IML NOTES: QUANTUM CORRELATIONS

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Abstract. The first section is intended to give operator algebraists a brief introduction to entanglement and some of its uses. The second section is to show the connections between purification notions in physics and dilation theory in operator theory. Those only interested in quantum correlations can skip directly to section 3.

1. Composite Systems, Entanglement and Joint Probabilities

Let $\mathcal{H}_A$ be the state space for Alice's lab and $\{X_k\}_k$, $\sum_k X_k^* X_k = I_{\mathcal{H}_A}$ be a measurement system on $\mathcal{H}_A$ and let $\mathcal{H}_B$ be the state space for Bob's lab and $\{Y_\ell\}_\ell$, $\sum_\ell Y_\ell^* Y_\ell = I_{\mathcal{H}_B}$ be a measurement system on $\mathcal{H}_B$. Suppose $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that $\|\psi\| = 1$. If $p_A(k)$ and $p_B(\ell)$ respectively denote the probability that Alice gets outcome k in the combined lab and the probability that Bob gets outcome ℓ in the combined lab, then

$$p_A(k) = \|(X_k \otimes I)\psi\|^2 \text{ and } p_B(\ell) = \|(I \otimes Y_\ell)\psi\|^2.$$

If Alice's outcome is k, then the state becomes

$$\frac{(X_k \otimes I)\psi}{\|(X_k \otimes I)\psi\|}.$$

Similarly if Bob's outcome is ℓ, then the state becomes

$$\frac{(I \otimes Y_\ell)\psi}{\|(I \otimes Y_\ell)\psi\|}.$$

The *joint probability* of getting outcome k for Alice and outcome ℓ for Bob, denoted by $p_{A,B}(k,\ell)$, is given by

$$p_{A,B}(k,\ell) = \|(X_k \otimes Y_\ell)\psi\|^2.$$

We can also use the notion of conditional probabilities in the quantum setting. The *conditional probability* that Bob gets outcome ℓ, given that Alice got outcome k is given by

$$p(B = \ell | A = k) = \frac{\|(I \otimes Y_\ell)(X_k \otimes I)\psi\|^2}{\|(X_k \otimes I)\psi\|^2} = \frac{p(B = \ell, A = k)}{p(A = k)},$$

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since if Alice has already got the outcome k, that is, $A = k$ then the state is $\frac{(X_k \otimes I)\psi}{\|(X_k \otimes I)\psi\|}$, so that the probability of getting outcome ℓ for Bob given that $A = k$ is computed as in the usual definition of conditional probability.

Note: We use $A = k$ to mean "A gets the outcome k."

Definition 1.1. A state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ is said to be *separable* if it is of the form $\psi = \gamma \otimes \phi$ for some $\gamma \in \mathcal{H}_A$ and $\phi \in \mathcal{H}_B$; that is to say, ψ is an elementary tensor (without loss of generality, by scaling, each of γ and ϕ are norm 1). If ψ is not of this form, we say that ψ is *entangled*.

It is worth noticing how separable states behave in a combined lab. Indeed, if $\psi = \gamma \otimes \phi$ is separable with $\|\gamma\| = \|\phi\| = 1$, then one has

$$p_A(k) = \|(X_k \otimes I)(\gamma \otimes \phi)\|^2 = \|X_k\gamma \otimes \phi\|^2 = \|X_k\gamma\|^2\|\phi\|^2 = \|X_k\gamma\|^2,$$

while the joint probability becomes

$$\begin{aligned} p(B = \ell, A = k) &= \|(X_k \otimes Y_\ell)(\gamma \otimes \phi)\|^2 \\ &= \|X_k\gamma \otimes Y_\ell\phi\|^2 = \|X_k\gamma\|^2\|Y_\ell\phi\|^2 \\ &= p_A(k) \cdot p_B(\ell). \end{aligned}$$

Recall that in probability, events E_1, E_2 are *independent* if $\text{Prob}(E_1 \cap E_2) = \text{Prob}(E_1) \cdot \text{Prob}(E_2)$, so we infer that $A = k$ and $B = \ell$ are independent then and only then

$$\begin{aligned} p(B = \ell | A = k) &= \frac{\|(X_k \otimes Y_\ell)(\gamma \otimes \phi)\|^2}{\|(X_k \otimes I)(\gamma \otimes \phi)\|^2} \\ &= \frac{\|X_k\gamma\|^2\|Y_\ell\phi\|^2}{\|X_k\gamma\|^2} \\ &= \|Y_\ell\phi\|^2 \\ &= p(B = \ell). \end{aligned}$$

Thus in case of separable states, the quantum probabilities exactly reflect independent classical probabilities.

Definition 1.2. The state $\psi = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 + e_1 \otimes e_1) \in \mathbb{C}^2 \otimes \mathbb{C}^2$ is called the *Einstein-Poldosky-Rosen (EPR) state*.

Example 1.3 (Quantum Teleportation)**.** Let E_{00}, E_{11} be the diagonal matrix units in M_2. Then $E_{00}^*E_{00} + E_{11}^*E_{11} = E_{00}^2 + E_{11}^2 = I_{\mathcal{H}_A}$ where $\mathcal{H}_A = \mathbb{C}^2$, so this is a measurement system. Suppose Bob has the same measurement system (in a different lab) so that $\mathcal{H}_A = \mathcal{H}_B = \mathbb{C}^2$. In the combined lab, the probabilities are as follows:

$$\begin{aligned} p_A(0) = p_A(A=0) &= \|(E_{00}\otimes I)\psi\|^2 \\ &= \|(E_{00}\otimes I)(\frac{1}{\sqrt{2}})(e_0\otimes e_0 + e_1\otimes e_1)\|^2 \\ &= \frac{1}{2}\|e_0\otimes e_0\|^2 = \frac{1}{2}. \end{aligned}$$

Similarly, one can check that $p_A(1) = \frac{1}{2}$, while $p_B(0) = p_B(1) = \frac{1}{2}$.

If A observes the outcome 0, then the state changes to

$$\frac{(E_{00}\otimes I)(\frac{1}{\sqrt{2}})(e_0\otimes e_0 + e_1\otimes e_1)}{\|(E_{00}\otimes I)(\frac{1}{\sqrt{2}})(e_0\otimes e_0 + e_1\otimes e_1)\|} = e_0\otimes e_0.$$

Now suppose B performs a measurement. B has measurement operators $I\otimes E_{00}$ and $I\otimes E_{11}$. We know that $(I\otimes E_{11})(e_0\otimes e_0) = 0$, so B cannot possibly measure the outcome 1. Therefore, if A measures 0, then B must measure 0 with probability 1. The same analysis works if A measures 1. This demonstrates that entangled systems, to some degree, behave like dependent events. We confirm this with the computations below.

$$\begin{aligned} p(B=0|A=0) &= \|(I\otimes E_{00})(e_0\otimes e_0)\|^2 = \|e_0\otimes e_0\|^2 = 1 \neq p_B(0) = \frac{1}{2}, \\ p(B=1|A=0) &= \|(I\otimes E_{11})(e_0\otimes e_0)\|^2 = 0 \neq p_B(1) = \frac{1}{2}. \end{aligned}$$

This shows that there is a large amount of dependence here. This is the basis for "spooky action at a distance", or "quantum teleportation".

Example 1.4 (Super Dense Coding)**.** The idea is the following: If Alice has states in $\mathbb{C}^2$, we know that we can only make two states perfectly distinguishable. Consider the EPR state $\psi = \frac{1}{\sqrt{2}}(e_0\otimes e_0 + e_1\otimes e_1)$. Consider the matrices that were used to form the 1-Pauli's, given by

$$I = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, Y = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}, \text{ and } Z = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Then Alice, after applying these operations on the EPR state, has

$$\begin{aligned} (I\otimes I)\psi &= \frac{1}{\sqrt{2}}(e_0\otimes e_0 + e_1\otimes e_1) \\ (X\otimes I)\psi &= \frac{1}{\sqrt{2}}(e_1\otimes e_0 + e_0\otimes e_1) \\ (Y\otimes I)\psi &= \frac{1}{\sqrt{2}}(e_0\otimes e_1 - e_1\otimes e_0) \\ (Z\otimes I)\psi &= \frac{1}{\sqrt{2}}(e_0\otimes e_0 - e_1\otimes e_1) \end{aligned}$$

four outcomes that are orthonormal and hence perfectly distinguishable!

Discussion: Suppose Alice and Bob start with the EPR state $\psi = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 + e_1 \otimes e_1)$ and Alice performs one of the four above mentioned operations on the state of the photon in its lab and sends that single photon to Bob via a quantum channel. Bob now has access to both the photons (actually the access to two states, one EPR and one which he received from Alice, since the above four states are perfectly distinguishable and hence there exists a measurement system that tells Bob which state he received.) This allows Bob to know precisely which operation Alice performed. Moral: Alice only needed to send one photon to communicate four pieces of information. Similarly, in $\mathbb{C}^d$, with basis $e_0, \dots, e_{d-1}$, consider the EPR state $\psi = \frac{1}{\sqrt{d}}(e_0 \otimes e_0 + \cdots + e_{d-1} \otimes e_{d-1})$. Then there exist d^2 unitaries, U_i, in $\mathbb{C}^d$ such that $(U_i \otimes I)\psi$ is orthogonal to $(U_j \otimes I)\psi$ for any $i \neq j$. Again, if B keeps half of the photons, then A can communicate d^2 pieces of information.This example shows the existence of a way (entanglement) to boost the capacity of the quantum channel in question.

Definition 1.5. An ensemble of states $\{\psi_i, p_i\}$, $\psi_i \in \mathcal{H}_A \otimes \mathcal{H}_B$, $p_i > 0$ and $\sum_i p_i = 1$ is called *separable* if each $\psi_i \in \mathcal{H}_A \otimes \mathcal{H}_B$ is separable. A density matrix $P \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is called *separable* if it is the density matrix of a separable ensemble.

Proposition 1.6. *Let $P \in \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a density matrix. Then the following are equivalent.*

1. *P is separable.*
2. *There exist density matrices $R_i \in \mathcal{L}(\mathcal{H}_A)$ and $Q_i \in \mathcal{L}(\mathcal{H}_B)$ and $p_i > 0$ with $\sum_i p_i = 1$ such that $P = \sum_i p_i R_i \otimes Q_i$.*
3. *There exist $E_i \in \mathcal{L}(\mathcal{H}_A)$ and $F_i \in \mathcal{L}(\mathcal{H}_B)$ that are rank one projections, along with $p_i > 0$ with $\sum_i p_i = 1$, such that $P = \sum_i p_i E_i \otimes F_i$.*

2. Dilations: State Purification, POVM's vs. PVM's

The idea of dilation is to make things simpler by representing them on a larger space.

Example 2.1. Suppose we want to find a formula for $\cos(\alpha+\beta)$. The easiest way is to think of $e^{i\alpha} = \cos(\alpha) + i\cos\beta$ so that $\cos(\theta) = \mathrm{Re}(e^{i\theta})$. (Here, we are in a sense dilating to a form of $\mathbb{R}^2$, namely $\mathbb{C}$.) Now the problem is very easy, since

$$\begin{aligned}\cos(\alpha+\beta) &= \mathrm{Re}(e^{i(\alpha+\beta)}) \\ &= \mathrm{Re}(e^{i\alpha}e^{i\beta}) \\ &= \mathrm{Re}((\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta.\end{aligned}$$

Example 2.2. Consider the Fibonacci numbers given by $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$. Then these numbers are really satisfying the equation

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix}.$$

Let $R = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Then $R^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix}$. Now, $R = R^*$ is diagonalizable with eigenvalues $\frac{1 \pm \sqrt{5}}{2}$. Hence, for some unitary U, we have $R = U^* \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} U$. It follows that

$$\begin{pmatrix} f_{n+1} \\ f_{n+2} \end{pmatrix} = U^* \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{pmatrix} U \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This gives a nice explicit formula!

Example 2.3. *(Halmos Dilation)* Let $C \in \mathcal{L}(\mathcal{H})$ with $\|C\| \leq 1$. Consider the operator matrix $\begin{pmatrix} C & \sqrt{I - CC^*} \\ \sqrt{I - C^*C} & -C^* \end{pmatrix} \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$. One can check that this is a unitary, while C is just a corner of the matrix.

2.1. **State Purification.** In the quantum setting, this idea of dilation is essentially what Physicists refer to as state purification. To be more precise, consider a measurement system $\{X_i\}_{i=1}^m$ with each $X_i : \mathcal{H}_A \to \mathcal{K}$, so that $\sum_{i=1}^m X_i^* X_i = I_A$. Consider an ensemble $\{v_k, p_k\}_{k=1}^r$ where $v_k \in \mathcal{H}_A$ with $\|v_k\| = 1$ and $\sum_{k=1}^r p_k = 1$. Let $\rho = \sum_{k=1}^r p_k v_k v_k^*$ be the density matrix of the ensemble. Recall that the probability of getting outcome i is

$$\sum_{k=1}^r p_k \|X_i v_k\|^2 = tr(X_i \rho X_i^*).$$

Suppose we are only interested in the probabilities. Then we can replace ρ by a pure state on a larger space. We will see this in two ways.

The first way to see the above is by letting

$$v = \begin{pmatrix} \sqrt{p_1} v_1 \\ \vdots \\ \sqrt{p_r} v_r \end{pmatrix} \in \mathcal{H}_A \otimes \mathbb{C}^r \text{ where } r = rank(\rho).$$

One can see that v is a unit vector. Now replace X_i by $\tilde{X}_i : \mathcal{H}_A \otimes \mathbb{C}^r \to \mathcal{K} \otimes \mathbb{C}^r$, with

$$\tilde{X}_i = \begin{pmatrix} X_i & & \\ & \ddots & \\ & & X_i \end{pmatrix} = X_i \otimes I_r.$$

Then it follows that

$$\sum_{i=1}^{m} \tilde{X}_i^* \tilde{X}_i = I_{\mathcal{H}_A} \otimes I_r = \begin{pmatrix} I_{\mathcal{H}_A} & & \\ & \ddots & \\ & & I_{\mathcal{H}_A} \end{pmatrix}.$$

Hence, $\{\tilde{X}_i\}_{i=1}^m$ is a measurement system on $\mathcal{H}_A \otimes \mathbb{C}^r$. Moreover, it is readily checked that

$$\|\tilde{X}_i v\|^2 = \sum_{k=1}^{r} p_k \|X_i v_k\|^2,$$

which is the same probability as before (for getting outcome i). Hence, if we only care about probabilities, then we may replace our ensemble with a pure state.

The second way to obtain the above is more canonical. Let $\mathcal{H}_A = \mathbb{C}^n$ and $\mathcal{L}(\mathcal{H}_A) = M_n \simeq \mathbb{C}^{n^2}$ which is a Hilbert space. Given $X_i \in \mathcal{L}(\mathcal{H}_A) = M_n$, we may obtain a "new map" $\tilde{X}_i : M_n \to M_n$, defined by

$$\tilde{X}_i(Y) = X_i Y = \begin{pmatrix} X_i y_1 & \vdots & \dots & \vdots & X_i y_n \end{pmatrix},$$

where $Y = \begin{pmatrix} y_1 & \vdots & \dots & \vdots & y_n \end{pmatrix} \in M_n$. Then

$$\tilde{X}_i Y \simeq \begin{pmatrix} X_i & & \\ & \ddots & \\ & & X_i \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Given a density matrix $\rho \in M_n$, the probability of obtaining outcome i is

$$tr(X_i \rho X_i^*) = tr((X_i \rho^{\frac{1}{2}})(\rho^{\frac{1}{2}} X_i^*)) = tr((\rho^{\frac{1}{2}} X_i^*)(X_i \rho^{\frac{1}{2}})) = \langle \tilde{X}_i(\rho^{\frac{1}{2}}) | \tilde{X}_i(\rho^{\frac{1}{2}}) \rangle_{M_n}.$$

Since $\rho^{\frac{1}{2}}$ is a vector in M_n, we obtain

$$\|\rho^{\frac{1}{2}}\|_{M_n}^2 = \rho^{\frac{1}{2}} | \rho^{\frac{1}{2}} ra_{M_n} = tr((\rho^{\frac{1}{2}})^* (\rho^{\frac{1}{2}})) = tr(\rho) = 1.$$

It follows that $\rho^{\frac{1}{2}}$ is a unit vector, hence a pure state in the Hilbert space M_n. (So, when physicists replace ρ by $\sqrt{\rho}\sqrt{\rho}$, they actually consider $\sqrt{\rho}$ to be a pure state in space M_n.)

3. POVM's and PVM's

With this in mind, we will now talk about positive operator-valued measures (POVM's). If we are only interested in probabilities in the context of a measurement system $\{X_i\}_i$ and a state h, then the probability of obtaining outcome i is $\|X_i h\|^2 = \langle X_i h | X_i h \rangle = \langle h | X_i^* X_i h \rangle$. The probability, then only really depends on $R_i = X_i^* X_i \geq 0$, while $I = \sum_{i=1}^m X_i^* X_i = \sum_{i=1}^m R_i$.

Definition 3.1 (POVM)**.** An *m-outcome positive operator-valued measure* (POVM) on a Hilbert space $\mathcal{H}_A$ is a set $\{R_i\}_{i=1}^m \subseteq \mathcal{L}(\mathcal{H}_A)$ with $R_i \geq 0$ and $\sum_{i=1}^m R_i = I$.

Definition 3.2 (PVM)**.** An *m-outcome projection-valued measure* (PVM) on a Hilbert space $\mathcal{H}_A$ is a set of projections $\{P_i\}_{i=1}^m$ in $\mathcal{L}(\mathcal{H}_A)$ such that $\sum_{i=1}^m P_i = I_{\mathcal{H}_A}$.

Obviously, every PVM is a POVM. Note that each PVM corresponds to decomposing the Hilbert space $\mathcal{H}$ into a direct sum of its subspaces. Namely, $\mathcal{H} = \mathcal{H}_1 + \cdots + \mathcal{H}_n$, where $\mathcal{H}_i$ is a subspace of $\mathcal{H}$ and $\mathcal{H}_i \perp \mathcal{H}_j$, for $i \neq j$.

Proposition 3.3. *Let $\{R_i\}_{i=1}^m$ be a POVM on $\mathcal{H}$. Then there is a PVM $\{P_i\}_{i=1}^m$ on $\mathcal{H}\otimes\mathbb{C}^m$ and an isometry $V : \mathcal{H} \to \mathcal{H}\otimes\mathbb{C}^m$ such that $R_i = V^*P_iV$ for all $1 \leq i \leq m$.*

Proof. As usual, we identify

$$\mathcal{H} \otimes \mathbb{C}^m \cong \underbrace{\mathcal{H} \oplus \cdots \oplus \mathcal{H}}_{m \text{ times}}.$$

Let $P_i = I_{\mathcal{H}} \otimes E_{ii}$, the $m \times m$ matrix over $\mathcal{L}(\mathcal{H})$ with $I_{\mathcal{H}}$ in the (i, i)-position and 0 everywhere else. That is, each P_i is the projection onto the i-th copy of $\mathcal{H}$ in $\mathcal{H} \otimes \mathbb{C}^m$ which means

$$P_i\left(\begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix}\right) = \begin{bmatrix} 0 \\ \vdots \\ h_i \\ \vdots \\ 0 \end{bmatrix}.$$

Then clearly $P_i = P_i^* = P_i^2$ and $\sum_{i=1}^m P_i = I_{\mathcal{H}} \otimes I_m = I_{\mathcal{H}\otimes\mathbb{C}^m}$. Hence, $\{P_i\}_{i=1}^m$ is a PVM. Define $V : \mathcal{H} \to \mathcal{H} \otimes \mathbb{C}^m$ by $Vh = \sum_{i=1}^m (R_i^{1/2}h) \otimes e_i = (R_1^{\frac{1}{2}}h, ..., R_m^{\frac{1}{2}}h)$. This is linear, with

$$\|Vh\|^2 = \sum_{i=1}^m \langle R_i^{\frac{1}{2}}h|R_i^{\frac{1}{2}}h\rangle = \sum_{i=1}^m \langle h|R_ih\rangle = \langle h, h\rangle = \|h\|^2.$$

Therefore, V is an isometry. Finally, we see that

$$\begin{aligned} \langle h|V^*P_iVh\rangle &= \langle Vh|(0, ..., 0, \underbrace{R_i^{\frac{1}{2}}h}_{i\text{-th slot}}, 0, ..., 0)\rangle \\ &= \langle (R_1^{\frac{1}{2}}h, ..., R_i^{\frac{1}{2}}h, ..., R_m^{\frac{1}{2}}h)|(0, ..., 0, R_i^{\frac{1}{2}}h, 0, ..., 0)\rangle \\ &= \langle R_i^{\frac{1}{2}}h|R_i^{\frac{1}{2}}h\rangle \\ &= \langle h|R_ih\rangle, \end{aligned}$$

which implies that $R_i = V^*P_iV$. □

Regarding the previous proposition, a few notes are in order:

(1) If $h \in \mathcal{H}$, then

$$\langle h|R_ih\rangle_{\mathcal{H}} = \langle h|V^*P_iVh\rangle_{\mathcal{H}} = \langle Vh|P_i(Vh)\rangle_{\mathcal{H}\otimes\mathbb{C}^m},$$

so with $\tilde{h} = Vh$, we have

$$\|\tilde{h}\| = \|h\| \text{ and } \mathrm{prob}(i) = \langle h|R_i h\rangle = \langle \tilde{h}|P_i\tilde{h}\rangle.$$

So, to summarize, when we are given a POVM $\{R_i\}_{i=1}^n$ on $\mathcal{H}$ and a state $h \in \mathcal{H}$, we may regard the pair as the PVM $\{P_i\}_{i=1}^n$ on a larger Hilbert space $\mathcal{H}\otimes\mathbb{C}^n$ together with the state $\tilde{h} = Vh \in \mathcal{H}\otimes\mathbb{C}^n$ with the same probabilities of outcomes.

(2) Note that R_i, R_j, R_i^2 can be anything and there is no reason to have $R_iR_j = R_jR_i$. But, for a PVM, the fact that the projections add to the identity implies that their ranges are pairwise orthogonal, so that $P_iP_j = 0$ for $i \neq j$. Hence, the computations with a PVM are *much better* than the computations with a POVM.

Remark 3.4. This proposition actually is a special case of the Stinespring's dilation theorem. Let $l_n^\infty = C(\{1, \ldots, n\})$ be the abelian C*-algebra generated by the functions δ_i given by $\delta_i(j) = \delta_{ij}$. Thus a complex-valued function f on $\mathbb{C}^n$ can be regarded as $(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i = f(i)$, and we identify $f = \sum_{i=1}^n \lambda_i\delta_i$. Hence each δ_i corresponds to the basis vector e_i for l_n^∞.

We can then define a completely positive map $\Phi\colon l_n^\infty \to \mathcal{L}(\mathcal{H})$ by $\Phi(\delta_i) = R_i$, so $\Phi((\lambda_1, \ldots, \lambda_n)) = \sum_{i=1}^n \lambda_i R_i$. There is also a map $\pi\colon l_n^\infty \to \mathcal{L}(\mathcal{H}\otimes\mathbb{C}^n)$ by $\pi(\delta_i) = P_i$, $\pi((\lambda_1, \ldots, \lambda_n)) = \sum_{i=1}^n \lambda_i P_i$. Since the P_i's are orthgonal projections,

$$\begin{aligned}\pi((\lambda_1, \ldots, \lambda_n)\cdot(\mu_1, \ldots, \mu_n)) &= \sum_{i=1}^n \lambda_i\mu_i P_i = (\sum_{i=1}^n \lambda_i P_i)(\sum_{j=1}^n \mu_j P_j)\\ &= \pi((\lambda_1, \ldots, \lambda_n))\pi((\mu_1, \ldots, \mu_n)).\end{aligned}$$

Readers can check that π preserves the unit and adjoints. Hence π is indeed a unital $*$-homomorphism. Moreover,

$$\begin{aligned}V^*\pi((\lambda_1, \ldots, \lambda_n))V &= V^*(\sum_{i=1}^n \lambda_i P_i)V = \sum_{i=1}^n \lambda_i V^* P_i V\\ &= \sum_{i=1}^n \lambda_i R_i = \Phi((\lambda_1, \ldots, \lambda_n)).\end{aligned}$$

3.1. **POVM's and PVM's in multiexperiment settings.** Now what if we have more than one measurement system? Suppose that Alice's state space is $\mathcal{H}_A$. Suppose that she has a whole family of experiments from which she could choose. Each one is represented by a POVM $\{R_{t,i}\}_{i=1}^m$ where $t \in T$ and T is the set of experiments.

(Note that in complicated situation each experiment in the family may have different number of outcomes. However, we can get rid of that by adding extra outcomes when necessary with 0 probabilities and assuming that all experiments have same number of outcomes; the number being that of the highest-outcome experiment.)

The next theorem tells us that we can again dilate this family of POVM's into a family of PVM's simultaneously; that is, using only one isometry V that works for all $v \in \mathcal{V}$.

Later we will present another proof of this fact that invokes a theorem of Boca. But for now this will give us a concrete introduction to the constructions involved in the proof of Boca's theorem.

Theorem 3.5. *Let* $\{R_{t,i}\}_{i=1}^m$ *be a family of POVM's on* $\mathcal{H}$*, indexed by* $t \in T$ *with* $|T| = n$ *(we only consider the case where* $|T|$ *is finite). Then there is a Hilbert space* $\mathcal{K}$ *and a family of PVM's* $\{P_{t,i}\}_{i=1}^m$ *on* $\mathcal{K}$ *for* $t \in T$*, and an isometry* $V : \mathcal{H} \to \mathcal{K}$ *such that* $V^* P_{t,i} V = R_{t,i}$ *for all* i, t*. Moreover, if* $\dim(\mathcal{H}) < \infty$*, then* $\dim(\mathcal{K}) < \infty$*.*

Proof. We just proved the case for $n = 1$, so we proceed by induction. Assume that we can do this for $|T| = n$. Now suppose we have $n + 1$ experiments. We know that there exists a Hilbert space $\mathcal{K}_1$, an isometry $V_1 : \mathcal{H} \to \mathcal{K}_1$ and PVM's $\{P_{t,i}\}_{i=1}^m$ for $1 \le t \le n$ such that $V_1^* P_{t,i} V_1 = R_{t,i}$ for all i and for all $1 \le t \le n$. Let $\tilde{R}_{n+1,i} = V_1 R_{n+1,i} V_1^* \in \mathcal{L}(\mathcal{K}_1)$. Then $\tilde{R}_{n+1,i} \ge 0$ and

$$\sum_{i=1}^m \tilde{R}_{n+1,i} = V_1 \left(\sum_{i=1}^m R_{n+1,i} \right) V_1^* = V_1 V_1^*,$$

which is a projection. Adjust $\tilde{R}_{n+1,1}$ by setting $\tilde{R}_{n+1,1} = V_1 R_{n+1,1} V^* + (I - V_1 V_1^*)$, so that $\{\tilde{R}_{n+1,i}\}_{i=1}^m$ are a POVM on $\mathcal{K}_1$.

On $\mathcal{K}_1$, we have PVM's $\{P_{t,i}\}_{i=1}^m$ for $1 \le t \le n$ and a POVM $\{\tilde{R}_{n+1,i}\}_{i=1}^m$. Let $\mathcal{K} = \mathcal{K}_1 \otimes \mathbb{C}^m$, and define $V_2 : \mathcal{K}_1 \to \mathcal{K}$ by

$$V_2 k = ((\tilde{R}_{n+1,1})^{\frac{1}{2}} k, ..., (\tilde{R}_{n+1,m})^{\frac{1}{2}} k).$$

Then V_2 is an isometry. Set $P_{n+1,i} = I_{\mathcal{K}_1} \otimes E_{ii}$, for $1 \le i \le m$. Then $\{P_{n+1,i}\}_{i=1}^m$ is a PVM and $V_2^* P_{n+1,i} V_2 = \tilde{R}_{n+1,i}$. Now set $Q_{t,j} = V_2 P_{t,j} V_2^* \in \mathcal{L}(\mathcal{K})$ for $2 \le j \le m$, and set $Q_{t,1} = V_2 P_{t,1} V_2^* + (I - V_2 V_2^*)$, for all $1 \le t \le n$. It is easy to see that $V_2^* Q_{t,j} V_2 = P_{t,j}$. We need to see that $\{Q_{t,i}\}_{i=1}^m$ for $1 \le t \le n$ are PVM's and not just POVM's. Note that $\{P_{n+1,i}\}_{i=1}^m$ is a PVM. For the other ones, we will see that $Q_{t,j}^2 = Q_{t,j}$ and hence they must be PVM's (for $i \ge 2$). Note that

$$Q_{t,j}^2 = (V_2 P_{t,j} V_2^*)(V_2 P_{t,j} V_2^*) = V_2 P_{t,j}^2 V_2 = V_2 P_{t,j} V_2 = Q_{t,j}.$$

Finally,

$$\begin{aligned} Q_{t,1}^2 &= [V_2 P_{t,1} V_2^* + (I - V_2 V_2^*)][V_2 P_{t,1} V_2^* + (I - V_2 V_2^*)] \\ &= V_2 P_{t,1} V_2^* + (I - V_2 V_2^*) = Q_{t,1}. \end{aligned}$$

□

To motivate the concept of quantum probabilities, we will talk about finite input-output games in the next subsection.

3.2. **Classical Densities.** We suppose that Alice and Bob share some probability space (Ω, μ) and that for each input $a \in I_A$, Alice has a function $f_a : \Omega \to \mathcal{O}_A$ such that $\mu(\{w : f_a(w) = x\})$ is the probability that Alice produces output x, given that she received input a. Similarly, we suppose that for each $b \in I_B$ Bob has a function $g_b : \Omega \to \mathcal{O}_B$ such that $\mu(\{w : g_b(w) = y\}$ is the probability that Bob produces output y, given that he received input b. In this case, we have

$$p(x, y|a, b) = \mu(\{w : f_a(w) = x,\ g_b(w) = y\}).$$

The set of all such $p(x, y|a, b)$ is called the set of all *local densities.* When $I_A = I_B, \mathcal{O}_A = \mathcal{O}_B, |I_A| = n$ and $|\mathcal{O}_B| = k$, then these are n^2k^2-tuples of real numbers. We denote this set by $LOC(n, k) = C_{\text{loc}}(n, k)$.

3.3. **Quantum Densities.** The idea is that for each input, Alice has a different experiment with $|\mathcal{O}_A|$ outcomes. There is a state space $\mathcal{H}_A(\dim \mathcal{H}_A < \infty)$, and for each $a \in I_A$, there is a POVM, $\{E_{a,x}\}_{x \in \mathcal{O}_A}$ (so $E_{a,x} \geq 0$ and $\sum_x E_{a,x} = I$), such that if we are in state $\psi \in \mathcal{H}_A$, then $\langle \psi | E_{a,x} \psi \rangle = p_A(x|a)$. Similarly, for each input $b \in I_B$, Bob also has a quantum experiment with $|\mathcal{O}_B|$ outcomes. These correspond to POVM's $\{F_{b,y}\}_{y \in \mathcal{O}_B}$ (where $F_{b,y} \geq 0$ and $\sum_y F_{b,y} = I$) on $\mathcal{H}_B(\dim \mathcal{H}_B < \infty)$.

The strategy is the following: Pick a state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ (often this will be entangled) such that $p(x, y|a, b) = \langle \psi | E_{a,x} \otimes F_{y,b}(\psi) \rangle$. The set of all such tuples when $I_A = I_B, |I_A| = n, \mathcal{O}_A = \mathcal{O}_B$, and $|\mathcal{O}_B| = k$ is denoted by $Q(n, k) = C_q(n, k)$.

Other Versions of Quantum Densities There is more than one definition of a quantum density. Other versions include (but may not be limited to):

(1) The same as above except drop the requirement of $\mathcal{H}_A$ and $\mathcal{H}_B$ being finite-dimensional, that is, drop the condition $\dim(\mathcal{H}_A), \dim(\mathcal{H}_B) < \infty$ and let the unit vector ψ belong to the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Recall that this space is the *completion* of the algebraic tensor product of the two spaces and that given orthonormal bases $\{e_\alpha\}$ for $\mathcal{H}_A$ and $\{f_\beta\}$ for $\mathcal{H}_B$ the vector ψ has a square summable series expansion
$$\psi = \sum_{\alpha,\beta} z_{\alpha,\beta} e_\alpha \otimes f_\beta.$$
Such vectors can have an infinite amount of entanglement. The resulting set is denoted by $C_{qs}(n, k)$.

(2) Another axiomatic description is that there is a universal state space $\mathcal{H}$ and all of the $E_{a,x}, F_{b,y}$ act on this space and commute with each other, for all choices of a, x, b, y. (This reflects the fact that the labs are different. Indeed, in the case of different labs, the operators would look like $E_{a,x} \otimes I$ and $I \otimes F_{b,y}$ and these always commute.) In this case, there is a state $\psi \in \mathcal{H}$ such that $p(x, y|a, b) =$

$\langle\psi|E_{a,x}F_{b,y}\psi\rangle$. The set of all such quantum densities will be denoted by $C_{qc}(n,k)$.

It is clear that

$$C_{\text{loc}}(n,k) \subseteq C_q(n,k) \subseteq C_{qs}(n,k) \subseteq C_{qc}(n,k) \subseteq \mathbb{R}^{n^2k^2}.$$

Thanks to the work of W. Slofstra, it is now known that $C_q(n,k)$ is not closed, so let us define

$$C_{qa}(n,k) := \overline{C_q(n,k)}.$$

Slofstra's proof showed that $C_q(n,k)$ is not closed for a relatively large value of n. Later results showed that it was not closed for relatively small values.

Next, suppose we had the commuting model with all the $E_{a,x}$'s and $F_{b,y}$'s being projections, and $\psi \in \mathcal{H}$ with $\|\psi\| = 1$, then the vectors $v_{a,x} = E_{a,x}\psi$ satisfy

$$v_{a,x} \perp v_{a,x'} \text{ for } x \neq x', \text{ while } \sum_x v_{a,x} = \left(\sum_x E_{a,x}\right)\psi = \psi.$$

Similarly, with $w_{b,y} = F_{b,y}\psi$ we have

$$w_{b,y} \perp w_{b,y'} \text{ for } y \neq y', \text{ while } \sum_y w_{b,y} = \psi.$$

Moreover, since $E_{a,x}$'s and $F_{b,y}$'s commute, we have,

$$\langle v_{a,x}|w_{b,y}\rangle = \langle E_{a,x}\psi|F_{b,y}\psi\rangle = \langle\psi|E_{a,x}F_{b,y}\psi\rangle = p(x,y|a,b) \geq 0.$$

Definition 3.6. The set $C_{\text{vect}}(n,k)$ is the set of all probability densities $p(x,y|a,b)$ of the form $p(x,y|a,b) = \langle v_{a,x}|w_{b,y}\rangle$ for some set of vectors $\{v_{a,x}, w_{b,y}\}$ satisfying

$$\sum_x v_{a,x} = \sum_y w_{b,y} := \psi, \ \forall a, b$$

where

- ψ is some vector in $\mathcal{H}$ with $\|\psi\| = 1$,
- $v_{a,x} \perp v_{a,x'}$ for $x \neq x'$,
- $w_{b,y} \perp w_{b,y'}$ for $y \neq y'$, and
- $\langle v_{a,x}, w_{b,y}\rangle \geq 0$.

Definition 3.7. We define $NSB(n,k) = C_{nsb}(n,k)$ as the set of all n^2k^2-tuples $p(x,y|a,b)$ such that

(1) $p(x,y|a,b) \geq 0$ for all x, y, a, b.
(2) $\sum_{x,y} p(x,y|a,b) = 1$.
(3) $\sum_y p(x,y|a,b) = \sum_y p(x,y|a,b')$, for all b, b'. (We denote this common value by $P_A(x|a)$.)
(4) $\sum_x p(x,y|a,b) = \sum_x p(x,y|a',b)$ for all a, a'. (We denote this common value by $P_B(y|b)$.

Probability densities with these properties are called *non-signalling boxes.* Axiom 3 and 4 are non-signalling conditions. Indeed, suppose we had $\sum_y p(x,y|a,b) \neq \sum_y p(x,y|a,b')$ for some x,b,b'. This means that if Alice runs experiment a and computed the probability of getting the outcome x, then she would get different probabilities depending on whether Bob runs experiment b or b'. This means that Bob could "signal" which experiment he ran, to Alice. (In most models, the axioms don't allow this to happen)

Theorem 3.8. *For each $k,n \in \mathbb{N}$, we have the sequence of inclusions*

$$C_{loc}(n,k) \subseteq C_q(n,k) \subseteq C_{qs}(n,k) \subseteq C_{qa}(n,k) \subseteq C_{qc}(n,k) \subseteq C_{vect}(n,k) \subseteq C_{nsb}(n,k) \subseteq \mathbb{R}^{n^2k^2}.$$

Next suppose we are given a finite input-output game $\mathcal{G} = (I_A, I_B, \mathcal{O}_A, \mathcal{O}_B, \lambda)$ where $|I_A| = |I_B| = n$, $|\mathcal{O}_A| = |\mathcal{O}_B| = k$, $\lambda : I_A \times I_B \times \mathcal{O}_A \times \mathcal{O}_B \to \{0,1\}$ and $t \in \{\text{loc}, q, qs, qa, qc, \text{vect}, nsb\}$, we say that $\mathcal{G}$ has a *perfect t-strategy* if there exists $p \in C_t(n,k)$ such that $\lambda(a,b,x,y) = 0 \implies p(x,y|a,b) = 0$.

Given a probability density $\pi : I_A \times I_B \to [0,1]$ and t as above, the *t-value* of $\mathcal{G}$ is defined as

$$\omega_t(\mathcal{G},\pi) = \sup\left\{ \sum_{x,y,a,b} \pi(a,b)p(x,y|a,b)\lambda(a,b,x,y) : p \in C_t(n,k) \right\}.$$

The idea is that we could distinguish among the sets $C_t(n,k)$ by either finding games with perfect strategies for one t but not another t, or we could show that the value of the game depends (non-trivially) on the choice of t.

Theorem 3.9 (JNPPSW, Ozawa)**.** *Connes Embedding Conjecture (from operator algebras) is true if and only if $C_{qa}(n,k) = C_{qc}(n,k)$ for all $k,n \in \mathbb{N}$.*

3.4. **Disambiguation Theorems.** In the literature, some authors define the sets $C_q(n,k)$, $C_{qc}(n,k)$ using POVM's; others define using PVM's. In this subsection, we aim to show that no matter which is used in defining these, we end up getting the same sets both ways.

These results appear in JNPPSW[**?**] and T. Fritz[**?**].

For the sake of proof, let us write $C_q(n,k)$ and $C_{qc}(n,k)$ to be the sets when we use POVM's in the definition, and let $\tilde{C}_q(n,k)$ and $\tilde{C}_{qc}(n,k)$ be the sets when we use PVM's in the definition. Since every PVM is also a POVM, we have $\tilde{C}_q(n,k) \subseteq C_q(n,k)$ and $\tilde{C}_{qc}(n,k) \subseteq C_{qc}(n,k)$.

Proposition 3.10. $\tilde{C}_q(n,k) = C_q(n,k)$.

Proof. Let $p(a,b|x,y) \in C_q(n,k)$. Then there exist $E_{x,a} \geq 0$ on $\mathcal{H}_A$ with $\dim(\mathcal{H}_A) < \infty$ and $\sum_a E_{x,a} = I$, and $F_{y,b} \geq 0$ on $\mathcal{H}_B$ with $\sum_b F_{y,b} = I$ (and $\dim(\mathcal{H}_B) < \infty$) and $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ such that $p(a,b|x,y) = \langle \psi | (E_{x,a} \otimes F_{y,b})\psi \rangle$ for each x,y,a,b.

By the dilation theorem, there is a Hilbert space $\tilde{\mathcal{H}}_A$ that is finite-dimensional and projections $P_{x,a}$ on $\tilde{\mathcal{H}}_A$ such that $\sum_a P_{x,a} = I$ and an isometry $V_A : \mathcal{H}_A \to \tilde{\mathcal{H}}_A$ such that $E_{x,a} = V_A^* P_{x,a} V_A$ for all x, a. One can do the same for Bob and obtain a Hilbert space $\tilde{\mathcal{H}}_B$ of finite dimension, along with projections $Q_{y,b}$ such that $\sum_b Q_{by,b} = I$ and an isometry $V_B : \mathcal{H}_B \to \tilde{\mathcal{H}}_B$ such that $V_B^* Q_{y,b} V_B = F_{y,b}$ for all b, y. Now define $\varphi = (V_A \otimes V_B)(\psi) \in \tilde{\mathcal{H}}_A \otimes \tilde{\mathcal{H}}_B$. Then $\langle \varphi, P_{x,a} \otimes Q_{y,b} \varphi \rangle \in \tilde{C}_q(n,k)$, while

$$\begin{aligned} \langle \varphi, P_{x,a} \otimes Q_{y,b} \varphi \rangle &= \langle (V_A \otimes V_B)\psi, (P_{x,a} \otimes Q_{y,b})(V_A \otimes V_B)\psi \rangle \\ &= \langle \psi | (V_A^* P_{x,a} V_A) \otimes (V_B^* Q_{y,b} V_B)\psi \rangle \\ &= p(x, y|a, b). \end{aligned}$$

Therefore, $C_q(n,k) \subseteq \tilde{C}_q(n,k)$. □

Proposition 3.11. $\tilde{C}_{qs}(n,k) = C_{qs}(n,k)$.

Proof. The proof is the same as above, except that we may not have $\mathcal{H}_A$ and $\mathcal{H}_B$ finite-dimensional. We hence assume that $\mathcal{H}_A$ and $\mathcal{H}_B$ are not necessarily finite dimensional and mimick the proof. Interestingly, the spaces $\tilde{\mathcal{H}}_A$ and $\tilde{\mathcal{H}}_B$ will still exist, and the rest of the proof will be the same. □

The last equality if hardest and many proofs rely on a result of F. Boca, which we will discuss later. Here we sketch a direct proof.

Theorem 3.12. $\tilde{C}_{qc}(n,k) = C_{qc}(n,k)$

Proof. So let us assume that $p(a, b|x, y) \in C_{qc}(n,k)$ given by

$$p(a, b|x, y) = \langle \psi | E_{x,a} F_{y,b} \psi \rangle,$$

where the E's and F's are POVM's and $E_{x,a} F_{y,b} = F_{y,b} E_{x,a},\ \forall x, y, a, b$.

Let us fix one value x_1. Recall the way that we dilated $E_{x_1,a}$ to a PVM, on $\mathcal{H} \otimes \mathbb{C}^k$ gave us $\hat{E_{x_1,a}} := P_{x_1,a} = I_{\mathcal{H}} \otimes \delta_{a,a}$ as the projection onto the a-th copy of $\mathcal{H}$ and an isometry $V : \mathcal{H} \to \mathcal{H} \otimes \mathbb{C}^k$ with $Vh = \sum_a E_{x_1,a}^{1/2} h \otimes e_a$.

For each $F_{y,b}$ set $\hat{F_{y,b}} = \sum_a F_{y,b} \otimes \delta_{a,a}$, i.e., the diagonal operator matrix with $F_{y,b}$ for the constant diagonal entry.

Also for each $x \neq x_1$ set $\hat{E_{x,a}}$ to be the operators on $\mathcal{H} \otimes \mathbb{C}^k$ described in the proof of Theorem 3.5. Note that in this construction, all of the entries of the operator matrix $\hat{E_{x,a}}$ belong to the C*-algebra generated by the positive operators $E_{x,a}$ and so these entries all commute with each of $F_{y,b}$.

Now all one needs to observe is that because $F\hat{y}, b$'s are all constant diagonal matrices, these operator matrices commute with the operator matrices $\hat{E_{x,a}}$, that is, they all commute as operators on $\mathcal{H} \otimes \mathbb{C}^k$.

One then proceeds inductively, dilating a new value of x and lifting to commuting positive operators on the larger space. Keeping track of the fact that if somethinbg was a projection in one stage of the process, that its lift remains a projection.

□

We now will move towards showing that $C_{\text{loc}}(2,2) \neq C_q(2,2)$ (so these sets are different even in the small cases). We first need the Clauser-Horne-Shimony-Holt inequality.

Let $I_A = \{a, a'\}$ and $I_B = \{b, b'\}$. Let $\mathcal{O}_A = \mathcal{O}_B = \{+,-\}$, and let $p \in C_{\text{loc}}(2,2) \subseteq \mathbb{R}^{2^2 \cdot 2^2} = \mathbb{R}^{16}$, say $p(\pm,\pm|r,s)$ for $r = a, a'$ and $s = b, b'$. Set $\sigma_{a,b}(p) = p(+,+|a,b) + p(-,-|a,b) - p(+,-|a,b) - p(-,+|a,b)$. Given $f_a, f_{a'}, g_b, g_{b'} : \Omega \to \{-1,+1\}$ (where (Ω,μ) is a probability space) such that $p(+,+|a,b) = \mu(\{t : f_a(t) = 1,\ g_b(t) = -1\})$ and $p(-,-|a,b) = \mu(\{t : f_a(t) = -1,\ g_b(t) = -1\})$, then

$$\begin{aligned}\int_\Omega f_a(t) g_b(t)\, d\mu(t) &= p(+,+|a,b) + p(-,-|a,b) - p(+,-|a,b) - p(-,+|a,b) \\ &= \sigma_{a,b}(p).\end{aligned}$$

Theorem 3.13 (CHSH-inequality)**.** *Let $p \in C_{loc}(2,2)$. Then*

$$-2 \leq \sigma_{a,b}(p) + \sigma_{a,b'}(p) + \sigma_{a',b}(p) - \sigma_{a',b'}(p) \leq 2.$$

Proof. By the above work, letting $M = \sigma_{a,b}(p) + \sigma_{a,b'}(p) + \sigma_{a',b}(p) - \sigma_{a',b'}(p)$, we have

$$M = \int_\Omega (f_a(t)[g_b(t) + g_{b'}(t)] + f_{a'}(t)[g_b(t) - g_{b'}(t)])\, d\mu(t).$$

Note that $g_b(t) + g_{b'}(t) \in \{-2, 2\}$ if and only if $g_b(t) - g_{b'}(t) = 0$ (both take values in $\{-1,1\}$), while $g_{b'}(t) - g_{b'}(t) \in \{-2,2\}$ if and only if $g_b(t) + g_{b'}(t) = 0$. Looking the integrand and noting that $f_a(t), f_{a'}(t) \in \{-1,1\}$, it follows that

$$-2 \leq f_a(t)[g_b(t) + g_{b'}(t)] + f_{a'}(t)[g_b(t) - g_{b'}(t)] \leq 2,\ \forall t.$$

Since μ is a probability measure, integrating gives $-2 \leq M \leq 2$, as desired. □

Now let us look at some special elements of $C_q(2,2)$. Let $\psi = \frac{1}{\sqrt{2}}(e_0 \otimes e_0 + e_1 \otimes e_1) \in \mathbb{C}^2 \otimes \mathbb{C}^2$. Given an angle θ, let $v_\theta = (\cos\theta, \sin\theta)^t$. Then define

$$P_\theta = v_\theta v_\theta^* = \begin{pmatrix} \cos^2(\theta) & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2(\theta) \end{pmatrix}.$$

It follows that $I - P_\theta = P_\theta^\perp = \begin{pmatrix} \sin^2(\theta) & -\cos(\theta)\sin(\theta) \\ -\cos(\theta)\sin(\theta) & \cos^2(\theta) \end{pmatrix}$. Given $\theta_a, \theta_b, \theta_{a'}, \theta_{b'}$, define $E_{a,+} = P_{\theta_a}$, $E_{a,-} = P_{\theta_a}^\perp$, $E_{a',+} = P_{\theta_{a'}}$, $E_{a',-} = P_{\theta_{a'}}^\perp$, $F_{b,+} = P_{\theta_b}$, $F_{b,-} = P_{\theta_b}^\perp$, $F_{b',+} = P_{\theta_{b'}}$, $F_{b',-} = P_{\theta_{b'}}^\perp$. Then the probability arising from this for $+, +, a, b$ is

$$\begin{aligned} p(+,+|a,b) &= \langle\psi|(E_{a,+}\otimes F_{b,+}\psi\rangle \\ &= \frac{1}{2}\langle e_0\otimes e_0 + e_1\otimes e_1|(\cos^2\theta_a e_0 + \cos\theta_a\sin\theta_a e_1)\otimes(\cos^2\theta_b e_0 + \cos\theta_b\sin\theta_b e_1) \\ &\quad + (\cos\theta_a\sin\theta_a e_0 + \sin^2(\theta_a)e_1)\otimes(\cos\theta_b\sin\theta_b e_0 + \sin^2\theta_b e_1)\rangle \\ &= \frac{1}{2}[\cos^2\theta_a\cos^2\theta_b + 2\cos\theta_a\sin\theta_a\cos\theta_b\sin\theta_b + \sin^2\theta_a\sin^2\theta_b] \\ &= \frac{1}{2}(\cos\theta_a\cos\theta_b + \sin\theta_a\sin\theta_b)^2 \\ &= \frac{1}{2}\cos^2(\theta_a-\theta_b). \end{aligned}$$

Through a similar calculation we have $p(-,-|a,b) = \frac{1}{2}\cos^2(\theta_a-\theta_b)$, while

$$p(+,-|a,b) = p(-,+|a,b) = \frac{1}{2}\sin^2(\theta_a-\theta_b).$$

It follows that

$$\sigma_{a,b}(p) = \cos^2(\theta_a-\theta_b) - \sin^2(\theta_a-\theta_b) = \cos(2(\theta_a-\theta_b)).$$

Hence, we have

$$\begin{aligned} \sigma_{a,b}(p) + \sigma_{a,b'}(p) + \sigma_{a',b}(p) + \sigma_{a',b} - \sigma_{a',b'}(p) &= \cos(2\theta_a - 2\theta_b) + \cos(2\theta_a - 2\theta_{b'}) \\ &\quad + \cos(2\theta_{a'} - 2\theta_b) - \cos(2\theta_{a'} - 2\theta_b). \end{aligned}$$

Set $\theta_{a'} = \frac{\pi}{4}$, $\theta_{b'} = 0$ and $\theta_a = \theta_b = \frac{\pi}{8}$. Then the above expression becomes

$$\cos(0) + \cos\left(\frac{\pi}{4}\right) + \cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{2}\right) = 1 + \sqrt{2} > 2.$$

This violates the CHSH inequality, so that $p \notin C_{\text{loc}}(2,2)$. It follows that $C_{\text{loc}}(2,2) \subsetneq C_q(2,2)$.

Problem 3.14. *Find and justify*

$$\max\{\cos(\theta_a-\theta_b)+\cos(\theta_a-\theta_{b'})+\cos(\theta_{a'}-\theta_b)-\cos(\theta_{a'}-\theta_{b'}) : \theta_a, \theta_b, \theta_{a'}, \theta_{b'}\}.$$

Proof. **Homework problem 16**; due 15th March, Tuesday. □

Next recall that $p(x,y|a,b) \in C_{nsb}$ if and only if $\sum_y p(x,y|a,b) = \sum_y p(x,y|a,b') := p_A(x|a)$ and $\sum_x p(x,y|a,b) = \sum_x p(x,y|a',b) = p_B(y|b)$.

Problem 3.15. *Prove that if $p \in C_{vect}(2,2)$, then*

$$\sigma_{a,b} + \sigma_{a,b'} + \sigma_{a',b} - \sigma_{a',b'} < 4,$$

and conclude that $C_{nsb}(2,2) \neq C_{vect}(2,2)$. (Hint: Prove by the method of contradiction, i.e., assume that one of them has quantity 4 above and arrive at a contradiction.)

Problem 3.16. *Prove that $C_{vect}(n,k) \subseteq C_{nsb}(n,k)$ for every n, k.*

We stated in Theorem **??** that for each $k, n \in \mathbb{N}$, we have the sequence of inclusions

$$C_{\text{loc}}(n,k) \subseteq C_q(n,k) \subseteq C_{qs}(n,k) \subseteq C_{qa}(n,k) \subseteq C_{qc}(n,k) \subseteq C_{\text{vect}}(n,k) \subseteq C_{nsb}(n,k) \subseteq \mathbb{R}^{n^2k^2}.$$

We will try and prove some of these inclusions. But before that let's recall and state the key conjectures as problems:

Proposition 3.17. *For any $n, k \in \mathbb{N}$, $C_{loc}(n,k) \subseteq C_{qc}(n,k)$.*

Proof. Let $f_a, g_b : \Omega \to \{1, ..., k\}$ for $1 \le a, b \le m$ be such that $p(x, y|a, b) = \mu(\{t \in \Omega : f_a(t) = x,\ g_b(t) = y\})$ for all x, y. Let us define

$$\Omega_{a,i} = \{t : f_a(t) = i\}$$

and

$$\Omega'_{b,j} = \{t : g_b(t) = j\}.$$

We observe that

$$\bigcup_{i=1}^{k} \Omega_{a,i} = \bigcup_{j=1}^{k} \Omega'_{b,j} = \Omega.$$

We further define

$$\chi_{a,i}(t) = \begin{cases} 1 \text{ if } t \in \Omega_{a,i} \\ 0 \text{ if } t \notin \Omega_{a,i}. \end{cases}$$

Similarly, we define

$$\chi'_{b,j}(t) = \begin{cases} 1 \text{ if } t \in \Omega'_{b,j} \\ 0 \text{ if } t \notin \Omega'_{b,j}. \end{cases}$$

Note that

$$p(i,j|a,b) = \mu(\Omega_{a,i} \cap \Omega'_{b,j}) = \int_{\Omega_{a,i}\cap\Omega'_{b,j}} 1 \, d\mu = \int_{\Omega} \chi_{a,i}\chi'_{b,j} \, d\mu.$$

We then define $\mathcal{H} = L^2(\Omega, \mu)$. For this Hilbert space let $E_{a,i} = M_{\chi_{a,i}}$, which denotes multiplication by $\chi_{a,i}$ on $L^2(\Omega, \mu)$. Then $\sum_{i=1}^k E_{a,i}$ is multiplication by $\sum_{i=1}^k \chi_{a,i} = 1$, so that $\sum_{i=1}^k E_{a,i} = M_{\sum_{i=1}^k E_{a,i}} = M_1 = I$. Similarly, if $F_{b,j} = M_{\chi'_{b,j}}$, then $\sum_{j=1}^k F_{b,j} = I$. Whenever $h \in \mathcal{H}$, we have

$$\langle h|E_{a,i}h\rangle = \int_{\Omega} \overline{h}(E_{a,i}h)\, d\mu = \int_{\Omega_{a,i}} |h|^2 \, d\mu \ge 0,$$

so that $E_{a,i} \ge 0$. It is easy to see that $E_{a,i}^2 = M_{\chi_{a,i}^2} = M_{\chi_{a,i}} = E_{a,i}$ so each $E_{a,i}$ is a projection (similarly, the $F_{b,j}$'s are projections). All of these operators ($E_{a,i}$'s, $F_{b,j}$'s), being multiplication operators, commute with each other (and hence each $E_{a,i}$ commutes with each $F_{b,j}$). Finally, we set

$\psi = 1 \in \mathcal{H}$. Then $\|\psi\|^2 = \int_\Omega 1\, d\mu = \mu(\Omega) = 1$ so ψ is a state. We observe that for each a, b, i, j, we have

$$\begin{aligned}\langle \psi | E_{a,i} F_{b,j} \psi \rangle &= \int_\Omega \bar{1} (M_{\chi_{a,i}} M_{\chi'_{b,j}} \cdot 1)\, d\mu \\ &= \int_{\Omega_{a,i} \cap \Omega'_{b,j}} 1\, d\mu \\ &= \mu(\Omega_{a,i} \cap \Omega'_{b,j}) = p(i, j|a, b).\end{aligned}$$

It follows that $C_{\text{loc}}(n, k) \subseteq C_{qc}(n, k)$. □

This leads us to,

Corollary 3.18. *$C_{qc}(n, k) \subseteq C_{vect}(n, k)$.*

Proof. Let $p(x, y|a, b) \in C_{qc}(n, k)$. Then Theorem **??** guarantees the existence of projections $P_{a,x}$ and $Q_{b,y}$ on a Hilbert space $\mathcal{H}$ with $\sum_x P_{a,x} = \sum_y Q_{b,y} = I$ such that $P_{a,x} Q_{b,y} = Q_{b,y} P_{a,x}$ on $\mathcal{H}$ for all a, b, x, y. Moreover, there is $\psi \in \mathcal{H}$ with $\|\psi\| = 1$ such that $p(x, y|a, b) = \langle \psi | P_{a,x} Q_{b,y} \psi \rangle$. Set $h_{a,x} = P_{a,x}\psi$ and $k_{b,y} = Q_{b,y}\psi$. Then $\sum_x h_{a,x} = \psi$ for all a and $\sum_y k_{b,y} = \psi$ for all b. Since $P_{a,x} P_{a,x'} = 0$ (because they are orthogonal to each other as projections), we have $h_{a,x} \perp h_{a,x'}$ for $x \neq x'$, and similarly $k_{b,y} \perp k_{b,y'}$ for $y \neq y'$.

Finally, $\langle h_{a,x} | k_{b,y} \rangle \in C_{\text{vect}}(n, k)$ but $\langle h_{a,x} | k_{b,y} \rangle = \langle P_{a,x}\psi | Q_{b,y}\psi \rangle$, but this is equal to $\langle \psi | P_{a,x} Q_{b,y} \psi \rangle = p(x, y|a, b)$. Therefore, $C_{qc}(n, k) \subseteq C_{\text{vect}}(n, k)$. □

Notice that in the proof above, the product $P_{a,x} P_{a,x'}$ had no reason to vanish if we have had merely positive operators. This is where we made use of the fact that these are actually orthogonal projections.

We end this subsection by proving an important theorem for finite input-output games.

Theorem 3.19. *Let $\mathcal{G} = (I_A, I_B, \mathcal{O}_A, \mathcal{O}_B, \lambda)$ where λ is the rule function as before. Then $\mathcal{G}$ has a perfect loc-strategy if and only if $\mathcal{G}$ has a perfect deterministic strategy.*

Proof. If $\mathcal{G}$ has a perfect deterministic strategy, then this means that there are functions $f : I_A \to \mathcal{O}_A$ and $g : I_B \to \mathcal{O}_B$ such that $\lambda(a, b, f(a), g(b))$ never violates the rule, which implies that $\lambda(a, b, f(a), g(b)) = 1$ for all a, b. (The reason we are not able to see the probability space is because it has got only one point!) Let $\Omega = \{t_0\}$ and $f_a : \Omega \to \mathcal{O}_A$ and $g_b : \Omega \to \mathcal{O}_B$ be given by $f_a(t_0) = f(a)$ and $g_b(t_0) = g(b)$. Suppose that $\lambda(a, b, x, y) = 0$. We must show that $\mu(\{t : f_a(t) = x,\ g_b(t) = y\}) = 0$. Note that for t_0 we have $\lambda(a, b, f_a(t_0), g_b(t_0)) = \lambda(a, b, f(a), g(b)) = 1$ so that $\{t : f_a(t) = x,\ g_b(t) = y\} = \emptyset$, and has measure zero. It follows that $\mathcal{G}$ has a perfect loc-strategy.

Conversely, suppose that $\mathcal{G}$ has a perfect loc-strategy. Then there is a probability space (Ω, μ) and functions $f_a : \Omega \to \mathcal{O}_A$ and $g_b : \Omega \to \mathcal{O}_B$ such that whenever $\lambda(a,b,x,y) = 0$, we have $\mu(\{t : f_a(t) = x,\, g_b(t) = y\}) = 0$. Set $\Omega_{a,x} = \{t : f_a(t) = x\}$. These sets are pairwise disjoint with union equal to Ω. Similarly, if $\Omega'_{b,y} = \{t : g_b(t) = y\}$, then the sets $\{\Omega'_{b,y}\}$ are pairwise disjoint with union equal to Ω. Then $\mu(\Omega_{a,x} \cap \Omega'_{b,y}) = 0$. Define

$$\mathcal{N} = \bigcup_{\substack{a,b,x,y \\ \lambda(a,b,x,y)}} (\Omega_{a,x} \cap \Omega'_{b,y}),$$

which is a finite union of sets of measure zero, so that $\mu(\mathcal{N}) = 0$. Therefore, $\mathcal{N} \neq \Omega$. Pick $t_0 \in \Omega \setminus \mathcal{N}$. Define $f : I_A \to \mathcal{O}_A$ and $g : I_B \to \mathcal{O}_B$ by $f(a) = f_a(t_0)$ and $g(b) = g_b(t_0)$. By definition, since $t_0 \notin \mathcal{N}$, we must have $t_0 \notin \Omega_{a,x} \cap \Omega'_{b,y}$ for all a, x, b, y. Therefore, if we had $\lambda(a, b, f(a), g(b)) = 0$ for some a, b, then with $x = f_a(t_0)$ and $y = g_b(t_0)$, we would have $\lambda(a,b,x,y) = 0$. Hence, $t_0 \in \Omega_{a,x} \cap \Omega'_{b,y}$ which is a contradiction. Hence, $\lambda(a, b, f(a), g(b)) = 1$ for all a, b. Therefore, $\mathcal{G}$ has a perfect deterministic strategy. □

Like the situation where loc strategies don't fool the referee in any way (since the colouring number stays the same), if we allow any non-signalling box strategy, then the situation becomes much less interesting.

Proposition 3.20 (Paulsen-Todorov). *Let G be a graph on $n \geq 2$ vertices. Then $\chi_{nsb}(G) = 2$.*

Proof. For $1 \leq i, j \leq 2$, we set

$$p(i,j|v,v) = \begin{cases} \frac{1}{2} \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}.$$

For $v \neq w$, set

$$p(i,j|v,w) = \begin{cases} 0 \text{ if } i = j \\ \frac{1}{2} \text{ if } i \neq j. \end{cases}$$

We claim that $p \in C_{nsb}(n, 2)$ and p is a perfect *nsb*-strategy. Indeed, if $i \neq j$ then $p(i,j|v,v) = 0$ so the first rule is satisfied. For the second rule, whenever $v \sim w$ and $i = j$ we have $p(i,j|v,w) = 0$ so the second rule is satisfied. Therefore, p is a perfect strategy. It is easy to check that

$$\sum_{i,j} p(i,j|v,w) = 2 \cdot \frac{1}{2} = 1;$$

$$\sum_{j=1}^{2} p(i,j|v,w) = \frac{1}{2} \text{ for all } w, \text{ and}$$

$$\sum_{i=1}^{2} p(i,j|v,w) = \frac{1}{2} \text{ for all } v.$$

Hence, $p \in C_{nsb}(n, 2)$. This completes the proof. □

4. The Group Algebra Approach to Densities

Definition 4.1 (Unital $*$-Representation)**.** Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{H}$ be a Hilbert space. A *unital $*$-representation of $\mathcal{A}$ on $\mathcal{H}$* is a map $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that

(1) $\pi(1) = I$ (unital)
(2) π is linear
(3) $\pi(XY) = \pi(X)\pi(Y)$.
(4) $\pi(X^*) = \pi(X)^*$.

With all of these properties together, π is a $*$-homomorphism.

What follows next is a celebrated theorem from the theory of C^*-algebras. This is essential for moving further.

Theorem 4.2 (Gelfand-Naimark-Segal Construction)**.** *Let $\mathcal{A}$ be a unital C^*-algebra and $s : \mathcal{A} \to \mathbb{C}$ be a state. Then there is a Hilbert space $\mathcal{H}_s$, a vector $\varphi \in \mathcal{H}_s$ of norm 1 and a unital $*$-representation $\pi_s : \mathcal{A} \to \mathcal{B}(\mathcal{H}_s)$ such that*

(1) $s(X) = \langle \varphi | \pi_s(X)\varphi \rangle$.
(2) $\{\pi(X)\varphi : X \in \mathcal{A}\}$ *is dense in* $\mathcal{H}_s$.

This is called the GNS representation of s.

Theorem 4.3. *Let $\mathcal{G} = (I, \mathcal{O}, \lambda)$ be a synchronous game. Then $\mathcal{G}$ has a perfect qc-strategy if and only if there is a unital C^*-algebra generated by projections $\{E_{vi} : v \in I, i \in \mathcal{O}\}$, and a tracial state τ on this C^*-algebra such that*

(1) $\sum_i E_{vi} = I$ *for all* v.
(2) *If* $\lambda(v, w, i, j) = 0$ *then* $E_{v,i}E_{w,j} = 0$.

Proof. For this proof we assume that $|I| = n$ and $|\mathcal{O}| = k$. Suppose we have a unital C^*-algebra and tracial state with the properties above. Then if τ is our tracial state, by the previous theorem we may set $p(i, j|v, w) = \tau(E_{v,i}E_{w,j}) \in C^s_{qc}(n, k)$. If $\lambda(v, w, i, j) = 0$ then $p(i, j|v, w) = \tau(E_{v,i}E_{w,j}) = \tau(0) = 0$ by assumption. Hence, $\mathcal{G}$ has a perfect qc-strategy.

Conversely, suppose that $p(i, j|v, w) \in C^s_{qc}(n, k)$ is a perfect qc-strategy for $\mathcal{G}$. We saw that if we took the representation $p(i, j|v, w) = \langle \psi | E_{v,i}F_{w,j}\psi \rangle$ of $p(i, j|v, w)$ and let $\mathcal{A}$ be the C^*-algebra generated by $\{E_{v,i}\}_{v,i}$, then $p(i, j|v, w) = \tau(E_{v,i}E_{w,j})$ gives rise to a trace on $\mathcal{A}$. If $\lambda(v, w, i, j) = 0$ then $\tau(E_{v,i}E_{w,j}) = 0$, and we want to show that $E_{v,i}E_{w,j} = 0$. Take any $X \in \mathcal{A}$; then $X^*X \leq \|X\|^2 I$. Hence, $E^*_{w,j}E^*_{v,i}(X^*X)E_{v,i}E_{w,j} \leq \|X\|^2 E^*_{w,j}E^*_{v,i}E_{v,i}E_{w,j}$. Taking traces we obtain

$$0 \leq \tau(E^*_{w,j}E^*_{v,i}X^*XE_{v,i}E_{w,j}) \leq \|X\|^2 \tau(E^*_{w,j}E_{v,i}E_{w,j}) = \|X\|^2 \tau(E_{v,i}E_{w,j}) = 0.$$

Therefore, $\tau(E^*_{w,j}E^*_{v,i}X^*XE_{v,i}E_{w,j}) = 0$. Switching some variables around shows that

$$\tau(XE_{v,i}E_{w,j}E^*_{w,j}E^*_{v,i}X^*) = 0.$$

For convenience, let $A = XE_{v,i}E_{w,j}$, so that $AA^* = XE_{v,i}E_{w,j}E^*_{w,j}E^*_{v,i}X^*$. Then by the GNS construction, on some Hilbert space (and for some unit vector φ), we have

$$\begin{aligned}0 = \tau(AA^*) &= \langle\varphi|\pi_\tau(AA^*)\varphi\rangle \\ &= \|\pi_\tau(A^*)\varphi\| \\ &= \|\pi_\tau(E^*_{w,j}E^*_{v,i}X^*)\varphi\| \\ &= \|\tilde{E}_{w,j}\tilde{E}_{v,i}\pi_\tau(X^*)\varphi\|\end{aligned}$$

But using GNS, $\{\pi_\tau(X^*) : X \in \mathcal{A}\}$ is dense in the new Hilbert space, so that $\tilde{E}_{w,j}\tilde{E}_{v,i} = 0$. Hence, $\tilde{E}_{v,i}\tilde{E}_{w,j} = 0$ as desired. □

Note that in the above proof, $\tilde{E}_{v,i}$'s are not Alice's original projections.

Corollary 4.4. *Let G be a graph. Then $\chi_{qc}(G) \leq k$ if and only if there is a unital C^*-algebra $\mathcal{A}$ with trace and projections $\{E_{v,i}\}_{1\leq i\leq k,\, 1\leq v\leq n}$ in $\mathcal{A}$ with $\sum_{i=1}^k E_{v,i} = I$ for all v such that $v \sim w$ implies that $E_{v,i}E_{w,i} = 0$ for all $1 \leq i \leq k$.*

It is worthwhile noticing that $E_{v,i}$'s, in above corollary, can't be any projections in any C^*-algebra. They have to be projections a C^*-algebra which has got a trace.

We next discuss the characterization of a synchronous game with a perfect q-strategy. We need the following theorem about finite-dimensional C^*-algebra before proceeding into this discussion.

Theorem 4.5. *Let $\mathcal{A}$ be a finite-dimensional C^*-algebra. Then there are $n_1, ..., n_L \in \mathbb{N}$ and a unital $*$-isomorphism $\pi : \mathcal{A} \to M_{n_1} \oplus \cdots \oplus M_{n_L}$. That is to say, $\mathcal{A} \simeq M_{n_1} \oplus \cdots \oplus M_{n_L}$.*

5. Tsirelson's Correlation Matrices

Tsirelson's idea was to work with self-adjoint unitaries (popularly known as reflections) on the set set $\{+1, -1\}$ under the multiplication operation instead of working with projections on the set $\{0, 1\}$ under the addition operation. Recall that the spectrum of projections consists of 0 and 1, and the spectrum of self-adjoint unitaries consists of 1 and -1.

Definition 5.1 (Quantum Correlation Matrix)**.** An $n \times n$ matrix $(c_{s,t})^n_{s,t=1}$ of real numbers is called a *quantum correlation matrix* if there exists $A_s = A^*_s \in M_p$ with $-I \leq A_s \leq I$ and there exists $B_t = B^*_t \in M_q$ with $-I \leq B_t \leq I$, along with the existence of $\psi \in \mathbb{C}^p \otimes \mathbb{C}^q$ such that $\|\psi\| = 1$ and $c_{s,t} = \langle\psi|(A_s \otimes B_t)\psi\rangle$. Let $\mathrm{Cor}_q(n) \subseteq M_n(\mathbb{R})$ be the set of all such matrices. Similarly, let $\mathrm{Cor}_{qc}(n) \subseteq M_n(\mathbb{R})$ be the set of all matrices of the form $(c_{s,t}) \in M_n$, where $c_{s,t} = \langle\psi|(A_s \otimes B_t)\psi\rangle$ for some $A_s = A^*_s$, $B_t = B^*_t \in \mathcal{B}(\mathcal{H})$ with $-I \leq A_s \leq I$ and $-I \leq B_t \leq I$ and $\psi \in \mathcal{H}$ with $\|\psi\| = 1$, such that $A_sB_t = B_tA_s$ for all s, t.

Proposition 5.2 (Disambiguation)**.** *A matrix* $(c_{s,t})_{s,t=1}^n$ *is in* $Cor_q(n)$ *if and only if there are matrices* $A_s = A_s^* \in M_p$ *and* $B_t = B_t^* \in M_q$ *with* $A_s^2 = I$ *and* $B_t^2 = I$*, along with* $\psi \in \mathbb{C}^p \otimes \mathbb{C}^q$ *with* $\|\psi\| = 1$*, such that* $c_{s,t} = \langle \psi | A_s \otimes B_t \psi \rangle$*.*

Remark 5.3. The above proposition tells us that no matter whether we choose projections or self-adjoint unitaries, the sets we get is the same.

Proof. We use the Halmos trick. For $H = H^*$ and $-I \le H \le I$, we may set

$$U = \begin{pmatrix} H & \sqrt{I - H^2} \\ \sqrt{I - H^2} & -H \end{pmatrix}$$

so that $U = U^*$ and $U^2 = I$. We now set

$$U_s = \begin{pmatrix} A_s & \sqrt{I - A_s^2} \\ \sqrt{I - A_s^2} & -A_s \end{pmatrix} \text{ and } V_t = \begin{pmatrix} B_t & \sqrt{I - B_t^2} \\ \sqrt{I - B_t^2} & -B_t \end{pmatrix}.$$

Then $U_s^2 = I$ and $V_t^2 = I$. Let $\tilde{\psi} = \begin{pmatrix} \psi \\ 0 \end{pmatrix}$, so that $\langle \psi | A_s \otimes B_t \psi \rangle = \langle \tilde{\psi} | (U_s \otimes V_t) \tilde{\psi} \rangle$. The other inclusion of sets is obvious, so we are done. □

We will see that in the context of working with reflections, we actually have $\mathrm{Cor}_q(n) = \mathrm{Cor}_{qc}(n)$. First, we need a bit of background on Clifford unitaries.

Definition 5.4 (Clifford Unitaries)**.** A family of $n \times n$ self-adjoint unitary matrices that anti-commute is described as *Clifford unitaries.*

So, a set of matrices $\{X_1, ..., X_m\}$ is Clifford unitaries if we have $X_i = X_i^*$, $X_i^2 = I$ for every i and $X_i X_j = -X_j X_i$ for all $i \neq j$. For the sake of completeness, here is a construction of such a set of unitaries. Let

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so that

$$XZ = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } ZX = -XZ.$$

This set consists of $m = 2$ self-adjoint unitaries that anti-commute and hence is a set of Clifford unitaries. Suppose $m > 2$ and we want to construct a set of Clifford unitaries that has cardinality m. Then we let,

$$C_1 = Z \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{m-1 \text{ times}},$$

$$C_2 = X \otimes Z \otimes I_2 \otimes \cdots \otimes I_2.$$

Similarly, for any i we let

$$C_i = \underbrace{X \otimes \cdots \otimes X}_{i-1 \text{ times}} \otimes Z \otimes \underbrace{I_2 \otimes \cdots \otimes I_2}_{m-i \text{ times}}.$$

Then each $C_i = C_i^*$ and $C_i^2 = I$. If $i < j$, then

$$C_iC_j = X^2 \otimes \cdots \otimes X^2 \otimes ZX \otimes X \otimes \cdots \otimes X \otimes Z \otimes I ...$$

and

$$C_jC_i = X^2 \otimes \cdots \otimes X^2 \otimes XZ \otimes X \otimes ... \otimes X \otimes Z \otimes I ...$$

so that $C_iC_j = -C_jC_i$. (Note that all the C_i's are real matrices and hence $C_i = C_i^t$ for every i.)

Lemma 5.5. *If $A, B \in M_d$ and $\psi = \frac{1}{\sqrt{d}}(e_1 \otimes e_1 + \cdots + e_d \otimes e_d)$, then $\langle \psi | (A \otimes B)\psi \rangle = tr(AB^t)$. (Note: "tr" denotes normalized trace.)*

Proof. Let $A = (a_{ij})$ and $B = (b_{ij})$. Then

$$\begin{aligned}\langle \psi | (A \otimes B)\psi \rangle &= \frac{1}{d}\sum_{i,j=1}^{d} \langle e_i \otimes e_i | (A \otimes B) e_j \otimes e_j \rangle \\ &= \frac{1}{d}\sum_{i,j=1}^{d} \langle e_i | A e_j \rangle \langle e_i | B e_j \rangle \\ &= \frac{1}{d}\sum_{i,j=1}^{d} a_{ij} b_{ij} = \frac{1}{d} tr(AB^t).\end{aligned}$$

Hence, $\langle \psi | (A \otimes B)\psi \rangle = \mathrm{tr}(AB^t)$. □

Theorem 5.6 (Tsirelson, 1987)**.** *Let $C = (c_{s,t})_{s,t=1}^n$ be a real $n \times n$ matrix. The following are equivalent.*

1. *$C \in Cor_q(n)$.*
2. *There is m and vectors $x_s, y_t \in \mathbb{R}^m$ with $\|x_s\| \le 1$ and $\|y_t\| \le 1$ such that $C = (\langle x_s, y_t \rangle)$. Moreover, m can always be taken to be $m = 4n$.*

Consequently, $Cor_q(n) = Cor_{qc}(n) = \{(\langle x_s, y_t \rangle) : x_s, y_t \in \mathbb{R}^{4n}, \|x_s\| \le 1, \|y_t\| \le 1\}$.

Proof. Let us show that (i) implies (ii). Write $c_{s,t} = \langle \psi | (A_s \otimes B_t)\psi \rangle = \langle (A_s \otimes I)\psi | (I \otimes B_t)\psi \rangle$. Let $x'_s = (A_s \otimes I)\psi$ and $y'_t = (I \otimes B_t)\psi$. Then $\|x'_s\| \le 1$ and $\|y'_t\| \le 1$, while $c_{s,t} = \langle x'_s, y'_t \rangle$. Now, $\mathrm{span}\{x'_s, y'_t\}_{s,t=1}^n$ is m-dimensional where $m \le 2n$. Identify this subspace of $\mathbb{C}^{2n}$ with $\mathbb{C}^m$, and write $x'_s = (x'_s(1), ..., x'_s(m))$ and $y'_t = (y'_t(1), ..., y'_t(m))$. Then write $x'_s(k) = a_s(k) + ib_s(k)$ where $a_s, b_s \in \mathbb{R}$. Similarly, write $y'_t(k) = \alpha_t(k) + i\beta_t(k)$. Then define $x_s = (a_s(1), -b_s(1), ..., a_s(m), -b_s(m)) \in \mathbb{R}^{2m}$ and define $y_t = (\alpha_t(1), \beta_t(1), ..., \alpha_t(m), \beta_t(m)) \in \mathbb{R}^{2m}$. Then $\|x_s\| = \|x'_s\|$ and $\|y_t\| = \|y'_t\|$. Finally, since $\langle x'_s, y'_t \rangle \in \mathbb{R}$ we still have $\langle x'_s, y'_t \rangle = \langle x_s, y_t \rangle$. Note that $2m \le 4n$, but by adding 0's we can always make these vectors into $(4n)$-tuples, as desired.

Now we show that (ii) implies (i). Let $x_s, y_t \in \mathbb{R}^m$ for $s, t = 1, ..., n$ be such that $\|x_s\| \le 1$, $\|y_t\| \le 1$ and $c_{s,t} = \langle x_s, y_t \rangle$. Now we have to build self-adjoint operators and a unit vector. Write $x_s = (x_s(1), ..., x_s(m))$ and

$y_t = (y_t(1), ..., y_t(m))$. Take m Clifford unitaries $C_1, ..., C_m$ on $\mathbb{C}^d$, and set $A_s = \sum_{i=1}^m x_s(i)C_i$. Let $B_t = \sum_{i=1}^m y_t(i)C_i$, so that $A_s = A_s^*$ and $B_t = B_t^*$. Note that

$$\begin{aligned} A_s^2 &= \sum_{i,j=1}^m x_s(i)x_s(j)C_iC_j \\ &= \sum_{i=1}^m x_s(i)^2C_i^2 + \sum_{i<j}(x_s(i)x_s(j) - x_s(i)x_s(j))C_iC_j \\ &= \left(\sum_{i=1}^m x_s(i)^2C_i^2\right) \leq I. \end{aligned}$$

Hence, $\sigma(A_s) \subseteq [-1, 1]$ so that $-I \leq A_s \leq I$. Similarly, $-I \leq B_t \leq I$. Denote by D^T the transpose of a matrix D. Then

$$A_sB_t^T = \sum_{i,j=1}^m x_s(i)y_t(j)C_iC_j^T = \left[\sum_{i=1}^m x_s(i)y_t(i)C_iC_i^T\right] + \sum_{i\neq j} x_s(i)y_t(j)C_iC_j.$$

Thus, $A_sB_t^T = \sum_{i=1}^m x_s(i)y_t(i)I + \text{stuff}$. Taking the normalized trace gives $\sum_{i=1}^m x_s(i)y_t(i) + \text{stuff}$. Now note that if $i \neq j$, then $tr(C_iC_j) = tr(C_jC_i)$ and $tr(C_iC_j) = tr(-C_jC_i) = -tr(C_jC_i)$ so that $tr(C_iC_j) = 0$. Hence, $tr(A_sB_t^T) = \sum_{i=1}^m x_s(i)y_t(i) = \langle x_s, y_t\rangle$. By the lemma, if $\psi = \frac{1}{\sqrt{d}}(e_1 \otimes e_1 + \cdots + e_d \otimes e_d)$ then $\langle x_s, y_t\rangle = \langle\psi|(A_s \otimes B_t)\psi\rangle$, and this gives the desired result. □

Now, let us go back to the other situation and see how things differ. Let us consider the sets $C_q(n, 2)$ and $C_{qc}(n, 2)$. In this case Alice has PVM's $\{E_{s,0}, E_{s,1}\}_{1\leq s\leq m}$ with $E_{s0} + E_{s1} = I$, and Bob has PVM's $\{F_{t,0}, F_{t,1}\}_{1\leq t\leq n}$ with $F_{t0}+F_{t1} = I$. If we had $A_s = A_s^*$ with $A_s^2 = I$ and $B_t^* = B_t$ with $B_t^2 = I$, then $\sigma(A_s), \sigma(B_t) \subseteq \{\pm1\}$. In this case we let $E_{s0} = \frac{I+A_s}{2}$, $E_{s1} = \frac{I-A_s}{2}$, $F_{t0} = \frac{I+B_t}{2}$, $F_{t1} = \frac{I-B_t}{2}$, so that

$$E_{si} = \frac{I + (-1)^iA_s}{2},\ F_{tj} = \frac{I + (-1)^jB_t}{2}.$$

We obtain $p(i, j|s, t) \in C_q(n, 2)$ given by

$$p(i, j|s, t) = \langle\psi|(E_{si}\otimes F_{tj})\psi\rangle = \frac{1}{4}\langle\psi|(I+(-1)^iA_s\otimes I+(-1)^jI\otimes B_t+(-1)^{i+j}A_s\otimes B_t)\psi\rangle.$$

If we use the Clifford construction from earlier, then

$$\begin{aligned} p(i, j|s, t) &= \frac{1}{4}tr(I + (-1)^iA_s + (-1)^jB_t^T + (-1)^{i+j}A_sB_t^T) \\ &= \frac{1}{4}(1 + 0 + 0 + (-1)^{i+j}\langle x_s, y_t\rangle) = \frac{1}{4} + \frac{1}{4}(-1)^{i+j}\langle x_s, y_t\rangle, \end{aligned}$$

since A_s and B_t are linear combination of Clifford unitaries and we know that the Clifford unitaries constructed earlier are of trace zero thereby rendering the trace of A_s and trace of B_t zero.

The above discussion can be summarized into the following theorem.

Theorem 5.7. *Let x_s, y_t be vectors in $\mathbb{R}^m$ for $1 \leq s,t \leq n$ with $\|x_s\| \leq 1$ and $\|y_t\| \leq 1$, such that $\langle x_s, y_t\rangle \in \mathbb{R}$ for all s,t. Then there exists $p(i,j|s,t) \in C_q(n,2)$ such that $p(i,j|s,t) = \frac{1}{4}[1+(-1)^{i+j}\langle x_s, y_t\rangle]$.*

Note that if p is as in the theorem, then $p(0,0|s,t) = p(1,1|s,t)$ and $p(0,1|s,t) = p(1,0|s,t)$ for all s,t.

Theorem 5.8. *Let $p(i,j|s,t) \in C_{vect}(n,2)$ be such that $p(0,0|s,t) = p(1,1|s,t)$ and $p(0,1|s,t) = p(1,0|s,t)$ for all s,t. Then*

(1) *There exist vectors x_s, y_t for $1 \leq s,t \leq n$ such that $\|x_s\|, \|y_t\| \leq 1$ and $p(i,j|s,t) = \frac{1}{4}[1+(-1)^{i+j}\langle x_s, y_t\rangle]$.*
(2) *$p(i,j|s,t) \in C_q(n,2)$.*

Proof. The second claim follows from the first by the previous theorem. To show (1), note that since $p \in C_{\text{vect}}(n,2)$, there exist vectors $v_{s0}, v_{s1}, w_{t0}, w_{t1}$ such that $v_{s0} \perp v_{s1}$ and $w_{t0} \perp w_{t1}$ for all s,t, and there exists a vector ψ of norm 1 such that $v_{s0}+v_{s1} = w_{t0}+w_{t1} = \psi$ for all s,t, while $p(i,j|s,t) = \langle v_{si}, w_{tj}\rangle \geq 0$. (The idea behind this is: if we have two orthogonal projections applied to a state vector, then the image will consist of two vectors which will add up to the state vector.) Note that

$$\begin{aligned} 1 = \sum_{i,j\in\{0,1\}} p(i,j|s,t) &= 2(p(0,0|s,t)+p(1,0|s,t)) \\ &= 2[\langle v_{s0}, w_{t0}\rangle + \langle v_{s1}, w_{t0}\rangle] \\ &= 2[\langle \psi, w_{t0}\rangle] \\ &= 2\langle w_{t0}, w_{t0}\rangle. \end{aligned}$$

Hence, $\|w_{t0}\|^2 = \frac{1}{2}$. Similarly, $\|w_{t1}\|^2 = \|v_{s0}\|^2 = \|v_{s1}\|^2 = \frac{1}{2}$. Now set $x_s = v_{s0} - v_{s1}$ and set $y_t = w_{t0} - w_{t1}$. Then $\|x_s\|^2 = \|v_{s0}\|^2 + \|v_{s1}\|^2 = 1$, and similarly, $\|y_t\|^2 = 1$. To see that p is of the desired form, we can compute

$$\begin{aligned} \frac{1}{4}[1+\langle x_s, y_t\rangle] &= \frac{1}{4}[1+\langle v_{s0}-v_{s1}, w_{t0}-w_{t1}\rangle] \\ &= \frac{1}{4}[1+\langle v_{s0}, w_{t0}\rangle + \langle v_{s1}, w_{t1}\rangle - \langle v_{s1}, w_{t0}\rangle - \langle v_{s0}, w_{t1}\rangle] \\ &= \frac{1}{4}[1+2p(0,0|s,t) - (1-2p(0,0|s,t))] \\ &= p(0,0|s,t) = p(1,1|s,t). \end{aligned}$$

Therefore,

$$p(0,0|s,t) = p(1,1|s,t) = \frac{1}{4}[1+(-1)^{i+j}\langle x_s, y_t\rangle],$$

when $(i,j) = (0,0)$ or $(i,j) = (1,1)$.

Now suppose that $(i,j) = (1,0)$ or $(i,j) = (0,1)$. Then similarly, one can check that

$$\begin{aligned}\frac{1}{4}[1 - \langle x_s, y_t\rangle] &= \frac{1}{4}[1 + 2p(1,0|s,t) - (1 - 2p(1,0|s,t))]\\ &= p(1,0|s,t) = p(0,1|s,t)\end{aligned}$$

Therefore, (1) holds. $\square$

Theorem 5.9. *Let $\mathcal{G}$ be an n-input, XOR game, with rule function $V(i,j|s,t) \in \{0,1\}$. The following are equivalent.*

1. *$\mathcal{G}$ has a perfect q-strategy.*
2. *$\mathcal{G}$ has a perfect vect-strategy.*
3. *There are vectors x_s, y_t with $\|x_s\|, \|y_t\| \leq 1$ such that $\langle x_s, y_t\rangle \in \mathbb{R}$ and such that whenever $V(i,j|s,t) = 0$, then $\langle x_s, y_t\rangle = (-1)^{i+j+1}$.*

Proof. Clearly (1) implies (2). Assume that (2) holds, and let $p(i,j|s,t) \in C_{\text{vect}}(n,2)$ be perfect; i.e. whenever $V(i,j|s,t) = 0$ then $p(i,j|s,t) = 0$. Let

$$\tilde{p}(i,j|s,t) = \frac{1}{2}[p(i,j|s,t) + p(i+1,j+1|s,t)].$$

Since $p(i,j|s,t), p(i+1,j+1|s,t) \in C_{\text{vect}}(n,2)$, and $C_{\text{vect}}(n,2)$ is convex, we have $\tilde{p}(i,j|s,t) \in C_{\text{vect}}(n,2)$. If $\lambda(i,j|s,t) = 0$ then $\lambda(i+1,j+1|s,t) = 0$. Therefore, $\tilde{p}(i,j|s,t) = 0$ so that $\tilde{p}$ is a perfect vect-strategy. Moreover, $\tilde{p}(0,0|s,t) = \tilde{p}(1,1|s,t)$ and $\tilde{p}(1,0|s,t) = \tilde{p}(0,1|s,t)$. Thus, there are x_s, y_t with $\|x_s\|, \|y_t\| \leq 1$ such that $\tilde{p}(i,j|s,t) = \frac{1}{4}[1 + (-1)^{i+j}\langle x_s, y_t\rangle]$. If $\lambda(i,j|s,t) = 0$, then $\frac{1}{4}[1 + (-1)^{i+j}\langle x_s, y_t\rangle] = 0$. Thus, $\langle x_s, y_t\rangle = (-1)^{i+j+1}$. This gives (3).

If (3) holds, then given such vectors x_s, y_t, we may set $p(i,j|s,t) = \frac{1}{4}[1 + (-1)^{i+j}\langle x_s, y_t\rangle] \in C_q(n,2)$. One can check that this gives a perfect q-strategy by the same calculation as above. Hence, (1) holds. $\square$

Remark 5.10. Since $\|x_s\|, \|y_t\| \leq 1$ as above and $\langle x_s, y_t\rangle = (-1)^{i+j+1}$, we must have $\|x_s\| = \|y_t\| = 1$ and hence $y_t = (-1)^{i+j+1}x_s$ by the Cauchy-Schwarz inequality.

Lemma 5.11. *Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems and let $s : \mathcal{S} \otimes_{\text{c}} \mathcal{T} \to \mathbb{C}$ be a state. Then there exists a Hilbert space H, unital completely positive maps, $\phi : \mathcal{S} \to B(H)$, $\psi : \mathcal{T} \to B(H)$ with commuting ranges and a unit vector $\xi \in H$, such that $s(x \otimes y) = \langle \phi(x)\psi(y)\xi, \xi\rangle$.*

Proof. We have that $\mathcal{S} \otimes_{\text{c}} \mathcal{T} \subset_{\text{coi}} C^*_u(\mathcal{S}) \otimes_{\max} C^*_u(\mathcal{T})$. Thus, we may extend the state s to a state, still denoted by s, on this C*-algebra. The rest follows from considering the GNS representation of the states. $\square$

Lemma 5.12. *Let $\mathcal{A}_i$ and $\mathcal{B}_j$ be unital C*-algebras, $i = 1, \ldots, n$, $j = 1, \ldots, m$, and set $\mathcal{S} = \mathcal{A}_1 \oplus_1 \cdots \oplus_1 \mathcal{A}_n$, $\mathcal{T} = \mathcal{B}_1 \oplus_1 \cdots \oplus_1 \mathcal{B}_m$, $\mathcal{A} = \mathcal{A}_1 * \cdots * \mathcal{A}_n$, and $\mathcal{B} = \mathcal{B}_1 * \cdots * \mathcal{B}_m$. Then $\mathcal{S} \otimes_{\text{c}} \mathcal{T} \subseteq_{\text{coi}} \mathcal{A} \otimes_{\max} \mathcal{B}$.*

Proof. Let $\phi : \mathcal{S} \to \mathcal{B}(H)$ and $\psi : \mathcal{T} \to \mathcal{B}(H)$ be unital completely positive maps with commuting ranges. It suffices to show that there exists a unital completely positive map $\Gamma : \mathcal{A} * \mathcal{B} \to \mathcal{B}(H)$ extending the map $\phi \cdot \psi : \mathcal{S} \otimes_c \mathcal{T} \to \mathcal{B}(H)$ defined by $\phi \cdot \psi(x \otimes y) = \phi(x)\psi(y)$.

We identify $\mathcal{S}$ with the linear span of $\mathcal{A}_1, \ldots, \mathcal{A}_n$ inside $\mathcal{A}$; similarly, we identify $\mathcal{T}$ with the linear span of $\mathcal{B}_1, \ldots, \mathcal{B}_m$ inside $\mathcal{B}$. The map ϕ is determined by a family $(\phi_i)_{i=1}^n$ of unital completely positive maps (where ϕ_i maps $\mathcal{A}_i$ into $\mathcal{B}(H)$) via the rule $\phi(a_1 + \cdots a_n) = \phi_1(a_1) + \cdots \phi_n(a_n)$, $a_i \in \mathcal{A}_i$, $i = 1, \ldots, n$. By [**?**], there exists a unital completely positive map $\tilde{\phi} : \mathcal{A} \to \mathcal{B}(H)$ given by $\tilde{\phi}(a_{i_1} \cdots a_{i_k}) = \phi_{i_1}(a_{i_1}) \cdots \phi_{i_k}(a_{i_k})$, where $a_{i_l} \in \mathcal{A}_{i_l}$ for each l, and $i_1 \neq i_2 \neq \cdots \neq i_k$.

Similarly, ψ is determined by a family $(\psi_j)_{j=1}^m$, where ψ_j is a map from $\mathcal{B}_j$ into $\mathcal{B}(H)$; moreover, $\phi_i(\mathcal{A}_i)$ commutes with $\psi_j(\mathcal{B}_j)$ for every pair i, j of indices. Let $\tilde{\psi} : \mathcal{B} \to \mathcal{B}(H)$ be the unital completely positive map given by $\tilde{\psi}(b_{j_1} \cdots b_{k_k}) = \psi_{j_1}(b_{j_1}) \cdots \psi_{j_k}(b_{j_k})$, where $b_{j_l} \in \mathcal{B}_{j_l}$ for each l, and $j_1 \neq j_2 \neq \cdots \neq j_k$. Since the linear span of free words are dense in the corresponding free products $\mathcal{A}$ and $\mathcal{B}$, we have that $\tilde{\phi}(\mathcal{A})$ and $\tilde{\psi}(\mathcal{B})$ commute. It is now clear that the unital completely positive map $\Gamma : \mathcal{A} * \mathcal{B} \to \mathcal{B}(H)$ arising from $\tilde{\phi}$ and $\tilde{\psi}$ in a similar fashion [**?**] extends $\phi \cdot \psi$. ☐

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