pmatrix}, \begin{pmatrix} v_2^+ \\ v_2^- \end{pmatrix} \right\rangle_r := \langle v_1^+, v_2^+ \rangle_{L^2_r} + \langle v_1^-, v_2^- \rangle_{L^2_r}$$

Paradifferential Hamiltonian : $S(U)$ matrix of symbols in Γ_p^m

$$H(U) := \frac{1}{2}\langle \mathrm{Op}^{BW}(S(U))U, U\rangle_r$$

Then its gradient

$$\nabla H(U) = \mathrm{Op}^{BW}(S(U))U + R(U)U$$

where $R(U)$ is a real-to-real matrix of smoothing operators for any $\rho \geq 0$

$$dH(U)[V] = \langle \mathrm{Op}^{BW}(S(U))U, V\rangle_r + \Big\langle \underbrace{\mathrm{Op}^{BW}\big(\tfrac{1}{2}d_U S(U)[V]\big)U}_{=:L(U)V}, U\Big\rangle_r$$

$$\Rightarrow \quad \nabla H(U) = \mathrm{Op}^{BW}(S(U))U + L(U)^\top U$$

key property

Transposed map $L(U)^\top$ is a smoothing operator for any $\rho \geq 0$

Delort, Feola-Iandoli

Holds for more general

Spectrally localized maps $\mathcal{S}_p^m$

$$S(U)v = \sum_{(j_1,\ldots,j_p),j,k,(\sigma_1,\ldots,\sigma_p)} S^{\sigma_1,\ldots,\sigma_p}_{j_1,\ldots,j_p,j,k} u^{\sigma_1}_{j_1} \ldots u^{\sigma_p}_{j_p} v_j e^{ikx}$$

for some $\mu > 0$

$$|S^{\sigma_1,\ldots,\sigma_p}_{j_1,\ldots,j_p,j,k}| \lesssim \max_2(|j_1|,\ldots,|j_p|,|j|)^{\mu} \max(|j_1|,\ldots,|j_p|,|j|)^m$$

1. $|j_1|,\ldots,|j_p| \le \delta|j|$,
2. $k = \sigma_1 j_1 + \ldots + \sigma_p j_p + j$ (translation invariance)
3. $|j| \sim |k|$

$$S(U)v = \mathrm{Op}^{BW}(a(U;x,\xi))v$$

$$L(\underbrace{u,\dots,u}_{p})v =:= pS(v,\underbrace{u,\dots,u}_{p-1})u$$

Matrix entries

$$L(u)v = \sum_{k=j_1+\dots+j_q+j} L_{\vec{\jmath}_p,j,k}\, u_{j_1}\dots u_{j_p} v_j e^{\mathrm{i}kx}\,, \quad L_{\vec{\jmath}_p,j,k} = \big\langle L\big(e^{\mathrm{i}j_1x},\dots,e^{\mathrm{i}j_px}\big)\big[e^{\mathrm{i}jx}\big], e^{-\mathrm{i}kx}\big\rangle_r$$

Transpose

$$\begin{aligned}(L^\top)_{\vec{\jmath}_p,j,k} &= \big\langle L^\top\big(e^{\mathrm{i}j_1x},\dots,e^{\mathrm{i}j_px}\big)\big[e^{\mathrm{i}jx}\big], e^{-\mathrm{i}kx}\big\rangle_r \\ &= \big\langle e^{\mathrm{i}jx}, L\big(e^{\mathrm{i}j_1x},\dots,e^{\mathrm{i}j_px}\big)\big[e^{-\mathrm{i}kx}\big]\big\rangle_r \\ &= \big\langle L\big(e^{\mathrm{i}j_1x},\dots,e^{\mathrm{i}j_px}\big)\big[e^{-\mathrm{i}kx}\big], e^{\mathrm{i}jx}\big\rangle_r = L_{\vec{\jmath}_p,-k,-j}\end{aligned}$$

$$
\begin{aligned}
[L^\top]_{\vec{\jmath}_p, j, k} = L_{\vec{\jmath}_p, -k, -j} &= \big\langle L\big(e^{\mathrm{i} j_1 x}, \ldots, e^{\mathrm{i} j_p x}\big)\big[e^{-\mathrm{i} k x}\big], e^{\mathrm{i} j x}\big\rangle_r \\
&= p\big\langle S\big(e^{-\mathrm{i} k x}, e^{\mathrm{i} j_1 x}, \ldots, e^{\mathrm{i} j_{p-1} x}\big)\big[e^{\mathrm{i} j_p x}\big], e^{\mathrm{i} j x}\big\rangle_r \\
&= p\, S_{-k, j_1, \ldots, j_{p-1}, j_p, -j} \neq 0
\end{aligned}
$$

for indices satisfying

$$
\max\{|k|, |j_1|, \ldots, |j_{p-1}|\} \leq \delta |j_p|\,, \quad |j_p| \sim |j| \qquad \Rightarrow
$$

The operator $L^\top(U)$ is smoothing for any ρ

$\max(|j_1|, \ldots |j_{p-1}|, |j_p|, |j|) \lesssim \max(|j_p|, |j|) \sim \max_2(|j_p|, |j|) \lesssim \max_2(|j_1|, \ldots, |j_p|, |j|)$

Outline

Hamiltonian paradifferential Birkhoff normal form procedure:

1. Paralinearization of the PDE
2. **Paradifferential linearly Hamiltonian normal form** reduction
   transform the water waves equations to a
   diagonal, constant coefficients in x paradifferential system
   up to smoothing remainders
   preserving the Linearly Hamiltonian structure
3. **Symplectic correction up to homogeneity** N
4. Hamiltonian paradifferential Birkhoff normal form

Today: steps 1 and 2

Step 1: paralinearization of water waves: Alazard-Metivier, Alazard-Burq-Zuily, Berti-Delort, we add vorticity

$$\partial_t U = J_c \mathrm{Op}^{\mathrm{BW}}\Big(A_{\frac{3}{2}}(U;x)\omega(\xi)\Big)U + \frac{\gamma}{2}G(0)\partial_x^{-1}U$$
$$\underbrace{+J_c \mathrm{Op}^{\mathrm{BW}}\Big(A_1(U;x,\xi) + A_{\frac{1}{2}}(U;x,\xi) + A_0^{(2)}(U;x,\xi)\Big)U + R(U)U}_{Hamiltonian\ \ vector\ \ field\ \ J_c\nabla H}$$

where

- $\omega(j) = \sqrt{|j|\tanh(\mathtt{h}|j|)\Big(g + \kappa j^2 + \frac{\gamma^2}{4}\frac{G(j)}{j^2}\Big)}$

$$A_{\frac{3}{2}}(U;x) := \begin{pmatrix} \underbrace{f(U;x)}_{even\ \ in\ \ \xi} & \underbrace{1+f(U;x)}_{real} \\ 1+f(U;x) & f(U;x) \end{pmatrix}, \quad f(U;x) = O(\|U\|)$$

$$A_1(U;x,\xi) := \begin{pmatrix} \underbrace{\mathrm{i}B(U;x)|\xi|}_{even\ \ in\ \ \xi} & \underbrace{-V(U;x)\xi}_{real} \\ V(U;x)\xi & -\mathrm{i}B(U;x)|\xi| \end{pmatrix}$$

$A_{\frac{1}{2}}(U;x,\xi)$ are symbols of order $1/2$
$A_0^{(2)}(U;x,\xi)$ are symbols of order 0 and $R(U)$ are smoothing operators

- Most complicated step paralinearization of Dirichlet-Neumann operator
- We have to recognize the linearly Hamiltonian structure of the sysmbols

$$W = \begin{pmatrix} w \\ \bar{w} \end{pmatrix} = \Phi(U)U\,,\ \underbrace{\Phi(U) : \dot{H}^s(\mathbb{T}, \mathbb{C}^2) \to \dot{H}^s(\mathbb{T}, \mathbb{C}^2)}_{linear\ \ operator}\,,\ U = \begin{pmatrix} u \\ \bar{u} \end{pmatrix}\ ,\ \|W(t)\|_s \sim_s \|U(t)\|_s$$

Symmetrization and Reduction to constant coefficient symbols up to smoothing remainders (Alinhac good unknown, paracomposition, ...)

$$\partial_t w = \underbrace{\mathrm{i}\,\mathrm{Op}^{BW}\Big(m_{\frac{3}{2}}(W;\xi)\Big)}_{paradifferential\ \ operator} w + \underbrace{R(W)}_{smoothing}\ w$$

$$m_{\frac{3}{2}}(W;\xi) := -\underbrace{(1+\zeta(W))\omega(\xi) - \frac{\gamma}{2}\frac{\mathtt{G}(\xi)}{\xi}}_{modification\ \ of\ \ dispersion} - \underbrace{\mathtt{V}(W)\xi}_{transport} - b_{\frac{1}{2}}(W)|\xi|^{\frac{1}{2}} - \underbrace{b_0(W;\xi)}_{0\ \ order\ \ terms}$$

- $m_{\frac{3}{2}}(W;\xi)$ is x-independent
- $m_{\frac{3}{2}}(W;\xi)$ is real (up to a 0-order symbol, technical reasons)

Φ is a linearly symplectic map

$$\Phi(U)^\top E_c \Phi(U) = E_c + E_{>N}(U), \quad E_c := \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad E_{>N}(U) = O(\|U\|^{N+1})$$

[remark: symplectic means $(d_U(\Phi(U)U))^\top E_c\, d_U(\Phi(U)U) = E_c$] $\Rightarrow$

the transformed system is linearly Hamiltonian (but **not** Hamiltonian)

$$\partial_t w = \underbrace{\mathrm{i}\mathrm{Op}^{BW}\Big(m_{\frac{3}{2}}(W;\xi)\Big)}_{self-adjoint} w + R(W)w, \quad \text{i.e.} \quad m_{\frac{3}{2}}(W;\xi) \ \text{is real}$$

- Local existence energy estimate for $\|w\|_s$
- The PDE neglecting R

$$\partial_t w = \mathrm{i}\mathrm{Op}^{BW}\Big(m_{\frac{3}{2}}(W;\xi)\Big) w$$

preserves all H^s norms $\| \ \|_s$

Question: What happens adding $R(W)$?
Make Birkhoff normal form transformations on the symbols and on the smoothing remainder

The $\Phi(U)$ are time 1-flow $\Psi^\tau(U)$ of a linearly Hamiltonian paradiff operator

$$\partial_\tau \Psi^\tau(U) = J_c \, \mathrm{Op}^{\mathrm{BW}}(B(\tau, U; x, \xi))\Psi^\tau(U), \qquad \Psi^0(U) = \mathrm{Id}\,,$$

where $\mathrm{Op}^{\mathrm{BW}}(B) = \mathrm{Op}^{\mathrm{BW}}(B)^\top$ of 1-order (linear hyperbolic PDEs). For example

$$B(\tau, U; x, \xi) := \begin{pmatrix} 0 & b(\tau, U; x)\xi \\ -b(\tau, U; x)\xi & 0 \end{pmatrix}, \quad b(\tau, U; x) := \frac{\beta(U; x)}{1 + \tau\beta_x(U; x)}\,,$$

which is the flow of the transport equation $\partial_\tau u = \mathrm{Op}^{BW}(\mathrm{i} b(\tau, U; x)\xi) u$

- $\Psi^\tau(U) : H^s \to H^s, \ \forall s$
- $\Psi^\tau(U)$ is linearly symplectic;
- $\Psi^\tau(U)$ preserve paradifferential structure (next slide)

BUT $\Psi^\tau(U)U$ it is NOT symplectic $\Longrightarrow$ we have LOST the Hamiltonian structure

Paradifferential equation

$$\partial_t U = \mathrm{Op}^{BW}(A(U;x,\xi))U + R(U)U$$

Paradifferential flow change of variable:

$$W = \Psi^\tau(U)U \quad \Longleftrightarrow \quad U = (\Psi^\tau(U))^{-1}W$$

$$\|W\|_{H^s} \sim \|U\|_{H^s}$$

New PDE (it is still in paradifferential form)

$$\partial_t W = \underbrace{\Psi^\tau(U)\Big[\mathrm{Op}^{BW}(A(U;x,\xi)) + R(U)\Big](\Psi^\tau(U))^{-1}}_{\textit{conjugation of space}} W + \underbrace{\partial_t\Psi^\tau(U)\big(\Psi^\tau(U)\big)^{-1}}_{\textit{conjugation of time}} W$$

$$= \Psi^\tau(U)\Big[\mathrm{Op}^{BW}(A(U;x,\xi))\Big](\Psi^\tau(U))^{-1}W + \partial_t\Psi^\tau(U)\big(\Psi^\tau(U)\big)^{-1}W + R(U)W$$

$$= \mathrm{Op}^{BW}(A_1(U;x,\xi))W + R(U)W$$

New PDE is still in paradifferential form : Lie expansion or Egorov type analysis

Model WW

$$u_t = \mathrm{iOp}^{BW}((1+\zeta(U;x))\omega(\xi) + + \ldots)u + R(u)[u]$$

1. $\omega(\xi) = \sqrt{\kappa}|\xi|^{3/2} + \ldots$
2. $\zeta(U;x) \in \mathbb{R}$

We show how to reduce it to constant coefficients

Reduction to constant coefficients in x at the highest order

$$u_t = \mathrm{iOp}^{BW}((1+\zeta(U;x))|\xi|^{3/2})u$$

Idea: use a change of variable $x \mapsto x + \beta(U;x)$, a diffeomorphism of $\mathbb{T}^1$, so that $\xi \mapsto (1+\beta_x)\xi$, and the PDE transforms into

Transformed PDE

$$u_t = \mathrm{iOp}^{BW}((1+\beta_x(U;x))^{3/2}(1+\zeta(U;x))\sqrt{\kappa}|\xi|^{3/2} + \ldots)u$$

Choice of $\beta(U;x)$

$$(1+\beta_x(U;x))^{3/2}(1+\zeta(U;x)) = c(U)$$

$$\beta_x(U;x) = \Big(\frac{c(U)}{1+\zeta(U;x)}\Big)^{\frac{2}{3}} - 1 \implies \text{determines } c(U) \text{ and } \beta(U;x)$$

We can not use

Composition operator

$$\Phi_\beta : u(x) \mapsto u(x + \beta(x))$$

because the conjugated vector operator

$$(\Phi_\beta)^{-1} \circ \big(\mathrm{Op}^{BW}(1 + \zeta(U; x))|\xi|^{3/2})\big) \circ \Phi_\beta$$

would not be any more in paradifferential form

We use a "Paracomposition operator"

A definition of paracomposition operator

We regard the change of variable $u(x) \to u(x+\beta(x))$ as a flow

Homotopy

$$u(x) \to u(x+\tau\beta(x))\,, \quad \tau \in [0,1]$$

This path is the flow of

linear transport equation

$$\partial_\tau u = b(U;\tau,x)\partial_x u\,, \quad b(U;\tau,x) = \frac{\beta(U;x)}{1+\tau\beta_x(U;x)}$$

$$\partial_\theta u = \mathrm{Op}(\mathrm{i}b(U;\tau,x)\xi)u$$

Paracomposition operator $\Phi_\beta^\star U; := \Phi_\beta(1)$: time one flow of

$$\partial_\tau u = \mathrm{i}\mathrm{Op}^{BW}(b(U;\tau,x)\xi)u\,, \quad u(\tau) = \Phi_\beta(U;\tau)u(0)$$

Proposition, Berti-Delort

1. Assuming $\|\beta\|_{H^{s_0}} < 1/2$ then

$$\Phi_\beta^\star : H^s \to H^s \, , \; \forall s \, , \quad \|\Phi_\beta^\star u\|_s \leq C \|u\|_s$$

2. **Paradifferential analogue of Egorov theorem**

$$(\Phi_\beta^\star)^{-1} \big(\mathrm{Op}^{BW} a(U; x, \xi)\big) \Phi_\beta^\star = \mathrm{Op}^{BW} \big(\alpha(U; x, \xi)\big) + R(U)$$

where

$$\alpha(U; x, \xi) = a\big(U; x + \beta(U; x), \xi(1 + \breve{\beta}_y(U; y))_{|y = x + \beta(U;x)}\big) + \dots$$

$$y = x + \beta(U; x) \qquad \Longleftrightarrow \qquad x = y + \breve{\beta}(U; y)$$

and $R(U) \in \mathcal{R}^{-\rho}$

PROOF. The conjugated vector field

$$P(\tau) := \Phi_\beta(\tau) \circ \mathrm{Op}^{BW}\big(a(U;x,\xi)\big) \circ \Phi_\beta(\tau)^{-1}$$

satisfies the Heisenberg equation

$$\partial_\theta P(\tau) = \big[\mathrm{i}\mathrm{Op}^{BW}(b(U;\tau,x)\xi), P(\tau)\big], \quad P(0) = \mathrm{Op}^{BW}\big(a(U;x,\xi)\big)$$

Solution in decreasing symbols

$$P(U;\tau) = \mathrm{Op}^{BW}(q(U;\tau,x,\xi) + \ldots)$$

$$\partial_\tau q(U;\tau,x,\xi) = \{b(U;\tau,x)\xi, q(U;\tau,x,\xi)\}, \quad q(0,x,\xi) = a(U;x,\xi)$$

$$q(U;\tau,x,\xi) = a(x + \tau\beta(U;x), \xi(1 + \breve{\beta}_y(U;y)_{|y=x+\beta(U;x)}))$$

Weyl quantization is convenient

proposition

$$\Phi_\beta^\star \circ \partial_t \circ (\Phi_\beta^\star)^{-1} = \partial_t + \Phi_\beta^\star (\Phi_\beta^\star)^{-1}$$
$$= \partial_t + \mathrm{Op}^{BW}(\mathrm{i} g(U;\cdot))\xi + R(U)$$

where $R(U)$ is a smoothing operator in $\mathcal{R}^{-\rho}$.

- The conjugation with ∂_t gives a lower order term, transport order 1,
- All the transformations are determined by the spatial operator since $\omega(\xi) \sim |\xi|^{3/2}$ is superlinear

Flow and Taylor expansion

$$\Phi^\tau(U) : H^s \to H^s\,, \quad \|\Phi^\tau(U)V\|_s \sim \|V\|_s$$

but a Taylor expansion gives unbounded operators

$$\Phi^\tau(U) = \mathrm{Id} + \underbrace{\mathrm{Op}^{BW}(B(U))}_{\textit{order } 1} U + \frac{1}{2}\underbrace{\mathrm{Op}^{BW}(B(U))\mathrm{Op}^{BW}(B(U))}_{\textit{order } 2} U + \ldots$$

Example $\partial_\tau u = \mathrm{Op}^{BW}\big(\underbrace{\mathrm{i}b(\tau, U; x)\xi}_{=B(U)}\big)u$ of transport

Key example: composition

$$u(x+\beta(x)) = u(x) + \underbrace{u_x(x)}_{\partial_x}\ \underbrace{\beta(x)}_{\textit{smallness}} + \frac{1}{2}\underbrace{u_{xx}(x)}_{\partial_x^2}\ \underbrace{\beta^2(x)}_{\textit{smallness}^2} + \ldots$$

Next goal: recover nonlinear Hamiltonian structure of the paradifferential normal form

1. WW are quasi-linear PDEs $\Rightarrow$ require paradifferential calculus to prove energy estimates (for local existence theory)
2. Usual paradifferential calculus does not preserve Hamiltonian structure

Goal :

- Preserve Hamiltonian structure in paradifferential calculus, at least up to homogeneity N

Hamiltonian paradifferential normal form

There is a symplectic map up to homogeneity N

$$Z = \begin{pmatrix} z \\ \bar{z} \end{pmatrix} = (\mathrm{Id} + \underbrace{R_{\leq N}(\cdot)}_{smoothing}) \circ \underbrace{\Phi(U)U}_{=W}$$

such that

$$\partial_t z = -\mathrm{i}\Omega(D)z + \underbrace{\mathrm{Op}^{BW}\big(-\mathrm{i}(\breve{\mathtt{m}}_{\frac{3}{2}})_{\leq N}(Z;\xi)\big)z + R_{\leq N}(Z)Z}_{\mathrm{i}\nabla_{\bar{z}}H(Z)}$$

is *Hamiltonian up to homogeneity N*

Thanks to the fact that the symplectic corrector is

$$\mathrm{Id} + \underbrace{R_{\leq N}(\cdot)}_{smoothing}$$

the paradifferential PDE structure is the same $\Rightarrow$ good energy estimates

THANKS for the ATTENTION!!
next episode at the Workshop...