IML COURSE NOTES
TRACES AND COMMUTATORS

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ABSTRACT. This contains notes about tracial states on C*-algebras, especially for the non-experts.

1. TRACES AND COMMUTATORS

Given a unital C*-algebra A(or even *-algebra we let C denote the linear span of the commutators, $\{ab - ba : a, b \in A\}$ and let $\mathcal{C}$ denote its norm closure. We let

$$T(A) = \{\tau : A \to \mathbb{C} : \|\tau\| = 1,\ \tau(1) = 1,\ \tau(\mathcal{C}) = \{0\}\},$$

That is $T(A)$ denotes the set of tracial states.

The following result is due to Kirchberg-Rordam [4]. The proof we give here is motivated by talks with M. Kennedy.

Proposition 1.1. *Let A be a unital C*-algebra and let $\mathcal{C}$ denote the closed linear span of the commutators. If $x = x^* \in A \cap \mathcal{C}^c$, then there is a tracial state τ on A such that*

$$|\tau(x)| = \inf\{\|x - c\| : c \in \mathcal{C}\}.$$

Consequently,

$$\sup\{|\tau(x)| : \tau \in T(A)\} = \inf\{\|x - c\| : c \in \mathcal{C}\}.$$

Proof. Consider the normed quotient space $A/\mathcal{C}$ and set

$$d = \inf\{x - c\|; c \in \mathcal{C}\} = \|x + \mathcal{C}\|.$$

Then there is a linear functional of norm 1, $g : A/\mathcal{C} \to \mathbb{C}$ such that

$$g(x + \mathcal{C}) = \|x + \mathcal{C}\| := d.$$

Let $f : A \to \mathbb{C}$ be the composition of g with the quotient map. Let $f^*(y) = \overline{f(y^*)}$.. Since $\mathcal{C}$ is *-invariant, and $f(\mathcal{C}) = (0)$ we have that $f^*(\mathcal{C}) = (0)$ and $f^*(x) = f(x) = d$.. Let $\phi = (f + f^*)/2$, so that ϕ is a self-adjoint functional that vanishes on $\mathcal{C}$ with $\|\phi\| \leq 1$. However, since for $c \in \mathcal{C}$, $\phi(x + c) = d = \inf\{\|x + k\| : k \in \mathcal{C}\}$ we see that $\|\phi\| = 1$.

By the Jordan decomposition for functionals on a C*-algebra(which generalizes the Jordan decomposition of signed measures), we may write $\phi = \phi^+ - \phi^-$ with $\phi^\pm$ positive linear functionals and

$$1 = \|\phi\| = \phi^+(1) + \phi^-(1).$$

Note that for every unitary $u \in A$ we have that

$$u^*yu - y = (u^*y)u - u(u^*y) \in \mathcal{C}.$$

Thus, setting $\psi^\pm(y) = \phi^\pm(u^*yu)$ we have that

$$\phi(y) = \phi(u^*yu) = \psi^+(y) - \psi^-(y),$$

with $\psi^\pm$ positive linear functionals and $\|\phi\| = \psi^+(1) + \psi^-(1)$. Hence, by the uniqueness of the Jordan decomposition(see Pedersen's book), we have that $\phi^\pm = \psi^\pm$ and so $\phi^\pm(u^*yu) = \phi^\pm(y)$.

It follows that

$$\phi^\pm(uy - yu) = \phi^\pm(u(yu)u^* - (yu)) = 0.$$

Using the fact that every element of A is a linear combination of unitaries, we get that $\phi^\pm(ay - ya) = 0$, $\forall a, y \in A$, and hence $\phi^\pm(\mathcal{C}) = (0)$.

Setting $r_\pm = \phi^\pm(1) \geq 0$ we have that $r_+ + r_- = 1$ and $\phi^\pm = r_\pm \tau_\pm$, with $\tau^\pm$ states that vanish on $\mathcal{C}$. Hence, $\tau^\pm$ are tracial states. Since

$$d = r_+\tau^+(x) - r_-\tau_-(x) \leq \max\{\tau^+(x), \tau^-(x)\} \leq \|x + \mathcal{C}\| = d,$$

either $\tau^+(x) = d$ or $\tau^-(x) = -d$. $\square$

Given a unital C*-algebra, we let $T(A)$ denote the set of tracial states on A. This is a, possibly empty, *weak**-compact, convex subset of the space of states on A.

Recall the *Cartesian decomposition*, $Re(x) = (x + x^*)/2$, $Im(x) = (x - x^*)/2i$.

Corollary 1.2. *Let A be a unital C*-algebra. Then:*

- $\mathcal{C} = \cap_{\tau \in T(A)} ker(\tau)$,
- $T(A) \neq \emptyset$ *if and only if $\mathcal{C} \neq A$,*
- $\|1 + \mathcal{C}\|$ *is either 0 or 1.*

Proof. Since traces vanish on $\mathcal{C}$ we have that

$$\mathcal{C} \subseteq \cap_{\tau \in T(A)} ker(\tau).$$

For the other inclusion, note that if $x \in \mathcal{C}^c$ then either $Re(x)$ or $Im(x)$ is not in $\mathcal{C}$. Thus, by the above, we have a trace such that $\tau(x) \neq 0$.

The second statement is clear, by the above result.

Finally, if $\|1 + \mathcal{C}\| \neq 0$, then there exists a tracial state τ and

$$1 = \tau(1) \leq \|1 + \mathcal{C}\| \leq 1.$$

$\square$

A similar distance formula holds for arbitrary elements, but the proof uses a great deal of von Neumann algebra theory. I feel that it should be possible to give a C*-algebraic proof using the off-diagonal tricks.

I am grateful to Roger Smith for the following proof.

Theorem 1.3. *Let A be a unital C*-algebra and let $a \in A \cap \mathcal{C}^c$, then there is a tracial state τ such that*

$$|\tau(a)| = \inf\{\|a - c\| : c \in \mathcal{C}\}.$$

Proof. WLOG assume that $d(a, \mathcal{C}) = 1$. By the Hahn-Banach theorem there is a bounded linear functional f such that

$$\|f\| = 1, f(a) = 1, f(\mathcal{C}) = \{0\}.$$

Let $\hat{f} : A^{**} \to \mathbb{C}$, $\hat{f}(\psi) = \psi(f)$ be the w*-continuous functional on the vonNeumann algebra A^{**}. If $x \in A, y \in A^{**}$ then there is a net $y_\alpha \in A$ converging w* to y and hence,

$$\hat{f}(xy) = \lim_\alpha f(xy_\alpha) = \lim_\alpha f(y_\alpha x) = \hat{f}(yx).$$

Repeating this argument we get that $\hat{f}(xy) = \hat{f}(yx)$ for all $x, y \in A^{**}$.

Let $\mathcal{D}$ denote the w*-closure of the commutators in A^{**}. We have that $\hat{f}$ vanishes on $\mathcal{D}$. Since $\mathcal{C} \subseteq \mathcal{D}$, we have

$$d(a, \mathcal{D}) \le d(a, \mathcal{C}) = 1.$$

For each $d \in D$,

$$\hat{f}(a - d) = f(a) = d(a, \mathcal{C}),$$

so

$$\|a - d\| \ge d(a, \mathcal{C}) \implies d(a, \mathcal{D}) = 1.$$

Since $\mathcal{D}$ is w*-closed, the distance from a to $\mathcal{D}$ is attained, so choose $d \in \mathcal{D}$ with $\|a - d\| = d(a, \mathcal{D}) = 1$.

By the Dixmier approximation theorem, there is a central element z in the norm closed convex hull of the set

$$\{u^*(a - d)u : u \in A^{**}, uu^* = u^*u = 1\}.$$

Then $\|z\| \le 1$ and

$$\hat{f}(z) = \hat{f}(a - d) = 1 \implies \|z\| = 1.$$

Since z is central it commutes with z^*, and hence is normal. Identify the abelian C*-algebra $C^*(z, 1) = C(\sigma(z))$ and identify $\hat{f}|_{C^*(z,1)}$ with a complex measure, μ on $\sigma(z) \subseteq \mathbb{D}^-$.

Writing $d\mu = \phi d\nu$ where $\nu = |\mu|$ is a probability measure on $\sigma(z)$ and $|\phi(t)| \le 1, \forall t \in \sigma(z)$ we have that for any $w \in C^*(z, 1)$,

$$\hat{f}(w) = \int w\phi d\nu.$$

We have

$$1 = \int z(t)\phi(t)d\nu \implies supp(\phi) \subseteq \sigma(z) \cap \mathbb{T}.$$

Fix $\epsilon > 0$ and divide the $\mathbb{T}$ into n disjoint half open arcs of equal length so that if $e^{i\theta}$ and $e^{i\alpha}$ belong to the same arc, then $|e^{i\theta} - e^{i\alpha}| < \epsilon$. Let $p_i \in A^{**}$ denote the spectral projections for z onto these arcs.

Note that since each p_i is central, $p_i(xy - yx) = (p_i x)y - y(p_i x) = p_i(xy - yx)p_i \in D$. Hence, $\mathcal{D} = \sum_i p_i \mathcal{D} = \oplus p_i \mathcal{D} p_I$ and since $a = \sum_i p_i a p_i$ we have

$$d(a, \mathcal{D}) = \max\{d(p_i a p_i, p_i \mathcal{D} p_i)\}.$$

Pick the $p = p_i \neq 0$ such that

$$1 = d(a, \mathcal{D}) = d(pap, p\mathcal{D}p) = d(pap, \mathcal{D}).$$

Let I be the arc for p, and fix $e^{i\theta} \in I$, then $\|zp - e^{i\theta}p\| < \epsilon$ and

$$p_i \neq 0 \implies \sigma(z) \cap I \neq \implies \|zp\| = 1.$$

Now pick a second linear functional f_2 such that $f_2(pa) = 1 = d(pa, \mathcal{D})$ and $f_2(\mathcal{D}) = \{0\}$. If we take the same unitary conjugates and convex combinations of $a - d$ that converged to z and apply them to $p(a - d)$ then they will converge to pz and hence, $f_2(pz) = 1$.

Now define $\tau : A^{**} \to \mathbb{C}$ by $\tau(x) = f_2(pzx)$. Then $\|\tau\| \leq 1$ and $\tau(1) = f_2(pz) = 1$. Using that pz is in the center, we have $\tau(xy - yx) = f_2(pz(xy - yx)) = f_2((pzx)y - y(pzx)) = 0$. Hence, τ is a tracial state. Note that $\tau(p) = f_2(pzp) = f_2(pz) = 1$ Finally,

$$\tau(z) = f_2(pz^2) = f_2((pz - pe^{i\theta} + pe^{i\theta})^2) =$$

$$f_2((pz - e^{i\theta}p)^2 + 2e^{i\theta}(pz - pe^{i\theta}) + pe^{2i\theta}) \implies$$

$$|\tau(z) - e^{2i\theta}f_2(p)| \leq \|f_2\|(\|pz - e^{i\theta}p\|^2 + 2\|p(z - e^{i\theta})\| \leq \epsilon^2 + \epsilon,$$

and $|e^{i\theta}f_2(p) - 1| = |f_2(e^{i\theta}p - pz)| \leq \|e^{i\theta}p - pz\| \leq \epsilon$. Thus,

$$|\tau(z) - e^{i\theta}| = |\tau(z) - e^{2i\theta}f_2(p) + e^{i\theta}(e^{i\theta}f_2(p) - 1)| \leq \epsilon^2 + 2\epsilon.$$

Finally, since $\tau(a) = \tau(z)$ we have that $\tau(a) \simeq e^{i\theta}$ and

$$|\tau(a)| \geq (1 - \epsilon)^2 = [d(a, \mathcal{C}) - \epsilon]^2,$$

and the result follows.

$\square$

There is another way to look at the above result. If we let $Aff(T(A))$ denote the vector space of continuous affine functions on $T(A)$, equipped with the supremum norm, then this is a *-vector space, in fact it is a operator subsystem of the abelian C*-algebra, $C(T(A))$. To each $a \in A$ we associate the continuous affine function

$$\hat{a} : T(A) \to \mathbb{C} \ \hat{a}(\tau) = \tau(a).$$

Then the above theorems tell us that the kernel of this map is the closed linear span of the commutators and that the map

$$\Gamma : A/\mathcal{C} \to Aff(T(A)), \ \Gamma(a + \mathcal{C}) = \hat{a},$$

is isometric and *-preserving.

In fact one can say more.

Proposition 1.4. *Let A be a unital C*-algebra. Then the quotient operator system, $A/\mathcal{C}$ and $Aff(T(A))$ are unitally completely order isomorphic.*

This follows from the work of M. Kennedy et al, that shows that $T(A)$ is a type of convex set for which there is a unique operator system structure on $Aff(T(A))$.

2. Amenable Traces

There are several equivalent definitions of amenable traces, we use the one that is most convenient for our purposes. For mor background see the book of Brown-Ozawa.

Let $\mathcal{A} \subseteq B(H)$ be a unital C*-subalgebra, a state $\psi : B(H) \to \mathbb{C}$ is called an $\mathcal{A}$**-quasitrace** provided

$$\psi(ax) = \psi(xa), \forall x \in B(H), a \in \mathcal{A}.$$

Note that the restriction of a quasitrace to $\mathcal{A}$ is a tracial state on $\mathcal{A}$.

Definition 2.1. A tracial state τ on $\mathcal{A}$ is called **amenable** if there exists a Hilbert space H, a unital *-homomorphism $\pi : \mathcal{A} \to B(H)$ and a $\pi(\mathcal{A})$-quasitrace ψ on $B(H)$ such that $\tau = \psi \circ \pi$. We let $T_{am}(\mathcal{A}) \subseteq \mathcal{A}^*$ denote the set of amenable traces

Note that if $\pi : \mathcal{A} \to M_n = B(\mathbb{C}^n)$ is a unital *-homomorphism, then $\tau(a) = \frac{Tr(\pi(a))}{n}$ is an amenable trace on $\mathcal{A}$.

Proposition 2.2. *Let $\mathcal{A} \subseteq B(H)$ be a unital C*-subalgebra, then every amenable trace on $\mathcal{A}$ is the restriction of an $\mathcal{A}$-quasitrace on $B(H)$ to $\mathcal{A}$. In particular the set $T_{am}(\mathcal{A})$ is a w*-closed subset of the state space of $\mathcal{A}$.*

Proof. Given a unital *-homomorphism $\pi : \mathcal{A} \to B(H_1)$ and a $\pi(\mathcal{A})$-quasitrace ψ on $B(H_1)$, by Arveson's Extension Theorem, we may extend π to a UCP map $\Phi : B(H) \to B(H_1)$. Define a state by $\gamma : B(H) \to \mathbb{C}$ by $\gamma(x) = \psi(\Phi(x))$.

By Choi's theorem on multiplicative domains, for any $a \in A$, $x \in B(H)$, $\Phi(ax) = \pi(a)\Phi(x)$ and $\Phi(xa) = \Phi(x)\pi(a)$. Hence,

$$\gamma(ax) = \psi(\Phi(ax)) = \psi(\pi(a)\Phi(x)) = \psi(\Phi(x)\pi(a)) = \psi(\Phi(xa)) = \gamma(xa),$$

and so γ is an $\mathcal{A}$-quasitrace. $\square$

Let $\mathcal{A} \subseteq B(H)$ be a unital C*-subalgebra let $\mathcal{B} \subseteq B(H)$ denote the closed linear span of the set of all

$$ax - xa, \ a \in \mathcal{A}, \ x \in B(H).$$

Proposition 2.3. *Let $x = x^* \in B(H)$ then*

$$d(x, \mathcal{B}) = \sup\{|\psi(x)| : \psi \ an \ \mathcal{A}\text{-}quasitrace \ \}.$$

In particular, if $a = a^* \in \mathcal{A}$, then

$$d(a, \mathcal{B}) = \sup\{|\tau(a)| : \tau \text{ an amenable trace on } \mathcal{A}\},$$

and so this distance is independent of the particular faithful representation of $\mathcal{A}$ on a Hilbert space.

Proof. It will be enough to prove the first statement. Consider the quotient space $B(H)/\mathcal{B}$ and set

$$d = d(x, \mathcal{B}) = \|x + \mathcal{B}\|.$$

Since $\mathcal{B}$ is a self-adjoint subspace, arguing as before, there is a linear functional

$$\phi : B(H) \to \mathbb{C},$$

with $\|\phi\| = 1$, $\phi(x) = d$, $\phi(\mathcal{B}) = \{0\}$, that is self-adjoint, i.e., $\phi(y^*) = \overline{\phi(y)}$.

Apply the Jordan decomposition to write $\phi = \phi^+ - \phi^-$ with

$$1 = \|\phi\| = \phi^+(1) + \phi^-(1).$$

If we fix a unitary $u \in \mathcal{A}$ and set $\psi^\pm(y) = \phi^\pm(u^*yu)$ then, $\phi = \psi^+ - \psi^-$ and so by uniqueness of the Jordan decomposition, $\phi^\pm(y) = \psi^\pm(y) = \phi^\pm(u^*yu)$. This leads to $\phi\pm(uy - yu) = 0$ for all unitaries in $\mathcal{A}$ and so $\phi^\pm(ay - ya) = 0$ for all $a \in \mathcal{A}$.

The rest of the argument is as in the proof of Proposition 1.1. □

Corollary 2.4. *Let $\mathcal{A} \subseteq B(H)$ be a unital C*-subalgebra and let $Q(\mathcal{A})$ denote the set of $\mathcal{A}$-quasitraces. Then*

- *$\mathcal{B} = \cap_{\psi \in Q(\mathcal{A})} ker(\psi)$,*
- *$Q(\mathcal{A}) \neq \emptyset$ iff $\mathcal{B} \neq B(H)$,*
- *$d(1, \mathcal{B})$ is either 0 or 1*

Moreover, if $\pi : \mathcal{A} \to B(H_1)$ is any unital *-homomorphism and $d(1, \mathcal{B}) = 0$, then $Q(\pi(\mathcal{A})) = \emptyset$ and the closed linear span of $\{\pi(a)y - y\pi(a)\}$ is dense in $B(H_1)$.

I haven't checked if the analogues of Roger Smith's result is true.

Problem 2.5. *For $x \in B(H)$ is*

$$d(x, \mathcal{B}) = \sup\{|\psi(x)| : \psi \in Q(\mathcal{A})?$$

For $a \in \mathcal{A}$ is

$$d(a, \mathcal{B}) = \sup\{|tau(a)| : \tau \in T_{am}(\mathcal{A})\}?$$

Note that $\mathcal{C} \subseteq \mathcal{A} \cap \mathcal{B} = \cap_{\tau \in T_{am}(\mathcal{A})} ker(\tau)$.

Problem 2.6. *For $a \in \mathcal{A}$ is $d(a, \mathcal{B}) = d(a, \mathcal{A} \cap \mathcal{B})$? If not is there an interpretation of this other distance?*

Problem 2.7. *Is the operator system quotient $B(H)/\mathcal{B}$ unitally completely order isomorphic to $Aff(Q(\mathcal{A}))$? Is $\mathcal{A}/(\mathcal{A} \cap \mathcal{B})$ unitally completely order isomorphic to $Aff(T_{am}(\mathcal{A}))$?*

Problem 2.8. *Find a characterization of* $\mathbb{C}(\mathbb{F}(n,k))\cap\mathcal{B} = \{x \in \mathbb{C}(\mathbb{F}(n,k)|\tau(x) = 0 \,\forall\tau \in T_{am}(C^*(\mathbb{F}(n,k)).$

It might follow from Slofstra's work or MIP*=RE that membership in $\mathbb{C}(\mathbb{F}(n,k)) \cap \mathcal{B}$ is undecidable.

E. Kirchberg gave a tensor product characterization of amenable traces. First some preliminaries.

Recall that given a C*-algebra $\mathcal{A}$, then $\mathcal{A}^{op}$ is the same normed *-vector space as $\mathcal{A}$ but with a product defined by

$$a \circ b := ba.$$

Since $\|a^* \circ a\| = \|aa^*\| = \|a\|^2$, we see that $\mathcal{A}^{op}$ is also a C*-algebra.

Even though $\mathcal{A}$ and $\mathcal{A}^{op}$ are equal as normed spaces, one can check that

$$\|(a_{i,j})\|_{M_n(\mathcal{A}^{op})} = \|(a_{i,j})^t\|_{M_n(\mathcal{A})},$$

and

$$(a_{i,j}) \in M_n(\mathcal{A}^{op})^+ \iff (a_{i,j})^t \in M_n(\mathcal{A})^+.$$

It is easy to check that the transpose map, $T : M_n \to M_n$ is a *-isomorphism from M_n to M_n^{op}, from which it follows that the identity map from M_n to M_n^{op} is positive but not 2-positive and has cb-norm n. Moreover, the identity map from $B(H)$ to $B(H)^{op}$ is not completely bounded, even though they are *-isomorphic, again pick a basis and define a transpose.

Finally, recall that given two C*-algebras, $\mathcal{A} \subseteq B(H_1)$, $\mathcal{B} \subseteq B(H_2)$ the closed linear span of

$$\{a \otimes b \in B(H_1 \otimes H_2) : a \in \mathcal{A},\, b \in \mathcal{B}\},$$

is a C*-algebra denoted $\mathcal{A} \otimes_{min} \mathcal{B}$, and that this C*-algebra is independent of the particular faithful representations of $\mathcal{A}$ and $\mathcal{B}$ as operators on Hilbert spaces. Also among all tensor cross norms making $\mathcal{A} \otimes \mathcal{B}$ into a C*-algebra, this is the smallest C*-norm on the tensor product.

The largest C*-tensor norm, denoted $\mathcal{A} \otimes_{max} \mathcal{B}$ is the universal norm for all commuting pairs of representations of the two algebras as operators on a Hilbert space.

Given a tracial state $\tau : \mathcal{A} \to \mathbb{C}$, we let $\pi_\tau : \mathcal{A} \to B(H_\tau)$ denote its GNS representation and let ξ_τ denote the unit vector that is the image of 1, so that

$$\tau(a) = \langle \xi_\tau | \pi_\tau(a)\xi_\tau \rangle.$$

The map $J : \mathcal{H}_\tau \to \mathcal{H}_\tau$ given by $J(\pi_\tau(a)\xi_\tau) = \pi_\tau(a^*)\xi_\tau$ turns out to be well-defined, isometric, idempotent and anti-linear and so extends to H_τ. Setting $\pi_\tau^{op}(a) = J\pi_\tau(a^*)J$ turns out to be a linear representation of $\mathcal{A}^{op}$ on H_τ and $\pi_\tau^{op}(\mathcal{A}^{op})$ commutes with $\pi_\tau(\mathcal{A})$ and so induces a *-homomorphism $\pi_\tau \otimes \pi_\tau^{op} : \mathcal{A} \otimes_{max} \mathcal{A}^{op} \to B(H_\tau)$ with

$$\langle \xi_\tau | \pi_\tau(a) \otimes \pi_\tau^{op}(b)\xi_\tau \rangle = \langle \xi_\tau | \pi_\tau(a) J\pi_\tau(b^*)J\xi_\tau \rangle = \langle \xi_\tau | \pi_\tau(a)\pi_\tau(b)\xi_\tau \rangle = \tau(ab).$$

All of this is used to show that for *any* tracial state, the map

$$\sum_i a_i \otimes b_i \to \sum_i \tau(a_i b_i),$$

defines a state on $\mathcal{A} \otimes_{max} \mathcal{A}^{op}$.

Theorem 2.9 (Kirchberg). *Let $\mathcal{A}$ be a unital C*-algebra and let $\tau : \mathcal{A} \to \mathbb{C}$ be a tracial state. Then τ is amenable if and only if the map*

$$f(\sum_i a_i \otimes b_i) = \sum_i \tau(a_i b_i),$$

extends to be a bounded linear functional on $\mathcal{A} \otimes_{min} \mathcal{A}^{op}$.

He used this in his new equivalence of Connes Embedding Problem.

Theorem 2.10 (Kirchberg). *The following are equivalent:*
   (1) *Connes' Embedding Problem has a positive answer,*
   (2) $\forall n, k, C^*(\mathbb{F}(n,k)) \otimes_{min} C^*(\mathbb{F}(n,k)) = C^*(\mathbb{F}(n,k)) \otimes_{max} C^*(\mathbb{F}(n,k)),$
   (3) $\forall n, k,$ *every tracial state on $C^*(\mathbb{F}(n,k))$ is amenable.*

Kirchberg's first(and easier!) result was used in the proof of the following result of Kim-P-Schafhauser. Recall that $\mathbb{F}(n,k)$ denotes the group that is the free product of n copies of the cyclic group of order k and that $C^*(\mathbb{F}(n,k))$ is generated by orthogonal projections $\{e_{x,a} : 1 \le x \le n, 0 \le a \le k-1\}$ corresponding to the spectral projections of the n order k unitary generators $u_x^k = 1$ of the group algebra $\mathbb{C}(\mathbb{F}(n,k))$.

Theorem 2.11. *Let $p(a,b|x,y) \in C_{qa}^s(n,k),$ then*
   (1) *p is in the closure of $C_q^s(n,k)$,*
   (2) *there is an amenable trace τ on $C^*(\mathbb{F}(n,k))$ such that*

$$p(a,b|x,y) = \tau(e_{x,a} e_{y,b}),$$

   (3) *there is a unital *-homomorphism into an ultrapower of the hyperfinite II_1-factor $\mathcal{R}$, $\pi : C^*(\mathbb{F}(n,k)) \to \mathcal{R}^\omega$, such that*

$$p(a,b|x,y) = \tau_\omega(\pi(e_{x,a})\pi(e_{y,b})).$$

Consequently, CEP is true if and only if $C_{qa}^s(n,k) = C_{qc}^s(n,k), \forall n, k.$ Thanks to MIP*=RE, we now know that CEP has a negative answer so that these things are all false. In MIP*=RE, they use the second, harder result of Kirchberg to show that CEP has a negative answer. However, MIP*=RE shows the existence of a synchronous game with a perfect qc-strategy but no perfect qa-strategy and to get from that fact to a refutation of CEP only needs the easier implications of all of these equivalences. For a nice write up of this more direct path, see Alexander Frei [2].

There are two other families of traces that we know very little about in the setting of quantum correlations. See N. Brown's article[1] for more about these.

To motivate the following definitions, I should mention another characterization of a trace being amenable. A tracial state τ on $\mathcal{A}$ is amenable if and only if there exists a sequence of UCP maps $\phi_n : \mathcal{A} \to M_{k_n}$ such that $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\|_2 \to 0$ and $tr_{k_n} \circ \phi_n \to \tau$ in the w*-topology. See the book of Brown-Ozawa for this characterization. In fact, some authors take this as the definition of amenable rather than as a theorem.

Definition 2.12. A tracial state τ on $\mathcal{A}$ is **uniformly amenable** if there exists a sequence of UCP maps $\phi_n : \mathcal{A} \to M_{k_n}$ such that $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\|_2 \to 0$ and $\|tr_{k_n} \circ \phi_n - \tau\|_{\mathcal{A}^*} \to 0$.

Definition 2.13. A tracial state on $\mathcal{A}$ is a **quasidiagonal trace** if there exists a sequence of UCP maps $\phi_n : \mathcal{A} \to M_{k_n}$ such that $\|\phi_n(ab) - \phi_n(a)\phi_n(b)\| \to 0$ and $tr_{k_n} \circ \phi_n \to \tau$ in the w*-topology on $\mathcal{A}^*$.

Problem 2.14. *Characterize the set of $p \in C_{qc}^s(n,k)$ such that $p(a,b|x,y) = \tau(e_{x,a}e_{y,b})$ for τ a **uniformly amenable** trace.*

Problem 2.15. *Same problem but for τ a **quasidiagonal** trace.*

It is not hard to show that both of these sets contain $C_q^s(n,k)$. So the real question is if they are all of $C_{qa}^s(n,k)$ or are strictly smaller.

Finally, if an n input, k output synchronous game $\mathcal{G}$ has a perfect qa-strategy, say given by the density $p(a,b|x,y) = \tau(e_{x,a}e_{y,b})$ with τ an amenable trace on $C^*(\mathbb{F}(n,k))$, then by forming a quotient, this trace defines a trace $\hat{\tau}$ on the C*-algebra of the game $C^*(\mathcal{G})$.

Problem 2.16. *Is $\hat{\tau}$ an amenable trace on $C^*(\mathcal{G})$?*

N. Brown proves that uniformly amenable traces pass to quotients. So if p arises from a uniformly amenable trace τ on $C^*(\mathbb{F}(n,k))$ then $\hat{\tau}$ is a uniformly amenable trace on $C^*(\mathcal{G})$.

3. Applications to Synchronous Values of Games

Recall that given an n input k output game, where Alice and Bob's input sets and output sets are the same, $\mathcal{G} = (X, A, V)$ and a probability density π on input pairs for $t = loc, q, qa, qc$, the synchronous values of the game are defined by

$$\omega_t^s(\mathcal{G}, \pi) = \sup\{ \sum_{x,y,a,b} \pi(x,y)p(a,b|x,y)V(a,b,x,y) : p \in C_t^s(n,k)\}.$$

These were defined in Dykema-P and studied in Helton-Mousavi-Nezhadi-P-Russell and in Levine-P. Given a game as above and define elements in $\mathbb{C}(\mathbb{F}(n,k))$ by

$$P_{\mathcal{G},\pi} = \sum_{x,y,a,b} \pi(x,y)e_{x,a}e_{y,b}e_{x,a} \text{ and } R_{\mathcal{G},\pi} = \sum_{x,y,a,b} \pi(x,y)e_{x,a}e_{y,b}.$$

Note that since for any trace $\tau(e_{x,a}e_{y,b}e_{x,a}) = \tau(e_{x,a}^2 e_{y,b}) = \tau(e_{x,a}e_{y,b})$ we have that $\tau(P_{\mathcal{G},\pi}) = \tau(R_{\mathcal{G},\pi})$. The expression $P_{\mathcal{G},\pi}$ has the advantage of being a positive element.

Theorem 3.1. *Let $\mathcal{G}, \pi$ be as above and let $C^*(\mathbb{F}(n,k)) \subseteq B(H)$. Then*

$$\omega_{qc}^s(\mathcal{G}, \pi) = \sup\{\tau(P_{\mathcal{G},\pi}) : \tau \in T(C^*(\mathbb{F}(n,k)))\} =$$
$$d(P_{\mathcal{G},\pi}, \mathcal{C}) = \inf\{t : t1 - P_{\mathcal{G},\pi} \in SOS + C\}$$

where $SOS \subseteq \mathbb{C}(\mathbb{F}(n,k))$ is the set of elements that can be written as a sum of hermitian squares and $C \subseteq \mathbb{C}(\mathbb{F}(n,k))$ is the linear span of commutators. Moreover,

$$\omega_q^s(\mathcal{G}, \pi) = \omega_{qa}^s(\mathcal{G}, \pi) = \sup\{\tau(P_{\mathcal{G},\pi}); \tau \in T_{am}(C^*(\mathbb{F}(n,k)))\} = d(P_{\mathcal{G},\pi}, \mathcal{B}).$$

The work in MIP*=RE shows that finding ω_{qa}^s is of a different computational complexity. This is reflected somewhat in the above formula, since it involves the distance to $\mathcal{B}$, which needs a faithful representation of $C^*(\mathbb{F}(n,k))$ to define, while $\omega_{qc}^s(\mathcal{G}, \pi)$ can be done in the group algebra.

Problem 3.2. *Are there other, possibly "better" formulas for $\omega_{qa}^s(\mathcal{G}, \pi)$?*

In MIP*=RE it is shown that a synchronous game $\mathcal{G}$ with density π exists such that
$$\omega_{qa}^s(\mathcal{G}, \pi) \leq \omega_{qa}(\mathcal{G}, \pi) \leq 1/2 < \omega_{qc}^s(\mathcal{G}, \pi) = 1.$$
Thus, for this game,
$$d(P_{\mathcal{G},\pi}, \mathcal{B}) \leq 1/2 < d(P_{\mathcal{G},\pi}, \mathcal{C}) = 1.$$

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