IML COURSE, MARCH 2026

VERN I. PAULSEN

ABSTRACT. These notes are intended to accompany and parallel my lectures at Institute Mittag-Leffler. These notes are based on my earlier Copenhagen notes. These notes go into more detail than I will be able to provide in the lectures. They assume some background in C*-algebras and operators on a Hilbert space.

CONTENTS

This material is based upon work supported by the Swedish Research Council while the author was in residence at Institut Mittag-Leffler in Djursholm, Sweden during the Spring of 2026.

1. Two Person Cooperative Games

The types of games that we shall be interested in are *two person games*, which are *cooperative* and *memoryless.* Generally, the two players are referred to as Alice and Bob. Intuitively, in such a game the two players are playing cooperatively to give correct pairs of answers to pairs of questions posed by a third party often called the *Referee* or *Verifier.* Whether the pair of answers returned by the players is satisfactory or not depends not just on the individual answers but on the 4-tuple consisting of the question-answer pair.

Such a game is described by two input sets I_A, I_B, two output sets O_A, O_B, and a function

$$V : O_A \times O_B \times I_A \times I_B \to \{0, 1\},$$

often called the *rules* or *verification function*, where

$$W := \{(a, b, x, y) : V(a, b, x, y) = 1\},$$

is the set of *correct* or *winning* 4-tuples and

$$L := \{(a, b, x, y) : V(a, b, x, y) = 0\},$$

is the set of *incorrect* or *losing* 4-tuples.

For each *round* of the game Alice and Bob receive an input pair (x, y) and return an output pair (a, b) is referred to as a *round* of the game.

When we say that the players are not allowed to *communicate* or that the game is *non-communicating*, this means that Alice must return her answer without knowing the question y that Bob was asked and without knowing the answer b that Bob gave. Similarly, Bob does not know Alice's question-answer pair.

Thus, a game $\mathcal{G}$ is specified by $\mathcal{G} = (I_A, I_B, O_A, O_B, V)$. Before the game begins Alice and Bob have all of the above information. Even though Alice and Bob are not allowed to communicate during the game they are allowed to communicate before the game and decide on some type of strategy.

A *deterministic strategy* for such a game is a pair of functions $f : I_A \to O_A,\ g : I_B \to O_B$ such that whenever Alice and Bob receive input pair (x, y) they respond with output pair $(f(x), g(y))$.

A deterministic strategy is called *perfect*, if

$$V(f(x), g(y), x, y) = 1,\ \forall x, y,$$

i.e., if it always returns correct answers.

When we want to talk about the *probability of winning such a game* we also need to specify a probability density on input pairs, i.e., a function $\pi : I_A \times I_B \to [0,1]$ such that

$$\sum_{x \in I_A, y \in I_B} \pi(x,y) = 1.$$

For games with densities, Alice and Bob also know the density before the start of the game.

We write $(\mathcal{G}, \pi) = (I_A, I_B, O_A, O_B, V, \pi)$ and call these games with densities. Many authors define a game to mean a game with density, but we are often interested in keeping the rules fixed and varying the density.

Given a game with a density if Alice and Bob adopt the deterministic strategy $f : I_A \to O_A$ and $g : I_B \to O_B$ then the probability that they will win the game is given by:

$$\sum_{x,y,a,b} \pi(x,y) V(f(x), g(y), x, y).$$

The **value of the game,** is the supremum of this probability over all possible strategies, i.e.,

$$\omega(\mathcal{G}, \pi) := \sup\{ \sum_{x,y,a,b} \pi(x,y) V(f(x), g(y), x, y) | f : I_A \to O_A, \; g : I_B \to O_B \}.$$

We will sometimes write $\omega_{det}(\mathcal{G}, \pi) := \omega(\mathcal{G}, \pi)$ to help distinguish this value of a game from other notions of the value of a game.

In general, finding the value of a game involves searching over all possible pairs of functions.

Here is a very simple, famous example:

1.1. **The CHSH Game.** Takes its name from [JCMHASRH69]. Famous as a game with a quantum advantage, see [RCPHBTJW04].

Here $I_A = I_B = O_A = O_B = \mathbb{Z}_2$–the binary field. The function λ is most easily described by saying that given the input pair (x, y) they win if their output pair (a, b) satisfies

$$a + b = xy.$$

Let's say that they also know that the probabilty is given by $\pi(x, y) = 1/4$, i.e., the uniform distribution on the 4 possible input pairs.

If they choose the deterministic strategy of always returning 0 no matter what input they receive then $a + b = 0$ and so they will win unless the input pair was $(1, 1)$. Thus the expected value of this deterministic strategy is $3/4$.

Exercise 1.1. Show that among all deterministic strategies, this is the one with the greatest expected value, so that $\omega(CHSH, \pi) = 3/4$ and that there is exactly one other deterministic strategy with value $3/4$.

Exercise 1.2. Suppose that we fix $0 < t \leq 1$ and change the input probability to

$$\pi_t(0,0) = \pi_t(1,0) = \pi_t(0,1) = t/3, \ \pi_t(1,1) = 1 - t.$$

What can you say about the best deterministic strategy in this case? Is there more than one optimal deterministic strategy?

Here are some of the most widely studied games. Too avoid too many subscripts, we shall from non on write $X = I_A, Y = I_B, A = O_A, B = O_B$.

1.2. **Synchronous Games.** Concept introduced in [VPSSDS$^+$16].

Definition 1.3. A game $\mathcal{G} = (X, Y, A, B, V)$ is called a **synchronous** game, if $X = Y, A = B$ and $V(a, b, x, y) = 0, \forall a \neq b, \forall x, y$. A game is called **symmetric** if $X = Y, A = B$ and $V(a, b, x, y) = V(b, a, y, x)$.

Note that if a pair of functions f, g are perfect for a synchronous game, then $f = g$.

Given a game $\mathcal{G}$ with $X = Y, A = B$, and density π, we define the **synchronous value** of the game by

$$\omega^s(\mathcal{G}, \pi) := \sup\{\sum_{x,y,a,b} \pi(x,y)V(f(x), f(y), x, y) | f : X \to A\}.$$

1.3. **Bisynchronous Games.** A game is called **bisynchrnous** if it is synchronous and $V(a, a, x, y) = 0 \ \forall x \neq y$. If a pair of functions f, g are perfect for a bisynchronous game, then $f = g$ and f is one-to-one.

These were introduced by P-Rahaman[VPMR21].

1.4. **XOR Games.** A game $\mathcal{G} = (X, Y, A, B, V)$ is called an XOR game if there is a function $F : X \times Y \to \mathbb{Z}_2$ such that

$$W = \{(a, b, x, y) : a - b = F(x, y)\}.$$

Thus, the CHSH game is an XOR game with $F(x, y) = xy$.

An XOR game is synchronous if and only if $F(x, x) = 0, \ \forall x$ and symmetric if $F(x, y) = F(y, x)$.

Thus, the CHSH game is an XOR game with $F(x, y) = xy$. It is a symmetric game but not synchronous.

1.5. **Unique Games.** A game $\mathcal{G} = (X, Y, A, B, V)$ is a **unique game** provided that for each triple x, y, a there is a unique b such that $V(a, b, x, y) = 1$ and for each x, y, b there is a unique a such that $V(a, b, x, y) = 1$.

The XOR games are all unique games.

It is easily checked that for a game to be unique, $|A| = |B|$ and if we identify $A = B$, then there is a function $F : X \times Y \to Perm(A)$, where $Perm(A)$ denotes the group of permutations of A, such that

$$V(a, b, x, y) = 1 \iff a = F(x, y)b.$$

Such a game is synchronous if and only if $X = Y$ and $F(x, x) = id, \ \forall x$ and symmetric if and only if $F(y, x) = F(x, y)^{-1}$.

A special family of unique games are the **group based games**, introduced in [RLVP]. These are games for which $A = B = \Gamma$, where Γ is a finite group and there is a function $F : X \times Y \to \Gamma$ such that

$$V(a, b, x, y) = 1 \iff a = F(x, y)b.$$

Thus, a group based game is a unique game with $A = B = \Gamma$ and the only permutations allowed are left multiplication by a group element.

A synchronous group based game has a perfect deterministic strategy f if and only if $F(x, y) = f(x)f(y)^{-1}$.

One generalization of the CHSH game is the group based game, where $X = Y = A = B = \mathbb{Z}_p$ and $F(x, y) = xy$. Here, $\Gamma = (\mathbb{Z}_p, +)$ and we are using the product on $\mathbb{Z}_p$. This has no perfect strategy since there is no function satisfying $f(x) - f(y) = xy, \forall x, y$.

1.6. **The Graph Homomorphism Game.** These games are studied extensively in [DR13, LMDR12, LMDR16a, LMDR16b].

Given two graphs $G_i = (V_i, E_i), i = 1, 2$ a *homomorphism* from G_1 to G_2 is a function $f : V_1 \to V_2$ with the property that if $(x, y) \in E_1$ then $(f(x), f(y)) \in E_2$. We write $G_1 \to G_2$ to indicate that there exists a graph homomorphism from G_1 to G_2.

Graph homomorphisms are convenient for capturing many of the parameters studied in graph theory. For instance, if K_k denotes the *complete graph* on k vertices, i.e., every pair is an edge, then it is not hard to see that G has a k-colouring if and only if $G \to K_k$. Thus, the chromatic number of a graph, $\chi(G)$ is the smallest k for which $G \to K_k$.

Similarly, a *clique* in G is a subset of vertices such that every pair is connected by an edge. It is not hard to see that G has a clique of size k if and only if $K_k \to G$.

A set of vertices in G is called *independent* if they contain no edges. If G^c denotes the *complement of* G, i.e., the graph with the same vertices, but with (x, y) an edge in G^c if and only if $x \neq y$ and (x, y) was NOT an edge in G. Thus, G has an independent set of size k if and only if $K_k \to G^c$.

The graph homomorphism game $Hom(G_1, G_2)$ is the synchronous game with $X = Y = V_1, A = B = V_2$ and

$$L = \{(a, b, x, y) : (x, y) \in E_1, (a, b) \notin E_2\} \cup \{(x, x, a, b) : x \in V_1, a \neq b\}.$$

Exercise 1.4. Show that a perfect deterministic strategy for this game is an actual graph homomorphism.

Often we set $Hom(G, K_k) := Col(G, k)$ and call this the graph colouring game.

Given a graph $G = (V, E)$ let π be the uniform density supported on the edge set, i.e.,

$$\pi(x, y) = \begin{cases} 1/|E| & (x, y) \in E \\ 0 & else \end{cases}.$$

Exercise 1.5. Show that $\omega(Col(G,2),\pi) = 1$ for any graph with an edge. Find a formula for $\omega^s(Col(G,2),\pi)$ in terms of the cut number of the graph.

1.7. **The Graph Isomorphism Game.** Introduced in [AALMDR$^+$19].

Given a graph $G = (V, E)$ we define a function

$$rel : V \times V \to \{-1, 0, +1\},$$

via

$$rel(x,y) = \begin{cases} -1, & (x,y) \in E, \\ 0, & x = y, \\ +1, & (x,y) \notin E \text{ and } x \neq y \end{cases}.$$

Most of you are probably familiar with the $V \times V$ *adjacency matrix* of a graph $A_G = (a_{x,y})$ with $a_{x,y} = 1 \iff (x,y) \in E$ and 0 otherwise.

The $V \times V$ matrix

$$S_G := (rel(x,y)),$$

is called the *Siedel adjacency matrix* and plays an equally interesting role in spectral graph theory. For example a graph is completely regular if and only if S_G has exactly two eigenvalues. Conjugating S_G by a diagonal matrix of ± 1's yields the Siedel adjacency matrix of a new graph, with the same eigenvalues, and such a graph is called *switching equivalent* to G.

Two graphs $G_i = (V_i, E_i), i = 1, 2$ are **isomorphic** if there is a one-to-one, onto function $f : V_1 \to V_2$ such that

$$rel(x,y) = rel(f(x), f(y)), \forall x, y \in V_1.$$

. In this case we write $G_1 \simeq G_2$. After identifying $V_1 = V_2$, this is equivalent to there existing a permutation matrix P such that $PS_{G_1} = S_{G_2}P$, or equivalently, $PA_{G_1} = A_{G_2}P$.

The *graph isomorphism game* $Iso(G_1, G_2)$ is the game with input set $I = V_1$, output set $O = V_2$, and we will define it in terms of its winning set

$$W = \{(a, b, x, y)) \in V_2 \times V_2 \times V_1 \times V_1 | rel(x,y) = rel(a,b)\}.$$

Exercise 1.6. Show that this is a bisynchronous game and that when $card(V_1) = card(V_2)$, then $f : V_1 \to V_2$ is a perfect deterministic strategy if and only if f is a graph isomorphism. More generally, show that f is a perfect deterministic strategy if and only if $card(V_1) \leq card(V_2)$ and G_1 is isomorphic to the induced subgraph of G_2 on the subset $f(V_1)$. (Some would call this an isomorphism onto the range.)

1.8. **Linear Constraint System Games.** .

Studied in [NM90]. Recall that if p is a prime number then the set of integers modulo p, $\mathbb{Z}_p$ equipped with addition modulo p and multiplication modulo p is a *field*. The most familiar of these is the *binary field* $\mathbb{Z}_2 = \{0, 1\}$, where $1 + 1 \equiv 0$.

In particular, for each $k \neq 0$ there is $j \neq 0$ such that

$$jk \equiv 1 (modp).$$

For example, in $\mathbb{Z}_{31}$, $2 \cdot 16 \equiv 32 \equiv 1(mod 32)$ and so $16 = 2^{-1}$, while $-2 \equiv +29$.

Suppose that we are given a system of n linear equations in m variables over $\mathbb{Z}_p$:

$$f_i(x_1, ..., x_m) := \sum_{j=1}^{m} a_{i,j} x_j = b_i, 1 \le i \le n,$$

or in matrix vector notation,

$$A\vec{x} = \vec{b}.$$

There are two versions of the linear constraint system game. One is not synchronous and the other is synchronous.

First we discuss the non-synchronous game. In this game Alice is given an equation, i.e., $I_A = \{1, ..., n\}$ and Bob is given a variable, $I_B = \{1, ..., m\}$. Alice must return values for each of the variables in her equation and Bob must give a value to his variable. They win if Alice's variable values satisfy the equation that she was given and if value that Bob assigned to his particular variable is the same as the value that Alice gave to that variable. We denote this game by $LCS(A, \vec{b})$.

For this game there is a group, called the *solution group* of the game. This group has a distinguished element J and it is known that this game has a perfect q-strategy if and only if this group has a finite dimensional representation that sends J^{p-1} to $-I$.

Here is the intuitive description of the synchronous game, denoted $SyncLCS(A, b)$. The input set is $I = \{1, ..., n\}$. Suppose that for inputs Alice receives i_1 and Bob receives i_2. To win Alice must return values for each of the variables in equation i_1 that has a non-zero coefficient that satisfy $f_{i_1}(\vec{x}) = b_{i_1}$ and Bob must return values for each of the variables in equation i_2 that has a non-zero coefficient and satisfy $f_{i_2}(\vec{y}) = b_{i_2}$. In addition if variable j has a non-zero coefficient in both equations, then Alice and Bob must have given the same value to that variable, i.e., $x_j = y_j$.

This isn't quite a "game" as we've defined them since there is not a fixed output set. We fix that by making the rules that: To win, Alice and Bob must return vectors $\vec{v}, \vec{w} \in (\mathbb{Z}_p)^k$ such that:

(1) $f_{i_1}(\vec{v}) = b_{i_1}$, and $f_{i_2}(\vec{w}) = b_{i_2}$,
(2) $a_{i_1,j} a_{i_2,j} \neq 0 \implies v_j = w_j$,
(3) $a_{i_1,j} = 0 \implies v_j = 0$ and $a_{i_2,j'} = 0 \implies w_{j'} = 0$

With these slightly modified rules, we can see that this is a synchronous game with output set $O = \mathbb{Z}_p^m$. Thus the output set has p^m elements.

In the next section we shall discuss perfect t-strategies. It is known that these two version of the game are equivalent in the sense that one version has a perfect t-strategy if and only if the other version has a perfect t-strategy for $t = q, qa, qc$.

This brings up the interesting question of how does one prove that two games with such different descriptions have such similar behavior? We will see one way to approach such problems when we discuss the "algebra of a game".

Exercise 1.7. Show that for both versions, this game has a perfect deterministic strategy if and only if the system of equations has a solution.

There is an interesting connection between solutions to systems of equations over $\mathbb{Z}_2$ and graph isomorphisms due to Arkhipov. Given a binary system of linear equations $A\vec{x} = \vec{b}$, Arkhipov constructs a graph $G_{A,\vec{b}}$ with the property that

$$G_{A,\vec{b}} \sim G_{A,\vec{0}} \iff A\vec{x} = \vec{b} \text{ has a solution }.$$

This gives a third game for studying linear systems. (Is it known if the algebra of this game is the same as other other two?)

1.9. **Products of Games.** Given games $\mathcal{G}_i = (X_i, Y_i, A_i, B_i, V_i), i = 1, ..., n$ with probability densities $\pi_i : X_i \times Y_i \to [0, 1]$, the **product** of these games, denoted $\prod_{i=1}^n \mathcal{G}_i := \mathcal{G}_1 \times \cdots \times \mathcal{G}_n$ is the game with input sets $X := \prod_{i=1}^n X_i := X_1 \times \cdots \times X_n$, $Y := \prod_{i=1}^n Y_i$, output sets $A := \prod_{i=1}^n A_i$ and $B := \prod_{i=1}^n B_i$ and value function $V := \prod_{i=1}^n V_i$ where by this last product we mean the pointwise product of the value functions, i.e.,

$$V(a, b, x, y) = V_1(a_1, b_1, x_1, y_1) \cdots V_n(a_n, b_n, x_n, y_n),$$
$$a = (a_1, ..., a_n), b = (b_1, ..., b_n), x = (x_1, ..., x_n), y = (y_1, ..., y_n),$$

where the product is in $\mathbb{Z}_2$. Notice that, in words, an n-tuple wins the product game if and only if it wins each game. The product density is given by $\pi(x, y) = \pi_1(x_1, y_1) \cdots \pi_n(x_n, y_n)$.

When $\mathcal{G}_1 = \cdots = \mathcal{G}_n := \mathcal{G}$ we write $\mathcal{G}^{(n)}$ for the product instead.

It is easy to see that the product game has a perfect deterministic strategy if and only if each game has a perfect deterministic strategy.

It is known the the value is supermultiplicative, i.e.,

$$\omega(\mathcal{G}_1 \times \mathcal{G}_2, \pi_1 \times \pi_2) \geq \omega(\mathcal{G}_1, \pi_1) \cdot \omega(\mathcal{G}_2, \pi_2).$$

Nevertheless, a theorem of Raz[RR98], shows that if $\omega(\mathcal{G}, \pi) \neq 1$, then

$$\lim_n \omega(\mathcal{G}^{(n)}, \pi^{(n)}) = 0,$$

and the convergence is exponentially fast.

1.10. **Direct Sums.** There is another operation on games called the **direct sum**. The input and output sets and density are the same as for the product. The only difference is that the verification function is given by

$$V(x, y) = \sum_i V_i(x_i, y_i),$$

where the sum is in $\mathbb{Z}_2$. Thus, to win the direct sum game, one must win an odd number of games. This game is denoted $\mathcal{G}_1 \oplus \cdots \mathcal{G}_n$.

The direct sum plays an important role for XOR games.

1.11. **Random Strategies.** Since games are *memoryless*, that is, if Alice and Bob receive the same input pair (x, y) at two different rounds of the game, then there is no penalty if they return different output pairs at different rounds.

This allows for the possibility of strategies that produce the answer pairs randomly.

A *random strategy* for such a game yields a *conditional probability density*,

$$p(a, b|x, y),\ \ x \in X,\ y \in Y,\ a \in A,\ b \in B,$$

which gives the conditional probability that Alice and Bob return output pair (a, b), given that they received input pair (x, y).

We will often refer to the density as the strategy, without discussing how the density arises.

Given a game $\mathcal{G} = (X, Y, A, B)$ along with a density π and a random strategy p the probability that they will win, i.e., the *expected value* of the game is given by,

$$\mathbb{E}(p) = \sum_{a,b,x,y} \pi(x, y)p(a, b|x, y)V(a, b, x, y).$$

We now look at the main sets of such densites arising from quantum models.

2. Models for Quantum Correlations: Tsirelson's Problems

In the last section we mentioned that there were different models for quantum densities without really addressing what these models are. We remedy that problem here.

Suppose that Alice and Bob have separated, isolated labs and they can each perform one of n_A, respectively, n_B, quantum measurements and each measurement has, respectively, k_A and k_B outcomes. We let $p(a, b|x, y)$ denote the conditional probability density that Alice gets outcome a and Bob gets outcome b, when the perform measurements x and y, respectively. Such densities are also called **quantum correlations** and Tsirelson was interested in mathematical descriptions of the set of all such conditional densities.

It turns out that the axiomatic quantum theory allows for several possible mathematical descriptions of these sets of densities and Tsirelson was interested in whether these were all the same. So we start with the possible descriptions.

The basic quantum model assumes that Alice and Bob labs are described by finite dimensional state spaces, $\mathcal{H}_A, \mathcal{H}_B$ and that the state of their combined labs is given by a unit vector $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$. Alice's and Bob's measurements are each given by an (n, k)-POVM, $\{E_{x,a} : 1 \le x \le n_A, 1 \le a \le k_A\}$ and $\{F_{y,b} : 1 \le y \le n_B, 1 \le b \le k_B\}$, which means we have families of projections such that

$$\sum_{a=1}^{k_A} E_{x,a} = I_{\mathcal{H}_A}, \forall x, \text{ and } \sum_{b=1}^{k_B} F_{y,b} = I_{\mathcal{H}_B}, \forall y,$$

and

$$p(a,b|x,y) = \langle \psi | (E_{x,a} \otimes F_{y,b}) \psi \rangle.$$

We let $C_q(n_A, n_B, k_A, k_B)$ denote the set of all $p(a,b|x,y)$ that can be obtained as above, which we call the **quantum correlations** or **quantum densities**. Note that since $0 \le p(a,b|x,y) \le 1$ that we can always regard $C_q(n_A, n_B, k_A, k_B)$ as a subset of the compact set $[0,1]^{n_A n_B k_A k_B}$. Generally, we shall be interested in the case that $n_A = n_B = n$ and $k_A = k_B = k$, in which case we shorten this to $C_q(n,k)$.

A slightly more general model is to allow Ḩ$_A$ and $\mathcal{H}_B$ to be arbitrary Hilbert spaces in which case we denote this larger set by $C_{qs}(n_A, n_B, k_A, k_B)$ where the subscript stands for **quantum spatial**.

There is no reason that either of these sets needs to be closed. However, a nice result that uses C*-algebra theory is that they both have the same closure and we set

$$C_{qa}(n_A, n_B, k_A, k_B) := C_q(n_A, n_B, k_A, k_B)^- = C_{qs}(n_Q, n_B, k_A, k_B)^-.$$

These are called the **quantum approximate** correlations.

An even more general model is to assume that the combined state space of Alice and Bob does not decompose as a tensor product but instead that it is a single Hilbert space $\mathcal{H}$ so that they each have POVM's on this space,

$$\{E_{x,a} : 1 \le x \le n_A, 1 \le a \le k_A\} \ \{F_{y,b} : 1 \le y \le n_B, 1 \le b \le k_B\} \subseteq B(\mathcal{H}),$$

with the property that $E_{x,a} F_{y,b} = F_{y,b} E_{x,a}, \ \forall x, y, a, b$. We call this a **commuting model**. Note that it is only Alice's operators that must commute with Bob's operators. There is no requirement that Alice's operators commute among themselves for different inputs.

The set of all

$$p(a,b|x,y) = \langle \phi | E_{x,a} F_{y,b} \phi \rangle,$$

that can be obtained in this manner for some commuting model and some unit vector ϕ is denoted $C_{qc}(n_A, n_B, k_A, k_B)$ and called the **quantum commuting** correlations. This set is known to be closed but the proof needs some C*-algebra theory

The explanation for this commuting hypothesis is that the outcome should not depend on the order of applying their measurements. Note that in the tensor cases we have that

$$E_{x,a} \otimes F_{y,b} = (E_{x,a} \otimes I_{\mathcal{H}_B})(I_{\mathcal{H}_A} \otimes F_{y,b}) = (I_{\mathcal{H}_A} \otimes F_{y,b})(E_{x,a} \otimes I_{\mathcal{H}_B}).$$

so it is a commuting correlation. It is known that

$$C_q(n_A, n_B, k_A, k_B) \subseteq C_{qs}(n_A, n_B, k_A, k_B) \subseteq C_{qa}(n_A, n_B, k_A, k_B) \subseteq C_{qc}(n_A, n_B, k_A, k_B).$$

Most of these containments are fairly straightforward, except for $C_{qs} \subseteq C_{qa}$ which uses results about *residually finite dimensional(RFD)* C*-algebras. We will sketch this fact later.

In the case that $n_A = n_B = k_A = k_B = 2$, Tsirelson proved that these sets are all equal, and wondered if this could be true more generally. For more on why they are equal in the 2 input, 2 output case see the supplementary notes on correlations.

Work of Junge, Palazuelas, Perez-Garcia, Scholze, and Werner [MJMNCP$^+$11] and Ozawa [NO13] proved that whether or not two of these models, C_{qa} and C_{qc}, gave the same densities or not is equivalent to the famous *Connes' Embedding Problem(CEP)* having an affirmative answer.

Slofstra made the first breakthrough when he showed that $C_q(n,k) \neq C_{qa}(n,k)$, i.e., $C_q(n,k)$ is not a closed set for n,k sufficiently large. He did this by creating a LCS game $(\mathcal{G},\pi)$ with no perfect strategy in C_q but with $\omega_q(\mathcal{G},\pi) = 1$.

From [KDVPJP19], we now know that $C_q(n,k)$ is not a closed set for most values of n,k

Theorem 2.1. *The sets $C_q(n,k)$ are not closed for every $n \geq 5, k \geq 2$. Let $\frac{\sqrt{5}-1}{2\sqrt{5}} \leq t \leq \frac{\sqrt{5}+1}{2\sqrt{5}}$ and for $0 \leq a,b \leq 1,\ 1 \leq x,y \leq 5$ set*

$$p(0,0|x,x) = t, \qquad p(0,1|x,x) = p(1,0|x,x) = 0, \qquad p(1,1|x,x) = 1-t,$$

and for $x \neq y$, set

$$p(0,0|x,y) = \frac{1}{4}t(5t-1), \qquad p(0,1|x,y) = p(1,0|x,y) = \frac{5}{4}t(1-t),$$
$$p(1,1|x,y) = \frac{1}{4}(1-t)(4-5t).$$

Then $p \in C_{qa}(5,2)$ for all t in this interval, but $p \in C_q(5,2)$ only for t rational.

Note that this is a nice continuous path of correlations p_t but to "decide" if p_t belongs to $C_q(5,2)$ one must be able to decide if t is rational. For example it is still unknown if $e+\pi$ is rational. So if we take a rational multiple of $e+\pi$ that lands us in the above interval, then for such values of t it is still unknown if p_t belongs to $C_q(5,2)$. From this result it follows that $C_q(n,k)$ is not closed for every $n \geq 5,\ k \geq 2$.

By Tsirelson's results, $C_q(2,2)$ is closed, but it is still not known if $C_q(3,2)$ and $C_q(4,2)$ are closed. In fact, there is a famous inequality called the $I(3,3,2,2)$ inequality that if it s not attained would show that $C_q(3,2)$. See [KPTV10] for an introduction to work on $I(3,3,2,2)$.

Combining the above theorem with results from [SJKVPCS18], we know that $C_{qs}(n,k)$ is not closed for every $n \geq 5, k \geq 2$. In [ACJS18] it is shown that $C_{qs}(4,3)$ is not closed. See T. Russell's papers for other work on closure problems.

In MIP*=RE [ZJANTV$^+$20] the authors construct a non-local game to prove that $C_{qa} \neq C_{qc}$, i.e., that these sets of densities are different and that hence CEP has a negative answer.

They did this by producing a game with a perfect strategy in C_{qc} but no perfect strategy in C_{qa}. Currently, this is the only proof of this famous problem in operator algebras.

Einstein wondered if entanglement could be explained with a theory of *local, hidden variables*. Essentially this theory postulates that the randomness observed in quantum measurements is occurring because there is some *hidden* probability space (Ω, P) and for each $x \in I_A, y \in I_B$ random variables

$$f_x : \Omega \to O_A, \ g_y : \Omega \to O_B,$$

such that each time an experiment is conducted one is really evaluating these random variables at some unknown point $t \in \Omega$, i.e., if Alice conducts measurement x and Bob measurement y, then the values of the outcomes will be $a = f_x(\omega), b = g_y(\omega)$ and the only reason that we cannot predict a priori the values of the outcome is that we do not a priori know which point t we will be evaluating these random variables at when we perform the measurement. Bell pursued this idea and was able to show mathematically that $C_{loc}(2,2) \subsetneq C_q(2,2)$ and developed his sets of now-called *Bell inequalities* to describe the smaller sets.

Exercise 2.2. Prove that in this case

$$p(a,b|x,y) = P(\{\omega \in \Omega | f_x(\omega) = a, g_y(\omega) = b\}).$$

The set of all conditional probability densities that can be obtained in this fashion, as we vary the probability space and the random variables are called the **local densities** and is denoted by

$$C_{loc}(n_A, n_B, k_A, k_B).$$

Exercise 2.3. Show that

$$C_{loc}(n_A, n_B, k_A, k_B) \subseteq C_q(n_A, n_B, k_A, k_B).$$

This is what motivates the term **non-local games**, which is a bit of a misnomer. It really refers to the fact that we are allowing these two person cooperative games to be played using densities that are non-local.

Exercise 2.4. Prove that a game has a perfect local density if and only if it has a perfect deterministic strategy.

Exercise 2.5. Prove that $p(a,b|x,y)$ is an extreme point of $C_{loc}(n_A, n_B, k_A, k_B)$ if and only if there are functions $f : \{1, ..., n_A\} \to \{1, ..., k_A\}$ and $g : \{1, ..., n_B\} \to \{1, ..., k_B\}$ such that

$$p(a,b|x,y) = \begin{cases} 1 & \text{when } a = f(x),\ b = g(y), \\ 0 & \text{else.} \end{cases}$$

Before leaving this section, we want to mention one more family of correlations. This is the largest set of abstract conditional densities that obeys some natural axioms from probability.

A collection of numbers $p(a,b|x,y)$ is called a **non-signalling density**(or correlation) provided that:

- $p(a,b|x,y) \geq 0, \forall x,y,a,b$
- $1 = \sum_{a,b} p(a,b|x,y),\ \forall x,y$
- $\forall x,y,y',\ \sum_b p(a,b|x,y) = \sum_b p(a,b|x,y')$. This common value is denoted $P_A(a|x)$ and is called the *conditional probability that Alice gets outcome a given input x,*
- $\forall y,x,x',\ \sum_a p(a,b|x,y) = \sum_b p(a,b|x',y)/$ This common value is denoted $p_B(b|y)$ and is called the *conditional probability that Bob gets outcome b given input y.*

The set of all non-signalling densities is denoted C_{ns}.

For all of the games mentioned above, it is also interesting to determine if they have perfect non-signalling densities.

Exercise 2.6. Prove that all of the densities in C_{qc} are non-signalling, so that $C_{qc} \subseteq C_{ns}$.

2.1. **Other Values of Games, Products of Games.** For each of these sets of densities $t = loc, q, qs, qa, qc, ns$ we define a value of the game $(\mathcal{G},\pi)$ by

$$\omega_t(\mathcal{G},\pi) := \sup\{\sum_{x,y,a,b} \pi(x,y)p(a,b|x,y)V(a,b,x,y)|\, p \in C_t\}.$$

Exercise 2.7. Prove that $\omega_{loc}(\mathcal{G},\pi) = \omega(\mathcal{G},\pi)$.

If we let π denote the uniform density on input pairs for the CHSH game then it is known that

$$3/4 = \omega(CHSH,\pi) = \omega_{loc}(CHSH,\pi) < \omega_q(CHSH,\pi) = cos^2(\pi/8) \simeq .85.$$

Thus, the CHSH game separates C_{loc} and C_q.

In a similar fashion values of non-local games have been used to separate all of the other collections of densities.

H. Yuen proves that if $\omega_q(\mathcal{G},\pi) < 1$, then $\omega_q(\mathcal{G}^n,\pi^n) \to 0$, but the rate is only inverse polynomial. He asks if the full analogue of Raz's theorem is true:

Problem 2.8. *If* $\omega_q(\mathcal{G},\pi) < 1$*, then are there constants* $b, c > 0$ *such that* $\omega_q(\mathcal{G}^n,\pi^n) \leq be^{-cn}$*?*

For the other values it is not even known if these values tend to 0.

Problem 2.9. *If* $\omega_{qc}(\mathcal{G},\pi) < 1$*, then does* $\omega_{qc}(\mathcal{G}^n,\pi^n) \to 0$*? If so then can it also be made exponential of inverse polynomial?*

2.2. **Quantum Chromatic Numbers.** Remarkably, perfect quantum strategies exist for $Col(G,k)$ for values $k < \chi(G)$. This leads to the notion of various *quantum chromatic numbers* for graphs. If we fix $t = q, qs, qa, qc$, then the least k for which there exists a perfect density in C_t(to be defined later) for the game $Col(G,k)$ is called a *quantum chromatic number* of the graph and is denoted $\chi_t(G)$.

Not only can $\chi_q(G) < \chi(G)$, but the family of *Hadamard graphs* are known to have quantum chromatic numbers that are exponentially smaller than their chromatic numbers. The Hadamard graphs Ω_N are defined as follows:

- the vertex set is all N-tuples of ± 1, so that Ω_N has 2^N vertices,
- two vertices $x + (x_1, ..., x_N)$ and $y = (y_1, ..., y_N)$ are adjacent if and only if

$$x \cdot y := \sum_{i=1}^{N} x_i y_i = 0.$$

Note that if N is odd then $x \cdot y \neq 0$ so the only interesting case is for N even.

It is known that $\chi(\Omega_N) > (1.06)^N$, but exact values of the chromatic number are only known for a few values of N. On the other hand $\chi_q(\Omega_N) = N$, for all even N [DAJHYKYS06].

If we consider the graph G with uncountably many vertices,

$$V = \mathbb{T}^N := \{(\lambda_1, ..., \lambda_N) : \lambda_i \in \mathbb{C}, |\lambda_i| = 1\},$$

and $(\vec{\lambda}, \vec{\mu}) \in E \iff \sum_i \lambda_i \mu_i = 0$, then [VPIT15] show that this graph also has $\chi_q(G) = N$. Very little is known about the chromatic number of these graphs.

Theorem 2.10. *Let $V \subseteq \mathbb{C}^N$ with $|V| = n$, where each $v \in V$ is such that $v = (v(0), ..., v(N-1)) \in \mathbb{C}^N$ with $|v(j)| = 1$ for all j. Define $E = \{(v,w) : v, w \in V, v \perp w\}$ and let $G = (V,E)$. Then $\chi_q(G) \leq N$ (and hence $\chi_q(\Omega_N) \leq N$).*

Proof. Let $\xi = e^{\frac{2\pi i}{N}}$. For each $v \in V$, let $D_v = \text{diag}(v(0), ..., v(N-1))$ which is unitary. For each $0 \leq k \leq N-1$, let $R_k = \frac{1}{N}(\xi^{(\ell-j)k})_{\ell,j=0}^{N-1} \in M_N$. Set $h_k = \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} \xi^{\ell k} e_\ell$, so that $R_k = h_k h_k^* \geq 0$. This also shows that R_k is a rank one projection. Recall that $\sum_{k=0}^{N-1} (\xi^\ell)^k = 0$ for all $1 \leq \ell \leq N-1$ (it is equal to N when $\ell = 0, N$). Hence,

$$\sum_{k=0}^{N-1} R_k = \frac{1}{N} \left(\sum_{k=0}^{N-1} (\xi^{\ell-j})^k \right) = I.$$

Let $P_{v,k} = D_v^* R_k D_v$, so that $P_{v,k} \geq 0$ and $\sum_{k=0}^{N-1} P_{v,k} = I$. These are Alice's POVM's. Define $Q_{w,k} = D_w R_k^t D_w^*$. Since the transpose map is positive,

$Q_{w,k} \geq 0$ and $\sum_k Q_{w,k} = I$. These $Q_{w,k}$'s are Bob's POVM's. Now let $\eta = \frac{1}{\sqrt{N}} \sum_{p=0}^{N-1} e_p \otimes e_p \in \mathbb{C}^N \otimes \mathbb{C}^N$. Then we define

$$p(k, m|v, w) = \langle \eta, P_{v,k} \otimes Q_{w,m}\eta \rangle \in C_q(n, N).$$

Now check that this is a perfect density for $Col(G, N)$ for the above graph. □

The fact that $\chi_q(\Omega_N) \geq N$ follows from the above proof plus the fact that the usual Lovasz theta lower bound for the chromatic number is actually a lower bound for the quantum chromatic number. See [DR13].

Exercise 2.11 (P-Todorov)**.** Prove that for any graph with at least one edge, $\chi_{ns}(G) = 2$.

It is known that $\chi_{qs}(G) = \chi_q(G)$ for every G and there are examples known for which $\chi_q(G) > \chi_{qa}(G)$ [SJKVPCS18].

A result of S. Harris[SH22, SH24], combined with MIP*=RE, implies that there exists a graph for which $\chi_{qa}(G) > \chi_{qc}(G) = 3$.

There is an extensive literature on quantum chromatic numbers, see the works of L. Mancinska and D. Roberson[LMDR12, LMDR16a, LMDR16b] for a better entry to this area. One lovely example that they have is of a graph G so that the one-point suspension of $G, G^\dagger$ satisfies $\chi_q(G) = \chi_q(G^\dagger)$. Note that we always have that $\chi(G^\dagger) = \chi(G)+1$ since the suspension vertex always requires a new color.

2.3. **The Graph Isomorphism Game.** For $t = q, qs, qa, qc$ we write $G_1 \simeq_t G_2$ if and only if there exists a perfect density for this game in C_t. Because these are synchronous games, we will have a general theorems about when these exist.

The introduction of the graph isomorphism game and the study of these "quantum isomorphisms" of graphs is due to [AALMDR+19].

Lovasz proved that graphs G_1 and G_2 are isomorphic if and only if the number of graph homomorphisms from H to G_1 is equal to the number of graph homomorphisms from H to G_2 for all graphs H. A beautiful result of [LMDR20] shows that $G_1 \simeq_{qc} G_2$ if and only if the number of graph homomorphisms from H to G_1 is equal to the number of graph homomorphisms from H to G_2 for all planar graphs H.

Here's what is known in general for the graph isomorphism game between two graphs G, H with n vertices. All conditions involve an $n \times n$ collection of projections, $\{E_{x,a} : x \in V_G, a \in V_H\}$ satisfying,

- $\sum_a E_{x,a} = I, \forall x$,
- $\sum_x E_{x,a} = I, \forall a$,
- $A_G(E_{x,a}) = (E_{x,a})A_H$,

with

- $G \simeq_q H \iff E_{x,a}$ can be chosen in a matrix algebra,

- $G \simeq_{qa} H \iff E_{x,a}$ can be chosen in an ultrapower of the hyperfinite II_1-factor,
- $G \simeq_{qc} H \iff E_{x,a}$ can be chosen in a C*-algebra with a trace $\iff E_{x,a}$ can be chosen as operators on a Hilbert space.

The qa version follows from the work of Brannan et al [MBACKE$^+$20] as does the fact that if a solution exists via operators on a Hilbert space, then a solution exists in a C*-algebra with a trace..

2.4. **Linear Constraint System Games.** .

Recall that there are two versions of this game. It is known that for each t, the non-synchronous version of this game has a perfect strategy in C_t if and only if the synchronous version has a perfect strategy in C_t. This was first shown in [SJKVPCS18] for the case $p = 2$ and then later was extended to all primes in A. Goldberg[AG21], using the idea of the algebra of a synchronous game.

For this game there is a very different way to prove the existence of perfect strategies. Given a system of equations $A\vec{z} = \vec{b}$ over $\mathbb{Z}_p$, there exists a group $\Gamma_{A,b}$, called the **solution group** of the game, intorduced by Cleve, Mittal[RCRM14] and developed further by Cleve, Liu,Slofstra[RCLLWS17] and are the basis for Slofstra's work on the non-closure of C_q. This group is given by generators and relations and has a distinguished element J and it was shown that $LCS(A, \vec{b})$ has a perfect:

(1) loc-strategy if and only if there is an abelian representation of $\Gamma_{A,b}$ sending J^{p-1} to -1 if and only if $A\vec{x} = \vec{b}$ has a solution,
(2) q-strategy if and only if this group has a finite dimensional unitary representation that sends J^{p-1} to $-I$.
(3) qc-strategy if and only if this group has a unitary representation that sends J^{p-1} to $-I$
(4) qa-strategy if and only if this group has a unitary representation into an ultrapower of the hyperfinite II_1-factor that sends J^{p-1} to $-I$

It is known that these two version of the game are equivalent in the sense that $LCS(A, \vec{b})$ has a perfect t-strategy if and only if $SyncLCS(A, \vec{b})$ has a perfect t-strategy for $t = q, qa, qc$. The qa result follows from the work [SJKVPCS18]. See also Goldberg[AG21] for further clarification.

Problem 2.12. *From* [SJKVPCS18] *$SyncLCS(A, \vec{b})$ has a perfect q-strategy if and only if it has a perfect qs-strategy. Does $LCS(A, \vec{b})$ have a perfect qs-strategy if and only if it has a perfect q-strategy?*

This brings up the interesting question of how does one prove that two games with such different descriptions have such similar behavior? We will see one way to approach such problems when we discuss the "algebra of a game".

Remarkably, there exist systems of equations that have no actual solutions but which have perfect densities in one of the quantum correlation sets. A famous one of these is *Mermin's Magic Square*[NM90]. This system of binary equations can most easily be represented as follows:

$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & 1 \\ x_4 & x_5 & x_6 & 1 \\ x_7 & x_8 & x_9 & 1, \\ - & - & - & \\ 0 & 0 & 0 & \end{array}$$

where each horizontal row is supposed to 1 and each vertical column is supposed to 0.

A moments reflection shows that this system of equations has no solution. However, the LCS game has a perfect strategy in $C_q(6, 2^9)$.

Slofstra was able to prove that $C_q \neq C_{qa}$ by creating a system of roughly 200 equations, each involving 3 variables over the binary field that had a perfect density in C_{qa} but no perfect density in C_q.

There is a beautiful connection between linear system games over the binary field and graph isomorphisms.

Giiven a binary system of equations $A\vec{X} = \vec{b}$, Arkhipov[AALMDR$^+$19] constructs a graph $G_{A,b}$ is constructed with the property that $G_{A,b} \simeq G_{A,0}$ if and only the system of equations has a solution.

Moreover they prove that for $t = q, qc$,

$$G_{A,b} \simeq_t G_{A,0} \iff LCS(A, b) \text{ has a perfect density in } C_t.$$

Later it was shown that the same result holds for $t = qa$ [?KPS].

There is no reason that one needs to restrict attention to linear equations in the above analysis. Especially, over $\mathbb{Z}_2$, Boolean equations are described by non-linear equations. This gives us a small hint at how perfect quantum solutions to Boolean equations might lead to "new" logics and new computability classes.

3. POVM's vs PVM's, Free Products, Group Algebras and Boca's Theorem

One goal of this section is to introduce the C*-algebra viewpoint to quantum correlations and use it to see that the set of correlations is unchanged if one puts on the extra restriction that the (n, k)-POVM's are (n, k)-PVM's. This disambiguation of the two possible definitions of quantum correlations first appeared in [MJMNCP$^+$11] and also in [TF12].

In this section we show this disambiguation by using Boca's Theorem[FB91]. In the supplementary notes, there is a more direct proof given using ideas from state purification and dilation theory.

A *unitary representation* of a group G is a group homomorphism from G into the group of unitaries on some Hilbert space. Since unitaries satisfy $U^{-1} = U^*$, if $\rho : G \to B(\mathcal{H})$ is a unitary representation, then $\rho(g^{-1}) = \rho(g)^*$.

If we let $\sigma_k = (\mathbb{Z}_k, +)$ denote the cyclic group of order k, then every unitary representation of this group is determined by a unitary U with

$U^k = I_{\mathcal{H}}$ such that $\rho(j) = U^j$. Since $U^k = I$ it is easy to see that every eigenvalue of U must be a k-th root of unity. If we set $\omega = e^{2\pi i/k}$, then the projection E_j onto the eigenspace for ω^j is given by

$$E_j = \sum_{r=0}^{k-1} (\omega^{-j} U)^r,$$

and

$$U = \sum_{j=0}^{k-1} \omega^j E_j.$$

One key point here is that the eigenprojections are not elements of the group but are linear combinations of group elements.

This is one of the motivations for studying *group algebras.* Given a group G, the complex group algebra, denoted $\mathbb{C}(G)$ is a vector space with basis $\{u_g : g \in G\}$. We use u_g for the basis elements instead of just g as a reminder that these elements should correspond to unitaries. We define a product by the rule $u_g \cdot u_h = u_{gh}$. Thus, given $a = \sum_i \alpha_i u_{g_i} \in \mathbb{C}(G)$ and $b = \sum_j \beta_j u_{h_j} \in \mathbb{C}(G)$ where $\alpha_i, \beta_j \in \mathbb{C}$, we have that

$$a \cdot b = (\sum_i \alpha_i u_{g_i}) \cdot (\sum_j \beta_j u_{h_j}) = \sum_{i,j} (\alpha_i \beta_j) u_{g_i h_j}.$$

This makes it the case that whenever $\rho : G \to B(\mathcal{H})$ is a unitary representation, then definiing $\tilde{\rho} : \mathbb{C}(G) \to B(\mathcal{H})$ by

$$\tilde{\rho}(\sum_i \alpha_i u_{g_i}) = \sum_i \alpha_i \rho(g_i),$$

defines an algebra homomorphism.

Returning to general group algebras, recall that the u_g are like placeholders for a unitary, so it is natural to define $u_g^* := u_{g^{-1}}$ and extend this to $\mathbb{C}(G)$ be setting

$$(\sum_i \alpha_i u_{g_i})^* := \sum_i \overline{\alpha_i} u_{g_i^{-1}}.$$

Now if ρ is a unitary representation, then $\tilde{\rho}$ is also a *-homomorphism, i.e.,

$$[\tilde{\rho}(\sum_i \alpha_i u_{g_i})]^* = \tilde{\rho}((\sum_i \alpha_i u_{g_i})^*).$$

This makes $\mathbb{C}(G)$ into what is known as a **-algebra.*

Conversely, one can show that if $\gamma : \mathbb{C}(G) \to B(\mathcal{H})$ is a *-homomorphism then setting $\rho(g) = \gamma(u_g)$ defines a unitary representation of G and $\gamma = \tilde{\rho}$.

In summary, studying unitary representations of groups is the same as studying *-homomorphisms of the corresponding group algebra. The group algebra is often more convenient because it contains linear combinations of group elements which can't be discussed in the context of groups.

For the cyclic group of order k, $\sigma_k = (\mathbb{Z}_k, +)$, we see that $\mathbb{C}(\sigma_k)$ has two natural bases, $\{u_0, ..., u_{k-1}\}$ and $\{e_0, ..., e_{k-1}\}$ where

$$e_j := \sum_{r=0}^{k-1} (\omega^{-j} u_1)^r = \sum_{r=0}^{k-1} \omega^{-jr} u_r,$$

represent the projections onto the eigenspaces of u_1.

Given a k-PVM $\{E_0, ..., E_{k-1}\}$ we see that setting $U = \sum_{j=0}^{k-1} \omega^j E_j$ defines a unitary of k.

Next we want to see what studying a (n, k)-PVM corresponds to doing. For this we need the concept of the **free products** of groups and of algebras.

Given two groups G, H, twe would like to form a group that contains both G and H as subgroups, but in such a way that there is no assumption that the elements of G commute with the elements of H. This is achieved by their free product, denoted $G \star H$. This group consists of all possible *words* in an alphabet consisting of the set $G \cup H$. Thus,

$$G \star H = \{g, h, g_1 \star h_1, h_2 \star g_2, g_3 \star h_3 \star g_4, ...\}.$$

The rule for multiplying two such words is called *concatenation.* If two words end in letters from different groups then their concatenation is to just form the longer word. For example,

$$(g_1 \star h_1) \star (g_2 \star h_2 \star g_3) = g_1 \star h_1 \star g_2 \star h_2 \star g_3.$$

On the other hand if the first word ends with an element of the same group as the second word starts with, then we just multiply those elements. For example,

$$(g_1 \star h_1) \star (h_2 \star g_2) = g_1 \star (h_1 h_2) \star g_2.$$

Finally, the inverse is defined by taking the inverse of each element in the reverse order, e.g.,

$$(g_1 \star h_1 \star g_2)^{-1} = g_2^{-1} \star h_1^{-1} \star g_1^{-1}.$$

The reason for all this fuss and bother is the following universal property of the free product: Given groups G, H, K and homomorphisms, $\rho : G \to K$ and $\pi : H \to K$ there is a unique homomorphism

$$\gamma : G \star H \to K \text{ with } \gamma(g \star h) = \rho(g)\pi(h).$$

The homomorphism γ is generally denoted $\rho \star \pi$.

Finally, we should note that $G \star H = H \star G$ and $\rho \star \pi = \pi \star \rho$ and that the identity elements satisfy $e_G = e_H = e_G \star e_H = e_{G \star H} = (g \star h) \star (h^{-1} \star g^{-1})$ and many others.

Free products of algebras are defined in a similar fashion. Given two unital algebras $\mathcal{A}$ and $\mathcal{B}$ their **free product amalgamated over the identity** is a unital algebra $\mathcal{A} \star_1 \mathcal{B}$ with

- unital homomorphisms, $id_{\mathcal{A}} : \mathcal{A} \to \mathcal{A} \star_1 \mathcal{B}$ and $id_{\mathcal{B}} : \mathcal{B} \to \mathcal{A} \star_1 \mathcal{B}$ such that the union of their ranges generate $\mathcal{A} \star_1 \mathcal{B}$,

- whenever $\mathcal{C}$ is another unital algebra and $\pi : \mathcal{A} \to \mathcal{C}$ and $\rho : \mathcal{B} \to \mathcal{C}$ are both unital homomorphisms, then there exists a unital homomorphism,
$$\pi \star_1 \rho : \mathcal{A} \star_1 \mathcal{B} \to \mathcal{C},$$
with
$$\pi \star_1 \rho(a_1 \star b_1 + b_2 \star a_2) = \pi(a_1)\rho(b_1) + \rho(b_2)\pi(a_2).$$

We also have that $\mathcal{A} \star_1 \mathcal{B} = \mathcal{B} \star_1 \mathcal{A}$, and that if both algebras are *-algebras, then their amalgamated product is a *-algebra.

These two constructions play nicely together, namely given two groups, G, H, we have that
$$\mathbb{C}(G \star H) \simeq \mathbb{C}(G) \star_1 \mathbb{C}(H).$$

Why this all matters to us is the following. Given an (n, k)-PVM $\{E_{x,a} : 1 \le x \le n, 0 \le a \le k-1\}$ we get unitaries,
$$U_x = \sum_{a=0}^{k-1} \omega^a E_{x,a},$$
and each unitary corresponds to a representation of the cyclic group of order k. So these n unitaries correspond to a representation of the free product of n copies of the cyclic group of order k. We denote this group by $\mathbb{F}(n, k)$.

Thus, we have one group algebra
$$\mathbb{C}(\mathbb{F}(n, k)) \simeq \mathbb{C}(\sigma_k) \star_1 \cdots \mathbb{C}(\sigma_k) \text{ n copies },$$
generated by projections $\{e_{x,a} : 1 \le x \le n, 0 \le a \le k-1\}$, where these are the spectral projections of the x-th unitary, so that representations of this algebra correspond to (n, k)-PVM's.

Exercise 3.1. Let $\{P_a : 0 \le a \le k-1\} \subseteq B(H)$ be a k-POVM, then the map $\phi : \mathbb{C}(\sigma_k) \to B(H)$ given by $\phi(e_a) = P_a$ is completely positive.

Theorem 3.2 (Boca's Theorem). *Let $\mathcal{A}_i, i = 1, ..., n$ be unital C^*-algebras and let $\phi_i : \mathcal{A}_i \to B(H)$ be UCP maps, then there is a UCP map $\Phi : \mathcal{A}_1 \star_1 \cdots \star_1 \mathcal{A}_n \to B(H)$ such that*
$$\Phi(id_{\mathcal{A}_i}(a_i)) = \phi_i(a_i), \ 1 \le i \le n.$$
Moreover, if $X \in B(H)$ commutes with $\cup_{i=1}^n \phi_i(\mathcal{A}_i)$, then X commutes with the range of Φ.

See [FB91] for the original proof and [KDEK17] for a nice proof.

Corollary 3.3. *Let $\{P_{x,a} : 1 \le x \le n, \ 0 \le a \le k-1\} \subseteq B(H)$ be an (n, k)-POVM, then there is a UCP map $\Phi : C^*(\mathbb{F}(n, k)) \to B(H)$ with $\Phi(e_{x,a}) = P_{x,a}$. Moreover if X commutes with all of the elements of the (n, k)-POVM, then X commutes with the range of Φ*

Theorem 3.4 (Disambiguation). *For $t = loc, q, qs, qa, qc$ the set $C_t(n_A, n_B, k_A, k_B)$ remains the same if the definition was made using (n, k)-PVM's instead.*

Exercise 3.5. Use Stinespring's Dilation Theorem to prove the corollary and disambiguation theorem.

From this point on, we will regard the correlation sets as defined by PVM's.

This theorem also leads to the tensor product characterization of densities often given by operator algebraists.

We recall the **minimal** tensor product of C*-algebras, $\mathcal{A}, \mathcal{B}$ denoted $\mathcal{A} \otimes_{min} \mathcal{B}$. By a theorem of Takesaki, if $\pi : \mathcal{A} \to B(H_1)$ and $\rho : \mathcal{B} \to B(H_2)$ are faithful, unital *-homomorphisms, then the C*-algebra generated by

$$\mathcal{A} \otimes_{min} \mathcal{B} := \{\pi(a) \otimes \rho(b) \in B(H_1 \otimes H_2) : a \in \mathcal{A},\, b \in \mathcal{B}\},$$

is independent of the particular pair of faithful, unital *-homomorphisms. Consequently, for $x \in \mathcal{A} \otimes \mathcal{B}$,

$$\|x\|_{\mathcal{A} \otimes_{min} \mathcal{B}} = \sup\{\|\pi \otimes \rho(x)\|_{B(H_1 \otimes H_2)},$$

where the supremum is over all pairs of not necessarily faithful unital *-homomorphisms.

The **maximal** tensor product is universal for all pairs of commuting representations. Thus, any pair of unital *-homomorphisms, $\pi : \mathcal{A} \to B(H)$, $\rho : \mathcal{B} \to B(H)$ such that the set $\pi(\mathcal{A})$ commutes with the set $\rho(\mathcal{B})$ defines a unital *-homomorphism,

$$\pi \odot \rho : \mathcal{A} \otimes_{max} \mathcal{B} \to B(H).$$

For $x = \sum_i a_i \otimes b_i \in \mathcal{A} \otimes_{max} \mathcal{B}$, we have $\pi \odot \rho(x) = \sum_i \pi(a_i)\rho(b_i)$ and

$$\|x\|_{\mathcal{A} \otimes_{max} \mathcal{B}} = \sup\{\|\pi \odot \rho(x)\|_{B(H)}\},$$

where the supremum is over all commuting pairs.

Theorem 3.6 (JNPPSW)**.** *We have that*

$$C_{qa}(n_A, n_B, k_A, k_B) = \{s(e_{x,a} \otimes e_{y,b}) | s : C^*(\mathbb{F}(n_A, k_A)) \otimes_{min} C^*(\mathbb{F}(n_B, k_B) \to \mathbb{C} \;\; s \text{ a state } \},$$

and

$$C_{qc}(n_A, n_B, k_A, k_B) = \{s(e_{x,a} \otimes e_{y,b}) | s : C^*(\mathbb{F}(n_A, k_A)) \otimes_{max} C^*(\mathbb{F}(n_B, k_B)) \to \mathbb{C} \;\; s \text{ a state } \}.$$

Consequently, if these two tensor products agree, then $C_{qa} = C_{qc}$.

The operator algebra interest in this was inspired by work of Kirchberg[EK94].

Theorem 3.7 (Kirchberg)**.** *Connes' Embedding Problem has an affirmative answer if and only if These tensor products agree for all n, k.*

We often refer to equality of these two tensor products as **Kirchberg's problem** since its statement is quite different from Connes" original problem, which involved traces, and the proof of this equivalence is very nontrivial . Later Ozawa[NO13] would prove the converse.

Theorem 3.8 (Ozawa)**.** *If $C_{qa}(n, k) = C_{qc}(n, k)$ for all n, k, then the minimal and maximal tensor products agree.*

Thanks, to MIP*=RE, we now know that for some large n, k these sets of densities are different and that consequentally Kirchberg's problem has a negative answer. But we don't know if they are different for small values of n, k.

Nore that if we have an (n_A, k_A)-PVM $\{E_{x,a}\}$ and an (n_B, k_B) PVM $\{F_{y,b}\}$ and a unit vector η and set

$$v_{x,a} = E_{x,a}\eta,\ w_{y,b} = F_{y,b}\eta,$$

then

- $v_{x,a} \perp v_{x,a'}$
- $w_{y,b} \perp w_{y,b'}$
- $\sum_a v_{x,a} = \sum_b w_{y,b} = \eta,$
- $\langle v_{x,a} | w_{y,b} \rangle \geq 0.$

We let $C_{vect}(n_A, n_B, k_A, k_B)$ denote the set of all

$$p(a, b|x, y) = \langle v_{x,a} | w_{y,b} \rangle,$$

that arise as the inner product of such a set of vectors.

This set also arises in the NPA-hierarchy.

Exercise 3.9. Show that

$$C_{qc}(n_A, n_B, k_A, k_B) \subseteq C_{vect}(n_A, n_N, k_A, k_B) \subseteq C_{ns}(n_A, n_B, k_A, k_B).$$

Again nothing is known about products for the vect case.

Problem 3.10. *If $\omega_{vect}(\mathcal{G}, \pi) < 1$, then does $\omega_{vect}(\mathcal{G}^n, \pi^n) \to 0$? If so what about the rate as asked earlier?*

4. Synchronous Densities, Traces, and the Fundamental Orthogonality Relations

Given a synchronous game $\mathcal{G} = (X, A, V)$ with $n = card(X)$ and $k = card(A)$ we see that any perfect density $p(a, b|x, y)$ for this game must satisfy

$$p(a, b|x, x) = 0, \forall a \neq b, \forall x.$$

We call densities that satisfy this property **synchronous** and we use the superscript s to denote the subset of synchronous densities. So we have $C_t^s(n, k) \subseteq C_t(n, k)$, for $t = loc, q, qs, qa, qc, vect, ns$ and

$$C_{loc}^s(n, k) \subseteq C_q^s(n, k) \subseteq C_{qs}^s(n, k) \subseteq C_{qa}^s(n, k) \subseteq C_{qc}^s(n, k) \subseteq C_{vect}(n, k) \subseteq C_{ns}^s(n, k).$$

It turns out that the first five such densities arise from *traces* on C*-algebras, so we need to introduce and understand this concept.

Everyone is familiar with the concept of the trace of matrices:

$$Tr : M_n \to \mathbb{C},\ Tr((a_{i,j})) = \sum_{i=1}^{n} a_{i,i},$$

this has the property that if $A = (a_{i,j})$ is positive semidefinite, then $Tr(A) \geq 0$ and given any two matrices

$$Tr(AB) = Tr(BA).$$

Note that this last property implies that given any commutator,

$$[A, B] := AB - BA,$$

we have that $Tr(AB - BA) = 0$. Since the Tr is linear, it will also vanish on sums of commutators and since $Tr(I_n) = n \neq 0$ this gives us a very easy way to see that the identity matrix cannot be expressed as a sum of commutators!

An abstract **tracial state** on a unital C*-algebra $\mathcal{A}$ is defined to be any linear functional $\tau : \mathcal{A} \to \mathbb{C}$ such that

- $\tau(a^*a) \geq 0, \forall a \in \mathcal{A}$,
- $\tau(ab) = \tau(ba)$,
- $\tau(I_{\mathcal{A}}) = 1$, where $I_{\mathcal{A}}$ denotes the identity element.

We call the pair $(\mathcal{A}, \tau)$ a **tracial C*-algebra**.

Many authors say "trace" when they really mean "tracial state" and I will often too!

Exercise 4.1. Show that there is a unique trace $tr_n : M_n \to \mathbb{C}$ and that it is given by

$$tr_n(A) = \frac{1}{n} Tr(A).$$

Generally a C*-algebra can have many traces or no traces. In particular if the identity can be written as a sum of commutators, then it is impossible to have a trace, since one would need the trace of the identity to be both 1 and 0. Here are examples of a C*-algebra with no traces and one with a one parameter family of traces.

Exercise 4.2. Let $\ell^2(\mathbb{N})$ be the Hilbert space of square summable sequences. This space has an orthonormal basis $\{e_n : n \in \mathbb{N}\}$ where e_n is the vector that is 1 in the n-th coordinate and 0 elsewhere.(In the physicists notation $e_n = |n\rangle$). Prove that the identity operator on this space is a sum of commutators and, consequently, there can be no trace on $B(\ell^2(\mathbb{N}))$. (Hint: First consider the operator that maps $e_n \to e_{2n}$.)

Exercise 4.3. Let $\mathcal{A} \subseteq M_{n+k}$ consist of all block diagonal matrices of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ with A $n \times n$ and B $k \times k$. Prove that setting

$$\tau(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}) = t tr_n(A) + (1-t) tr_k(B), \ 0 \leq t \leq 1,$$

defines a trace on $\mathcal{A}$ and that every trace is of this form.

Here is the key theorem connecting traces and synchronous densities.

Theorem 4.4 (P-Severini-Stahlke-Todorov-Winter [?PSSTW]). (1) *$p \in C_{qc}^s(n,k)$ if and only if there exists a tracial C^*-algebra $(\mathcal{A}, \tau)$ and an (n,k)-PVM $\{e_{x,a} : 1 \le x \le n, 1 \le a \le k\}$ in $\mathcal{A}$ such that*

$$p(a,b|x,y) = \tau(e_{x,a}e_{y,b}).$$

(2) *$p \in C_q(n,k)$ if and only if in the above representation we can assume that $\mathcal{A}$ is finite dimensional.*

We sketch one of the key ideas of the proof. Suppose that we have written

$$p(a,b|x,y) = \langle \phi | E_{x,a}F_{y,b}\phi \rangle,$$

then

$$1 = \sum_{a,b=1}^{k} p(a,b|x,x) = \sum_{a=1}^{k} p(a,a|x,x) = \sum_{a=1}^{k} \langle E_{x,a}\phi | F_{x,a}\phi \rangle \le$$
$$\sum_{a=1}^{k} \|E_{x,a}\phi\| \cdot \|F_{x,a}\phi\| \le (\sum_{a=1}^{k} \|E_{x,a}\phi\|^2)^{1/2} (\sum_{a=1}^{k} \|F_{x,a}\phi\|^2)^{1/2} = 1.$$

Thus, the inequality is an equality and this in turn implies that $E_{x,a}\phi = F_{x,a}\phi$, $\forall x, a$.

Using this one shows that if we let $\mathcal{A}$ be the C*-algebra generated by $\{E_{x,a} : 1 \le x \le n, 1 \le a \le k\}$ and let $\tau : \mathcal{A} \to \mathbb{C}$ be the state given by $\tau(X) = \langle \phi | X\phi \rangle$, then $\tau(XY) = \tau(YX)$, i.e., τ is a trace.

This proves one direction of (1). The converse, that setting $p(a,b|x,y) = \tau(E_{x,a}E_{y,b})$ when τ is a trace defines an element of $C_{qc}(n,k)$ is a standard argument for experts in C*-algebras.

Note that if $p \in C_q^s(n,k)$ then the $E_{x,a}$ would all be matrices and so $\mathcal{A}$ would be a finite dimensional C*-subalgebra of this matrix algebra. However, this does not imply that τ is the usual trace, unlike full matrix algebras where the trace is unique, subalgebras can have many traces.

Exercise 4.5. Complete the proof that τ as defined above satisfies $\tau(XY) = \tau(YX)$.

The characterization of C_{qs}^s and C_{qa}^s came later and was quite a bit harder. We mention that in the Connes Embedding Problem there is one special tracial C*-algebra that plays a central role. It is called an ultrapower of the hyperfinite II_1-factor and is denoted $(\mathcal{R}^\omega, \tau_\omega)$. For those interested in a bit more background, see the supplemental notes on traces. For the rest of us, it is enough to know that this is just some important tracial C*-algebra.

Theorem 4.6 (Kim-P-Schafhauser). (1) *$C_{qs}^s(n,k) = C_q^s(n,k)$, $\forall n, k$,*

(2) *$p(a,b|x,y) \in C_{qa}^s(n,k)$ if and only if there exists an (n,k)-PVM $\{e_{x,a} : 1 \le x \le n, 1 \le a \le k\} \subseteq \mathcal{R}^\omega$ such that*

$$p(a,b|x,y) = \tau_\omega(e_{x,a}e_{y,b}).$$

(3) $p(a,b|x,y) \in C_{qa}(n,k)$ *if and only if there is an amenable trace* $\tau : C^*(\mathbb{F}(n,k))$ *such that*

$$p(a,b|x,y) = \tau(e_{x,a}e_{y,b}).$$

Note that if $p(a,b|x,y) \in C^s_{qc}(n,k)$ then

$$p(a,b|x,y) = \tau(e_{x,a}e_{y,b}) = \tau(e_{y,b}e_{x,a}) = p(b,a|y,x).$$

Exercise 4.7. If $p(a,b|x,y) \in C^s_{ns}(n,k)$, then does it follow that $p(a,b|x,y) = p(b,a|y,x)$? Prove or give a counterexample. Same question for $C^s_{vect}(n,k)$.

Problem 4.8. *There are notions of* **uniformly amenable traces** *and* **quasidiagonal traces***. We do not know characterizations of the densities that are of the form* $p(a,b|x,y) = \tau(e_{x,a}e_{y,b})$ *for* τ *either uniformly amenable or quasidiagonal.*

Definition 4.9. A density in $C_{ns}(n,n)$ is called **bisynchronous** if it is synchronous and $p(a,a|x,y) = 0, \forall x \neq y$. We use $C^{bs}_t(n,n)$ to denote the subset of bisynchronous traces in $_t(n,n)$.

Exercise 4.10. Show that $p \in C^{bs}_{qc}(n,n)$ if and only if there are projections $\{e_{x,a} : 1 \leq x, a \leq n\}$ in a tracial C*-algebra $(\mathcal{A}, \tau)$ such that $\sum_a e_{x,a} = \sum_x e_{x,a} = 1$ and $p(a,b|x,y) = \tau(e_{x,a}e_{y,b})$.

Such a set of projections defines a representation of the **quantum permutation group**.

P-Rahaman show that given a bisynchronous density of size n, the map

$$\phi : M_n \to M_n \ \phi(E_{x,y}) = \sum_{a,b} p(a,b|x,y)E_{a,b},$$

is CPTP and factorizable in the sense studied in the papers of Haagerup-Musat.

Problem 4.11. *Is* $C^{bs}_{qa}(n,n)$ *the closure of* $C^{bs}_q(n,n)$*?*

Here is the key definition and theorem about perfect strategies for synchronous games.

Given a synchronous game $\mathcal{G} = (X, A, V)$ with $card(X) = n$ and $card(A) = k$ we say that an (n,k)-PVM, $\{E_{x,a} : x \in I, a \in O\}$ satisfies the **fundamental orthogonality relations(FOR)** for $\mathcal{G}$ if and only if

$$V(a,b,x,y) = 0 \implies E_{x,a}E_{y,b} = 0.$$

Theorem 4.12 (Helton-Meyer-P-Satriano [JHKMVPMS19], Kim-P-Schafhauser [?KPS])**.** *Let* $\mathcal{G} = (X, A, V)$ *be a synchronous game. Then* $\mathcal{G}$ *has a perfect strategy in:*

(1) C_{qc} *if and only if there is an* (n,k)*-PVM satisfying the FOR in a tracial C*-algebra,*
(2) C_q *if and only if there is an* (n,k)*-PVM satisfying the FOR in a matrix algebra,*

(3) C_{qa} *if and only if there is an* (n,k)*-PVM satisfying the FOR in* $\mathcal{R}^\omega$.

Let's see what these relations are for a few games. First note that since every game is synchronous,

$$\lambda(x,x,a,b)=0, a\neq b \implies E_{x,a}E_{x,b}=0,$$

i.e., for each x, $\{E_{x,a} : 1 \le a \le k\}$ is an orthogonal family of projections summing to the identity. Since these relations hold for every game, I will often only mention the "extra" orthogonality relations.

4.1. **The Graph Colouring Game.** Given a graph $G=(V,E)$ we see that the only extra relations are that

$$(x,y)\in E \implies Ex,aE_{y,a}=0,\ \forall a.$$

Exercise 4.13. Suppose that we try to n-colour the complete graph on $n+1$ vertices. Write down the equations that must be satisfied. Show that it is impossible for these FOR to be satisfied by a set of operators on a Hilbert space. Conclude that this game cannot have a perfect t-strategy for $t=loc,q,qs,qa,qc$(Hint: Consider $\sum_a\sum_x E_{x,a}$ and $\sum_x\sum_a E_{x,a}$.)

Exercise 4.14. Does the above game have a perfect ns or $vect$ strategy?

4.2. **The Graph Isomorphism Game.** . Given graphs $G_i=(V_i,E_i), i=1,2$ the rules imply that

$$rel(x,y)\neq rel(a,b) \implies E_{x,a}E_{y,b}=0.$$

These were analyzed in the paper [?ALMRS] where it was shown that these relations are equivalent to the following conditions:

(1) For each $x\in V_1$, $\{E_{x,a} : a\in V_2\}$ is an orthogonal family of projections summing to the identity.
(2) For each $a\in V_2$, $\{E_{x,a} : x\in V_1\}$ is an orthogonal family of projections summing to the identity.
(3) For each $x\in V_1$ and $a\in V_2$,

$$\sum_{\{x_1:(x,x_1)\in E_1\}} E_{x_1,a} = \sum_{\{a_2:(a,a_2)\in E_2\}} E_{x,a_2}.$$

This last relation is best visualized as follows. If we let A_G denote the **adjacency matrix** of a graph, i.e., the matrix that is 1 in the (x_1,x_2) entry if and only if $(x_1,x_2)\in E$ and let $(E_{x,a})$ denote the matrix of projections that has the projection $E_{x,a}$ in its $x,a)$ entry, then the third relation is that

$$A_{G_1}(E_{x,a})=(E_{x,a})A_{G_2}.$$

For another direct derivation of these relations in language closer to these notes see [MBACKE$^+$20].

4.3. **Linear System Games.** Given a system of $n \times m$ equations $A\vec{x} = \vec{b}$ over $\mathbb{Z}_p$ the extra FOR equations can be summarized as follows:

(1) $E_{i,\vec{x}} = 0$ unless $\vec{x}$ satisifes $\sum_j a_{i,j}x_j = b_i$ and $a_{i,j} = 0 \implies x_j = 0$.

(2) Given $i \neq j$ and vectors $\vec{x}$ and $vecy$ satisfying (1) for i and j, respectively,

$$E_{i,\vec{x}}E_{j,\vec{y}} = 0 \text{ unless } a_{i,k}a_{j,k} \neq 0 \implies x_k = y_k.$$

We now sketch the proof of the above theorem. First we need the concept of a *faithful trace*. A tracial state $\tau : \mathcal{A} \to \mathbb{C}$ is called **faithful** provided that $\tau(x^*x) = 0 \implies x = 0$.

If a tracial state is not faithful, then $J := \{x \in \mathcal{A} : \tau(x^*x) = 0\}$ can be shown to be a 2-sided ideal and so we may form a quotient C*-algebra $\mathcal{A}/J$ and the functional $\tilde{\tau}(x + J) = \tau(x)$ is well-defined and faithful trace on this quotient C*-algebra. From this we can always reduce to the case where our density

$$p(a, b|x, y) = \tau(E_{x,a}E_{y,b}),$$

is given by a faithful trace. Next note that

$$\lambda(x, y, a, b) = 0 \implies 0 = \tau(E_{x,a}E_{y,b}) = \tau((E_{x,a}E_{y,b})^*(E_{x,a}E_{y,b})) \implies E_{x,a}E_{y,b} = 0,$$

provided that the trace is faithful.

Thus, the (n, k)-PVM must satisfy the FOR.

This proves (1).

To prove (2) one notes that the quotient of a finite dimensional C*-algebra is still finite dimensional. But every finite dimensional C*-algebra is a direct sum of matrix algebras and a trace on a direct sum of matrix algebras is just a convex combination of the (normalized) trace on each matrix summand. Now show that the restriction of the projections to each summand must satisfy the FOR.

Statement (3) follows from the fact that the projections can be taken to be in $\mathcal{R}^\omega$ and the fact that the trace τ_ω is known to be faithful.

Exercise 4.15. Prove that every trace on a direct sum of matrix algebras is a convex combination of the normalized trace on each matrix summand. Which convex combinations are faithful? Now show, the last claim, that if a POVM in a direct sum satisfies the FOR then each the restriction to each block satisfies the FOR.

The game in MIP*=RE that has a perfect strategy in C_{qc} but not in C_{qa} is actually a synchronous game! So the perfect strategy in C_{qc} must actually be a synchronous strategy and so we have that there is a tracial C*-algebra $(\mathcal{A}, \tau)$ containing a (n, k)-PVM that satisfies the FOR of the game. But since their game has no perfect strategy in C_{qa} there cannot exist a (n, k)-PVM in $\mathcal{R}^\omega$ satisfying these FOR.

Corollary 4.16. *By MIP*=RE, there exists a finite set of orthogonality relations*

$$V(a, b, x, y) = 0 \implies E_{x,a}E_{y,b} = 0,$$

that can be realized in a tracial C-algebra $(\mathcal{A}, \tau)$ but there are no projections in $\mathcal{R}^\omega$ satisfying these relations. Hence, there is no trace preserving *-homomorphism from $(\mathcal{A}, \tau)$ into $(\mathcal{R}^\omega, \tau_\omega)$.*

CEP asks for a trace preserving *-homomorphism of $(\mathcal{A}, \tau)$ into $(\mathcal{R}^\omega, \tau_\omega)$ for every tracial C*-algebra. The above shows that there can be no *-homomorphism of $\mathcal{A}$ into $\mathcal{R}^\omega$ and that the obstruction is just some FOR. This is a stronger negation of the CEP and this proof doesn't need Kirchberg's deep theorem that CEP and equality of the tensor products are equivalent.

5. The *-Algebra of a Synchronous Game

Given a synchronous game, $\mathcal{G} = (X, A, V)$ with n inputs and k outputs, we know that to have a perfect strategy we will need an (n, k)-PVM, which means that basically, we are starting with the algebra $\mathbb{C}(\mathbb{F}(n, k))$ such that the generators $\{e_{x,a}\}$ satisfy the FOR. The way that we can algebraically create an algebra that satisfies the FOR is to take all the products that we want to be 0, form the 2-sided ideal that they generate and take a quotient. However, we also want to preserve the *-structure, so for this reason we take the 2-sided *-ideal.

So we let the *ideal of the game* be the 2-sided ideal, denoted $\mathcal{I}_G$ generated by the set of elements,

$$\{e_{x,a}e_{y,b}, e_{y,b}e_{x,a} : V(a, b, x, y) = 0\} \cup \{e_{x,a}e_{y,b}, e_{y,b}e_{x,a} : V(b, a, y, x) = 0\}.$$

A typical element of $\mathcal{I}_G$ has the form,

$$\sum_i p_i e_{x_i,a_i} e_{y_i,b_i} q_i,$$

where p_i, q_i are arbitrary elements of $\mathbb{C}(\mathbb{F}(n, k))$ and each $e_{x_i,a_i}e_{y_i,b_i}$ belongs to this defining set.

Note that this ideal is *-closed and that

$$\mathcal{I}_G = \mathbb{C}(\mathbb{F}(n, k)) \iff 1 \in \mathcal{I}_G.$$

We define the **algebra of the game** to be the *-algebra that is the quotient,

$$\mathcal{A}(G) := \mathbb{C}(\mathbb{F}(n, k))/\mathcal{I}_G.$$

If we let $\widehat{e_{x,a}} := e_{x,a} + \mathcal{I}_G$ denote the image of $e_{x,a}$ in the quotient then these elements generate $\mathcal{A}(G)$ and satisfy:

(1) $V(a, b, x, y)V(b, a, y, x) = 0 \implies \widehat{e_{x,a}}\widehat{e_{y,b}} = 0,$
(2) $\widehat{e_{x,a}}^2 = \widehat{e_{x,a}}^* = \widehat{e_{x,a}},$
(3) $\sum_a \widehat{e_{x,a}} = \widehat{1}.$

Restating our theorem about the FOR we have:

Theorem 5.1. [JHKMVPMS19] *Let $\mathcal{G}$ be a synchronous game.*

(1) *$\mathcal{G}$ has a perfect deterministic strategy if and only if G has a perfect loc-strategy if and only if there exists a unital *-homomorphism from $\mathcal{A}(G)$ to $\mathbb{C}$.*
(2) *$\mathcal{G}$ has a perfect q-strategy if and only if G has a perfect qs-strategy if and only if there exists a unital *-homomorphism from $\mathcal{A}(G)$ to some matrix algebra.*
(3) *$\mathcal{G}$ has a perfect qa-strategy if and only if there exists a unital *-homomorphism from $\mathcal{A}(G)$ into $\mathcal{R}^\omega$.*
(4) *$\mathcal{G}$ has a perfect qc-strategy if and only if there exists a unital *-homomorphism from $\mathcal{A}(G)$ into some tracial C*-algebra.*

It is possible, and in fact happens quite often, that the identity element belongs to $\mathcal{I}_G$ in which case $\mathcal{I}_G = \mathbb{C}(\mathbb{F}(n,k))$ and hence the quotient collapses to be 0. In this case there can be no unital *-homomorphisms into anything with a unit and hence we know that these games have no perfect strategies of any flavor. This often gives us a simple algebraic test to show that the game cannot have a perfect strategy of any flavour.

Exercise 5.2. Let $\mathcal{G} = Col(K_3, 2)$ the game for 2-colouring the complete graph on three vertices. Prove that $1 \in \mathcal{I}_G$.

This is a bit harder:

Exercise 5.3. Prove that $1 \in \mathcal{I}_G$ for $\mathcal{G} = Col(K_4, 3)$.

Remarkably, only a machine assisted proof is known to show that $1 \notin \mathcal{I}_G$ when $\mathcal{G} = Col(K_5, 4)$, i.e., the game for 4-colouring the complete graph on five vertices. In [JHKMVPMS19] this was achieved by using a non-commutative Grobner basis program. The algorithm produced so many elements for the Grobner basis that it appears unlikely that a simple direct proof can be given of this fact.

Problem 5.4. *Find a non-machine assisted proof that $1 \notin \mathcal{I}_G$ for $\mathcal{G} = Col(K_5, 4)$.*

Thus, $\mathcal{A}(Col(K_5, 4)) \neq (0)$, but by an earlier exercise, one can show that this *-algebra cannot be represented on any Hilbert space, and in particular cannot have a perfect qc-strategy. Thus, the algebra $\mathcal{A}(G)$ appears to not give us a full picture of when perfect qc-strategies exist. It also shows that synchronous games can give us a method to produce quite esoteric *-algebras.

Provided that the *-algebra does have a representation as operators on a Hilbert space, we let $C^*(\mathcal{G})$ be its enveloping C*-algebra, i.e., the completion of the algebra $\mathcal{A}(\mathcal{G})$ in the norm that is the supremum over all representations on a Hilbert space.

Paddock-Slofstra[CPWS23] gave the first example of a synchronous game such that $\mathcal{A}(\mathcal{G})$ had representations on a Hilbert space, yet $\mathcal{G}$ did not have a perfect qc-strategy. So this game algebra is representable, yet possesses no traces.

In the general theory of *-algebras, there is a big gap between the universal *-algebra for three self-adjoint idempotents that sum to 1 versus the universal *-algebra for four self-adjoiont idempotents that sum to 1. In the case of 3, one can show directly that just as with projections on a Hilbert space, the generators must commute. But in the case of 4, they do not commute and so this algebra has no faithful representation on a Hilbert space. See [?MBACKE^+20]. For a concrete construction addressing this problem see Samolinko-Turovska[YSLT02]. H. Radjavi(personal communitcation) has also constructed 4 infinite matrices that are idempotent, define unboounded operators on a Hilbert space, sum to 1 and do not commute, from which the prior statement about *-algebras follows.

So what is $\mathcal{A}(G)$ good for? Here is some applications:

Corollary 5.5. [JHKMVPMS19] *Let $t = loc, q, qs, qa, qc$ and let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two synchronous games. Assume that there exists a unital *-homomorphism from $\mathcal{A}(G_1)$ to $\mathcal{A}(G_2)$. If G_2 has a perfect t-strategy, then G_1 also has a perfect t-strategy.*

In particular, even if the games might seem very different, whenever $\mathcal{A}(G_1)$ and $\mathcal{A}(G_2)$ are *-isomorphic algebras we see that they share existence/non-existence of perfect strategies of the various flavours.

For a second application, recall that for linear systems there were two versions of the game, one synchronous and another that was not synchronous. For the synchronous version we have the algebra of the game, $\mathcal{A}(SyncLCS(A, b))$. The theory for the non-synchronous version says that there is a group, called the *solution group* $\Gamma_{(A,b)}$ with a distinguished element J and perfect strategies are determined by whether or not this group has a representation that sends J^{p-1} to $-I$. This means that in the representation $J^{p-1} + I = 0$ and the relevant algebra to study is the quotient of the group algebra by this relation, $\mathbb{C}(\Gamma_{(A,b)})/ < J^{p-1} + I >$.

Theorem 5.6 (A. Goldberg [AG21]). *Given a linear system of equations $A\vec{x} = \vec{b}$ over the field $\mathbb{Z}_p$, the two algebras $\mathcal{A}(SyncLCS(A, b))$ and $\mathbb{C}(\Gamma_{(A,b)})/ < J^{P-1} + I >$ are unitally *-isomorphic.*

Thus, we finally have a language that allows us to prove that these two versions of the game are really formally identical.

Theorem 5.7 (S. Harris [?Ha]). *Let $\mathcal{G}_1$ be a synchronous game. Then there is a graph G_2 such that there are *-homomorphisms $\mathcal{A}(G_1) \to \mathcal{A}(Col(G_2, 3)))$ and $\mathcal{A}(Col(G_2, 3)) \to \mathcal{A}(\mathcal{G}_1)$. Consequently, $\mathcal{G}_1$ has a perfect t-strategy if and only if $Col(G_2, 3)$ has a perfect t-strategy.*

This shows, for example, that the synchronous game constructed in MIP*=RE that has a perfect qc-strategy but no perfect qa-strategy, can be assumed, without loss of generality, to be a 3-coloring game for some graph.

In algebra there is a notion of *Morita equivalence* of algebras, so it is natural to define two synchronous games $\mathcal{G}_i, i = 1, 2$ to be Morita equivalent, denoted $\mathcal{G}_1 \simeq_M \mathcal{G}_2$ if and only if there game algebras are. It can be shown that this preseves existence of perfect t-strategies for $t = loc, q, qs, qa, qc$.

Problem 5.8. *Find other NASC for $\mathcal{G}_1 \simeq_M \mathcal{G}_2$. Does this relation preserve existence of perfect vect-strategies?*

Problem 5.9. *Let $G_i, H_i, i = 1, 2$ be graphs. When is $Hom(G_1, H_1) \simeq_M Hom(G_2, H_2)$? Same problem for the graph isomorphism game.*

The result of Paddock-Slofstra[CPWS23] shows that the game algebra begin non-zero does not give a complete answer to which games have perfect qc-strategies. It also leaves open the question of whether or not there is some way to algebraically determine if games have perfect qc-strategies. This problem was answered recently by Adam Bene Watts, John William Helton, and Igor Klep[?WHK].

Let $\mathcal{G} = (X, A, V)$ be a synchronous game. By the **left ideal** generated by the relations we mean the set of all finite sums

$$\mathcal{L} := \{\sum_i p_i e_{x_i,a_i} e_{y_i,b_i} : p_i \in \mathbb{C}(\mathbb{F}(n,k)),\ \lambda(x,y,a,b) = 0 \text{ or } \lambda(y,x,b,a) = 0\},$$

and by the **right ideal** generated by the relations we mean the set of all finite sums,

$$\mathcal{R} := \{\sum_i e_{x_i.a_i} e_{y_i,b_i} p_I : p_i \in \mathbb{C}(\mathbb{F}(n,k)).\ \lambda(x,y,a,b) = 0 \text{ or } \lambda(y,x,b,a) = 0\},$$

and by the **commutator space** we mean the set of all finite sums of the form

$$\mathcal{C} = \{\sum_i p_i q_i - q_i p_i : p_I, q_i \in \mathbb{C}(\mathbb{F}(n,k))\}.$$

Theorem 5.10 (Watts-Helton-Klep)**.** *Let $\mathcal{G}$ be a synchronous game. Then $\mathcal{G}$ has a perfect qc-strategy if and only if*

$$1 \notin \mathcal{L} + \mathcal{R} + \mathcal{C}.$$

This gives a completely symbolic means of determining whether or not perfect qc-strategies exist.

6. Values of Games

In order to talk about the value of a game $\mathcal{G} = (X, Y, A, B, V)$ we need to also have a *prior distribution* on input pairs, i.e.,

$$\pi : I_A \times I_B \to [0, 1],$$

with $\sum_{x,y} \pi(x,y) = 1$. Given a conditional probability density $p(a,b|x,y)$, the probability of winning, i.e., the *expected value* of the given strategy $p(a,b|x,y)$ is given by

$$\omega(\mathcal{G}, \pi, p) = \sum_{x,y,a,b} \pi(x,y)\lambda(x,y,a,b)p(a,b|x,y) = \sum_{(x,y,a,b)\in W} \pi(x,y)p(a,b|x,y)).$$

Given a set S of conditional probability densities the **S-value** of the pair $(\mathcal{G},\pi)$ is

$$\omega_S(\mathcal{G},\pi) := \sup\{\omega(G,\pi,p) : p \in S\}.$$

Identifying $S \subseteq [0,1]^m$, since the value is clearly a convex function of p, the value will always be attained at one of the extreme points of the closed convex hull of S.

To simplify and unify notation, we set

$$\omega_t(G,\pi) = \omega_{C_t}(G,\pi),\ \ t = loc, q, qa, qc, vect.$$

Note that, since the value is a continuous function of the density, we have $\omega_q(G,\pi) = \omega_{qa}(G,\pi)$.

We remark that my notation is very non-standard. Generally, π is considered part of the game, so the game is just $\mathcal{G}$, not $(\mathcal{G},\pi)$.

Also since the loc densities are all convex combinations of deterministic densities, we have that $\omega_{loc}(G,\pi)$ is just the supremum over all deterministic strategies. Thus,

$$\omega_{loc}(\mathcal{G},\pi) = \omega(\mathcal{G},\pi).$$

We also have that

$$\omega_q(\mathcal{G},\pi) = \omega_{qa}(\mathcal{G},\pi),$$

since the value is a supremum.

Note that if the prior density has full support then

$$\omega_t(\mathcal{G},\pi) = 1 \iff \text{ a perfect t-density exists },$$

in which case, $\omega_t(\mathcal{G},\rho) = 1$ for any density.

But we will see later that there are games for which it is natural to study densities without full support.

An often interesting question for $\omega_q(\mathcal{G},\pi)$ is whether or not the value is actually attained by an element of C_q. For $t = loc, qa, qc, vect$ the value is always attained, since the corresponding sets of densities are closed and hence compact.

Computing ω and ω^* for various games has been a topic of interest in computer science for a while. These values and ω_{qc} for various games has generated a great deal of interest in the operator algebras community since it was shown by [MJMNCP+11] and [NO13] that if the *Connes' embedding conjecture* had an affirmative answer, then

$$\omega_q(G,\pi) = \omega_{qc}(G,\pi),$$

for all games and densities.

Recently, [ZJANTV+20] proved the existence of a game for which

$$\omega_q(\mathcal{G},\pi) < 1/2 < \omega_{qc}(\mathcal{G},\pi) = 1,$$

thus refuting the embedding conjecture.

For synchronous games, it is natural to restrict the allowed strategies to synchronous densities.

Given a game $\mathcal{G} = (X, A, V)$(synchronous or not) with density π we set

$$\omega_t^s(\mathcal{G}, \pi) = \omega_{C_t^S}(G, \pi),\ t = loc, q, qc.$$

These are the values that we are interested in computing in this lecture.

Often this number is more natural than $\omega_t(\mathcal{G}, \pi)$. For an example, let's look at a graph on n vertice $G = (V, E)$ and consider the game $Col(G, 2)$ with π the uniform density on edges, $\pi(x, y) = \frac{1}{|E|^2}, \forall x, y$. Thus, Alice and Bob receive a pair (x, y) if and only if $(x, y) \in E$(recall that also $(y, x) \in E$).

Note that $\omega(Col(G, 2), \pi) = 1$ for this density since a perfect deterministic strategy is given by Alice always returning one of the colors and Bob the other.

To compute the synchronous value we look at all functions $f : I_A = V \to \{0, 1\}$. Each function corresponds to partitioning the vertex set into two subsets, $V = S_0 \cup S_1$. Given a pair $(x, y) \in E$ we will win iff they belong to different subsets. So we would like to choose S_0, S_1 to maximize the number of edges that belong to different subsets.

This number is precisely what is meant by the **maximum cut number** of G, which we denote $Max - Cut(G)$. Except that graph theorists count each edge $(x, y), (y, x)$ only once, while we count them twice.

Thus,

$$\omega_{loc}^s((Col(G, 2), \pi)) = \frac{2Max - Cut(G)}{|E|^2}.$$

Thus, it is the synchronous value, not the ordinary value, that captures max-cut.

Exercise 6.1. Compute $\omega_{loc}^s(Col(G, 2), \pi))$ and $\omega_{loc}^s(Col(G, 2), \pi))$ in the case that $\pi(x, y) = \frac{1}{n^2}$ is the uniform density on all vertex pairs.

We now look at the tracial characterizations of these synchronous values.

Given a C*-algebra $\mathcal{A}$ with unit, by a **trace** on $\mathcal{A}$ we mean a linear functional $\tau : \mathcal{A} \to \mathbb{C}$ satisfying $\tau(I) = 1$, $p \geq 0 \implies \tau(p) \geq 0$ and $\tau(xy) = \tau(yx)$. The first two conditions characterize **states** on $\mathcal{A}$. When $\mathcal{A} = M_n$ the $n \times n$ matrices, it is known that there is a unique trace, namely,

$$tr_n((a_{i,j})) = \frac{1}{n} \sum_i a_{i,i} = \frac{1}{n} Tr((a_{i,j})).$$

Given a C*-algebra $\mathcal{A}$ with unit I, a **k-outcome projection valued measure(k-PVM)** is a set of k projections, $E_a = E_a^2 = E_a^*$ such that $\sum_{a=1}^k E_a = I$. A family of n k-PVM's is a set of projections $\{E_{x,a} : 1 \leq x \leq n, 1 \leq a \leq k\}$ with $\sum_a E_{x,a} = I, \forall x$.

Recall that if $p(a, b|x, y)$ is a synchronous density, then

$$p(a, b|x, y) = \tau(E_{x,a} E_{y,b}) = \tau(E_{y,b} E_{x,a}) = p(b, a|y, x).$$

such a density is called **symmetric**.

This result translates into the following result about synchronous values.

Theorem 6.2. *Let $\mathcal{G} = (X, Y, V)$ be an n input k output game and let π be a prior distribution on inputs. Then*

(1)
$$\omega^s_{loc}(\mathcal{G}, \pi) = \sup\{ \sum_{\substack{x,y \\ (x,y,f(x),f(y)) \in W}} \pi(x,y) \},$$
where the supremum is over all functions, $f : X \to A$ from inputs to outputs,

(2)
$$\omega^s_q(\mathcal{G}, \pi) = \omega^s_{qa}(G, \pi) = \sup\{ \sum_{(x,y,a,b) \in W} \pi(x,y) tr_m(E_{x,a} E_{y,b}) \},$$
where the supremum is over all families of (n,k)-PVM's in M_m and over all m,

(3)
$$\omega^s_{qc}(\mathcal{G}, \pi) = \sup\{ \sum_{(x,y,a,b) \in W} \pi(x,y) \tau(E_{x,a} E_{y,b}) \},$$
where the supremum is over all unital C-algebras $\mathcal{A}$, traces τ, and families of n k-PVM's in $\mathcal{A}$.*

As we remarked earlier, the second supremum may not be attained.

6.1. **Group Algebras and Values.** Given two groups G, H we let $G \times H$ denote their abelian product, i.e.,$(g_1, h_1) \cdot (g_2, h_2) = (g_1 g_2, h_1 h_2)$. Note that
$$\mathbb{C}(G \times H) \simeq \mathbb{C}(G) \otimes \mathbb{C}(H),$$
via the map that sends $u_{(g,h)} \to u_g \otimes u_h$.

Values of games can be interpreted in terms of properties of the maximal and minimal C*-tensor product of this algebra. There is also a sum of squares interpretation.

Given a *-algebra $\mathcal{A}$ we set
$$SOS = \{ \sum_i a_i^* a_i : a_i \in \mathcal{A} \}.$$
. By a result of Ozawa,
$$p \in \mathbb{C}(G) \cap C^*(G)^+ \iff \forall \epsilon > 0, \epsilon 1 + p \in SOS.$$

Given a game $\mathcal{G}$ and density π we set
$$P_{\mathcal{G},\pi} = \sum_{(x,y,a,b) \in W} \pi(x,y) e_{x,a} \otimes e_{y,b}.$$
Using the fact that norms of positive elements are attained by taking the supremum over states, and Ozawa's result, we have:

Theorem 6.3. *Given an n input, k output game $G = (X, Y, A, B, V)$ with density π,*

$$\omega_q(\mathcal{G}, \pi) = \|P_{G,\pi}\|_{C^*(\mathbb{F}(n,k))\otimes_{min}C^*(\mathbb{F}(n,k))},$$

and

$$\omega_{qc}(G, \pi) = \|P_{G,\pi}\|_{C^*(\mathbb{F}(n,k)\otimes_{max}C^*(\mathbb{F}(n,k)} = \inf\{r : r1 - P_{\mathcal{G},\pi} \in SOS\}.$$

The results of [ZJANTV$^+$20] gave the first proof that this minimal and maximal norms are different.

We now turn to the synchronous case.

The element $e_{x,a}e_{y,b}$ is not positive, but for any trace we have that

$$\tau(e_{x,a}e_{y,b}) = \tau(e_{x,a}e_{y,b}e_{x,a}),$$

and $e_{x,a}e_{y,b}e_{x,a} \geq 0$.

We set

$$R_{G,\pi} = \sum_{(x,y,a,b)\in W} \pi(x,y)e_{x,a}e_{y,b}e_{x,a}.$$

We also set $\mathcal{C} \subseteq \mathbb{C}(\mathbb{F}(n,k))$ equal to the linear span of all commutators, $xy - yx$.

Given any C*-algebra $\mathcal{A}$ we let $T(\mathcal{A})$ denote the set of traces on $\mathcal{A}$ and let $T_{fin}(\mathcal{A})$ denote the set of traces that *factor through matrix algebras,* i.e., are of the form

$$\tau(a) = tr_m(\pi(a)),$$

for some m and some unital *-homomorphism $\pi : \mathcal{A} \to M_m$. Recall that $T_{am}(\mathcal{A})$ denotes the set of amenable traces discussed in the notes on traces.

If we fix a faithful representation $C^*(\mathbb{F}(n,k)) \subseteq B(H)$ then we let $\mathcal{D} \subseteq B(H)$ denote the closed linear span of all commutators of the form $ax - xa,\ a \in C^*(\mathbb{F}(n,k)),\ x \in B(H)$.

Theorem 6.4. *Let $\mathcal{G} = (X, A, V)$ be an n input, k output game with density π. Then*

$$\begin{aligned}\omega^s_{qc}(\mathcal{G}, \pi) = \sup\{\tau(R_{G,\pi}) : \tau \in T(C^*(\mathbb{F}(n,k))\} &= \inf\{\|R_{G,\pi} - C\| : C \in \mathcal{C}\} \\ &= \inf\{r : r1 - R_{\mathcal{G},\pi} \in \mathcal{C} + SOS\}\end{aligned}$$

and

$$\begin{aligned}\omega^s_q(\mathcal{G}, \pi) &= \sup\{\tau(R_{G,\pi}) : \tau \in T_{fin}(C^*(\mathbb{F}(n,k))\} \\ &= \omega^s_{qa}(\mathcal{G}, \pi) = \sup\{\tau(R_{G,\pi}) : \tau \in T_{am}(C^*(\mathbb{F}(n,k))\} \\ &= \inf\{\|R_{G,\pi} - D\| : D \in \mathcal{D}\}.\end{aligned}$$

This last formula for $\omega^s_{qc}(\mathcal{G}, \pi)$ first appeared in [RLVP]. The distance formula for $\omega^s_{qa}(\mathcal{G}, \pi)$ seems to be new. It is interesting that this distance to $\mathcal{D}$, because it is the supremum over all amenable traces, is independent of the particular faithful representation of $C^*(\mathbb{F}(n,k))$.

The equality of the value with the distance to the space of commutators follows from [EKMR14, Theorem 2.9] where it is shown that for positive elements of a C*-algebra, the supremum over all traces is equal to the distance to the space $\mathcal{C}$. For a direct proof of this fact, see the supplemental notes on traces. The corresponding distance formula for the supremum over T_{fin} is proven in the supplemental notes.

Problem 6.5. *Find formulas for $\omega^s_{vect}(\mathcal{G},\pi)$? Does it have an interpretation as a distance?*

If we let $\mathcal{D} = \cap_{\tau \in T_{fin}} ker(\tau)$, then this set is a "kernel" in the sense studied in the theory of tensor products of operator systems and hence there is a quotient operator system $\mathcal{A}/\mathcal{D}$ and this operator system comes equipped with an induced norm. For many such quotients it is known that the operator space quotient norm and this induced operator system norm are different.

Problem 6.6. *Are the operator space quotient norm and the operator system quotient norm equal for $\mathcal{A}/\mathcal{D}$?*

For the example of a game constructed in [ZJANTV$^+$20], it is known that

$$\omega^s_q(G,\pi) < 1/2 < \omega^s_{qc}(G,\pi) = 1,$$

and consequently, their results also give the first proof that $T_{fin}(C^*(\mathbb{F}(n,k))$ is not dense in $T(C^*(\mathbb{F}(n,k))$. Perhaps even more remarkable is that this difference is witnessed by the element $R_{G\pi}$ for some game, which only involves words in the generators of order three.

However, the game of [ZJANTV$^+$20] is mostly given implicitly and estimates on the values of n and k to achieve their example are very large.

In summary, we see that the theory of values and synchronous values of these games gives us interesting information about C*-algebras. Thus, we are led to study these values for interesting sets of games.

7. Synchronous Values of Products of Games

This section is mostly from [JHHMSN$^+$24].

There is a great deal of research concerning products of games and especially their behaviour when one does many iterations of a fixed game.[RJAPPY14, MBTVHY17, HY16]. Many of these results are false for synchronous values of games.

Since $\omega^s_q(\mathcal{G},\pi) \le \omega_q(\mathcal{G},\pi)$, if $\omega_q(\mathcal{G},\pi) < 1$, the synchronous powers must tend to 0 as well.

So the main case of interest is when the support of π is not full, and $\omega_t(\mathcal{G},\pi) = 1$, like $Col(G,2)$ with support on the edges. Here is an odd example.

Problem 7.1. *If G is a graph that is not 2-colorable and we let π be the uniform density on the edges of G, then does*

$$\omega_t(Col(G,2)^n, \pi^n) \to 0,$$

along with the usual questions about rates.

Example 7.2. Let $\mathcal{G} = (G, \pi)$ be the game where Alice's and Bob's question and answer sets are $\{0, 1\}$ and let the distribution π be given by $\pi(0, 1) = \pi(1, 1) = 1/2$, so $\pi(0, 0) = \pi(1, 0) = 0$. The players win if their answer pair is $(1, 1)$ when asked question pair $(0, 1)$. They also win if their answer pair is $(0, 1)$ when they receive question pair $(1, 1)$. They lose in all other cases. Note that Bob receives 1 with probability 1 while Alice receives $0, 1$ with equal probability.

This game has a perfect non-synchronous strategy, namely, for Bob to always return 1 and for Alice given input $x \in \mathbb{Z}_2$ to always return $x + 1$. Thus,

$$\omega_{loc}(\mathcal{G}, \pi) = \omega_{qc}(\mathcal{G}, \pi) = 1,$$

and consequently,

$$\omega_{loc}(\mathcal{G}^n) = \omega_{qc}(\mathcal{G}^n) = 1.$$

Theorem 7.3. *Let $\mathcal{G} = (G, \pi)$ be the game of the above example. Then*

$$\omega^s_{loc}(\mathcal{G}^n) = \omega^s_{qc}(\mathcal{G}^n) = 1 - \frac{1}{2^n}.$$

Proof. The synchronous value of this game is at most $1/2$, since on question $(1, 1)$ a synchronous strategy will require them to return the same answer and lose. On the other hand, the deterministic strategy of Alice and Bob always returning 1 has a value of $1/2$. Hence, $\omega^s_{loc}(G) = \omega^s_q(G) = \frac{1}{2}$. In terms of traces and projections, this is given by setting $E_{0,1} = E_{1,1} = I$ and $E_{0,0} = E_{1,0} = 0$.

Now for the n-fold parallel repetition the questions are pairs $x, y \in \{0, 1\}^n$ and the answers are pairs $a, b \in \{0, 1\}^n$. But $\pi^n(x, y) = 0$ unless $y = (1, ..., 1) := 1^n$, while $\pi(x, 1^n) = \frac{1}{2^n}$, $\forall x \in \{0, 1\}^n$.

The only question pair where the synchronous restriction can be enforced is therefore $(1^n, 1^n)$, and on this question any synchronous strategy loses as before. Thus, $\omega^s_{qc}(\mathcal{G}^n) \leq 1 - \frac{1}{2^n}$.

On the other hand, consider the deterministic strategy where when the input string is 1^n they return 1^n but for every other input string $x \neq 1^n$, they return the output string $\overline{x} = x + 1^n$, where addition is in the vector space $\mathbb{Z}_2^n$, i.e., each bit of x is flipped. For every string $x \neq 1^n$ that Alice receives this strategy wins. Hence, $\omega^s_{loc}(\mathcal{G}^n) \geq 1 - \frac{1}{2^n}$. Therefore the synchronous value of the parallel repeated game is $\omega^s_{loc}(G^n) = \omega^s_{qc}(G^n) = 1 - \frac{1}{2^n}$.

Alternatively, this is the strategy that corresponds to choosing PVM's,

$$E_{1^n,1^n} = E_{x,\overline{x}} = I, \ \forall x \neq 1^n,$$

and all other projections equal to 0. ∎

Thus, not only does the synchronous value not tend to 0, but it is monotonically increasing. Also, we have that

$$\omega^s_t(\mathcal{G}^2) > \min\{\omega_t(\mathcal{G}), \omega^s_t(\mathcal{G})\},$$

so that this example violates the synchronous analogues of properties (2) and (4).

Two objections can be raised to this example. The game itself is not synchronous and the distribution is not symmetric. It is natural to wonder if this pathology persists even when restricting attention to this smaller family of synchronous games with symmetric prior densities.

In [LMVPITAW23] the first example is given of a symmetric synchronous game and symmetric distribution for which the synchronous values for $t = loc, q, qc$ are all supermultiplicative.

Problem 7.4. *If $\mathcal{G}_i = (G_i, \pi_i), i = 1, 2$ are symmetric synchronous games with symmetric densities π_i, then Is $\omega_t^s(\mathcal{G}_1 \times \mathcal{G}_2, \pi_1 \times \pi_2) \le \min(\omega_t^s(\mathcal{G}_1, \pi_1), \omega_t^s(\mathcal{G}_2, \pi_2))$?*

Problem 7.5. *If $\mathcal{G}$ is a symmetric, synchronous game with symmetric distribution π, can $\omega_t^s(\mathcal{G}^n, \pi^n)$ be monotone increasing?*

And the analogues of Raz's and H. Yuen's results are unknown:

Problem 7.6. *If $\mathcal{G}$ is a symmetric, synchronous game with symmetric density π with $\omega_t(\mathcal{G}, \pi) = 1$ and $\omega_t^s(\mathcal{G}, \pi) < 1$ does $\omega_t^s(\mathcal{G}^n, \pi^n) \to 0$?*

The hope is that one can use trace technology to prove things about synchronous values of products of games

References

[FAJHMO97] F. Alizadeh, J.A. Haeberly, and M.L. Overton, *Complementarity and nondegeneracy in semidefinite programming*, Mathematical Programming **77** (1997), no. 2, 111–128.

[CAD06] C. Anantharaman-Delaroche, *On ergodic theorems for free group actions on noncommutative spaces*, Probability Theory and Related Fields **135** (2006), 520–546.

[SACLRM+98] S. Arora, C. Lund, R. Motwani, M. Sudan, and M. Szegedy, *Proof verification and the hardness of approximation problems*, Journal of the ACM **45** (1998), no. 3, 501–555.

[SASS98] S. Arora and S. Safra, *Probabilistic checking of proofs: A new characterization of NP*, Journal of the ACM **45** (1998), no. 1, 70–122.

[AALMDR+19] A. Atserias, L. Mančinska, D.E. Roberson, R. Sámal, S. Severini, and A. Varvitsiotis, *Quantum and non-signalling graph isomorphisms*, Journal of Combinatorial Theory, Series B **136** (2019), 289–328.

[DAJHYKYS06] D. Avis, J. Hagesawa, Y. Kikuchi, and Y. Sasaki, *A quanutm protocol to win the graph coloring game on all hadamard graphs*, IEICE Trans. Fundam. electron. Commun. Comput. Sci., E89-A(5):1378-1381 (2006).

[LBLFCL91] L. Babai, L. Fortnow, and C. Lund, *Non-deterministic exponential time has two-prover interactive protocols*, Computational Complexity **1** (1991), 3–40.

[MBTVHY15] M. Bavarian, T. Vidick, and H. Yuen, *Anchored parallel repetition for nonlocal games*, preprint (arXiv:1509.07466v1) (2015).

[MBTVHY16] ———, *Parallel repetition via fortification: analytic view and the quantum case*, preprint (arXiv:1603.05349) (2016).

[MBTVHY17] ———, *Hardness amplification for entangled games via anchoring*, STOC'17: Proceedings of the 49th annual ACM SIGACT symposium on theory of computing, 2017, pp. 303–316.

[JB64] J.S. Bell, *On the Einstein-Podolsky-Rosen paradox*, Physics **1** (1964), no. 3, 195–200.

[MBOSGJKAW88] M. Ben-Or, S. Goldwasser, J. Kilian, and A. Wigderson, *Multi-prover interactive proofs: How to remove intractability assumptions*, ACM Symposium on Theory of Computing, 1988, pp. 113–131.

[ABWJHIK23] A. Bene Watts, J.W. Helton, and I. Klep, *Noncommutative nullstellensatze and perfect Games*, Annales Henri Poincaré **24** (2023), 2183–2239.

[FB91] F. Boca, *Free products of completely positive maps and spectral sets*, J. Funct. Anal. **97** (1991), 251-263.

[ABPKPS23] A. Bochniak, P. Kasprzak, and P.M. Soltan, *Quantum correlations on quantum spaces*, International Mathematics Research Notices **2023** (2023), no. 14, 12400–12440.

[MBACKE+20] M. Brannan, A. Chirvasitu, K. Eifler, S.J. Harris, V.I. Paulsen, X. Su, and M. Wasilewski, *Bigalois extensions and the graph isomorphism game*, Communications in Mathematical Physics **375** (2020), 1777–1809.

[MBPGSH22] M. Brannan, P. Ganesan, and S.J. Harris, *The quantum-to-classical graph homomorphism game*, Journal of Mathematical Physics **63** (2022), 112204.

[MBSHITLT23a] M. Brannan, S.J. Harris, I.G. Todorov, and L. Turowska, *Synchronicity for quantum non-local games*, Journal of Functional Analysis **284** (2023), no. 2, 109738.

[MBSHITLT23b] ———, *Quantum no-signalling bicorrelations*, preprint (arXiv:2302.04268) (2023).

[NBNO08] N. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.

[PCMNAM+07] P.J. Cameron, M.W. Newman, A. Montanaro, S. Severini, and A. Winter, *On the quantum chromatic number of a graph*, The electronic journal of combinatorics **14** (2007).

[VCML15] V. Capraro and M. Lupini, *Introduction to Sofic and Hyperlinear Groups and Connes' Embedding Conjecture*, Lecture Notes in Mathematics, vol. 2136, Springer Cham, Switzerland, 2015.

[KMCXWHY15] K.-M. Chung, X. Wu, and H. Yuen, *Parallel repetition for entangled k-player games via fast quantum search*, CCC '15: Proceedings of the 30th Conference on Computational Complexity, 2015, pp. 512–536.

[JCMHASRH69] J.F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt, *Proposed experiment to test local hidden-variable theories*, Physical Review Letters **23** (1969), no. 15, 880–884.

[RCPHBTJW04] R. Cleve, P. Hoyer, B. Toner, and J. Watrous, *Consequences and limits of nonlocal strategies*, 19th IEEE Annual Conference on Computational Complexity, 2004, pp. 236–249.

[RCLLVP17] R. Cleve, L. Liu, and V. Paulsen, *Perfect embezzlement of entanglement*, Journal of Mathematical Physics **58** (2017), 012204.

[RCLLWS17] R. Cleve, L. Liu, and W. Slofstra, *Perfect commuting-operator strategies for linear system games*, Journal of Mathematical Physics **58** (2017), no. 1, 012202.

[RCRM14] R. Cleve and R. Mittal, *Characterization of binary constraint system games*, International Colloquium on Automata, Languages, and Programming, 2014, pp. 320–331.

[RCWSFUSU06] R. Cleve, W. Slofstra, F. Unger, and S. Upadhyay, *Perfect parallel repetition theorem for quantum XOR proof systems*, Computational Complexity **17** (2006), no. 2, 282–299.

[AC20] A. Coladangelo, *A two-player dimension witness based on embezzlement, and an elementary proof of the non-closure of the set of quantum correlations*, Quantum **4** (2020), 282–300.

[ACJS18] A. Coladangelo and J. Stark, *Unconditional separation of finite and infinite-dimensional quantum correlations*, preprint (arXiv:1804.05116) (2018).

[AC76] A. Connes, *Classification of Injective Factors, Cases II_1, II_∞, III_λ,* $\lambda \neq 1$, Annals of Mathematics (2) **104** (1976), no. 1, 73–115.

[JC00] J.B. Conway, *A course in operator theory*, Graduate Studies in Mathematics, vol. 21, American Mathematical Society, Providence, RI, 2000.

[JCDKRLIT16] J. Crann, D.W. Kribs, R.H. Levene, and I.G. Todorov, *Private algebras in quantum information and infinite-dimensional complementarity*, Journal of Mathematical Physics **57** (2016), 015208.

[ECAM23] E. Culf and A. Mehta, *New approaches to complexity via quantum graphs*, preprint (arXiv:2309.12887) (2023).

[GDlCTDTN20] G. De las Cuevas, T. Drescher, and T. Netzer, *Quantum magic squares: dilations and their limitations*, J. Math. Phys. **61** (2020).

[KDEK17] K.R. Davidson and E.T.A. Kakariadis, *A proof of Boca's theorem*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics **149** (2017), 869–876.

[MD24] M. Daws, *Quantum graphs: different perspectives, homomorphisms and quantum homomorphisms*, Communications of the American Mathematical Society **4** (2024), 117–181.

[KDVP16] K. Dykema and V.I. Paulsen, *Synchronous correlation matrices and Connes' embedding conjecture*, Journal of Mathematical Physics **57** (2016), 015214.

[KDVPJP19] K. Dykema, V.I. Paulsen, and J. Prakash, *Non-closure of the set of quantum correlations via graphs*, Communications in Mathematical Physics **365** (2019), 1125–1142.

[AEBPNR35] A. Einstein, B. Podolsky, and N. Rosen, *Can quantum-mechanical description of physical reality be considered complete?*, Physical Review **47** (1935), 777–780.

[AEKPMK11] A. Einstein, K. Przibram, and M. Klein, *Letters on Wave Mechanics: Correspondence with H. A. Lorentz, Max Planck, and Erwin Schrödinger*, Philosophical Library/Open Road, 2011.

[DFAKVPIT18] D. Farenick, A.S. Kavruk, V.I. Paulsen, and I.G. Todorov, *Characterisations of the weak expectation property*, New York Journal of Mathematics **24a** (2018), 107–135.

[UFJK00] U. Feige and J. Kilian, *Two-prover protocols—low error at affordable rates*, SIAM Journal on Computing **30** (2000), no. 1, 324–346.

[PFVR87] P. Frankl and V. Rodl, *Forbidden intersections*, Trans. American Mathematical Society **300** (1987), 259-286.

[AF22] A. Frei, *Connes implies Tsirelson: a simple proof*, preprint (arXiv:2209.07940) (2022).

[TF12] T. Fritz, *Tsirelson's problem and Kirchberg's conjecture*, Reviews in Mathematical Physics **24** (2012), 1250012, 67pp.

[AG21] A. Goldberg, *Synchronous linear constraint system games*, Journal of Mathematical Physics **62** (2021), 032201.

[IGBH23] I. Goldbring and B. Hart, *A survey on the model theory of tracial von Neumann algebras*, Model theory of operator algebras, 2023, DOI doi:10.1515/9783110768282.

[AG53] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. Sao Paulo **8** (1953), 1–79.

[UHMM11] U. Haagerup and M. Musat, *Factorization and dilation problems for completely positive maps on von Neumann algebras*, Communications in Mathematical Physics **303** (2011), 555–594.

[UHMM15] ———, *An asymptotic property of factorizable completely positive maps and the Connes embedding problem*, Communications in Mathematical Physics **338** (2015), 721–752.

[SH17] S.J. Harris, *Connes' embedding problem and winning strategies for quantum XOR games*, Journal of Mathematical Physics **58** (2017), 122203.

[SH22] ———, *Synchronous games with $*$-isomorphic game algebras*, Quantum Information and Computation **22** (2022), no. 11 & 12, 924–946.

[SH24] ———, *Universality of graph homomorphism games and the quantum coloring problem*, Annales Henri Poincaré (2024).

[SHRLVP$^+$18] S.J. Harris, R.H. Levene, V.I. Paulsen, S. Plosker, and M. Rahaman, *Schur multipliers and mixed unitary maps*, Journal of Mathematical Physics **59** (2018).

[JHKMVPMS19] J.W. Helton, K.P. Meyer, V.I. Paulsen, and M. Satriano, *Algebras, synchronous games and chromatic numbers of graphs*, New York Journal of Mathematics **25** (2019), 328–361.

[JHHMSN$^+$24] J.W. Helton, H. Mousavi, S.S. Nezhadi, V.I. Paulsen, and T.B. Russell, *Synchronous values of games*, Annales Henri Poincaré (2024).

[TH09] T. Holenstein, *Parallel repetition: simplification and the no-signaling case*, Theory of Computing **5** (2009), no. 8, 141–172.

[RJAPPY14] R. Jain, A. Pereszlényi, and P. Yao, *A parallel repetition theorem for entangled two-player one-round games under product distributions*, 2014 IEEE 29th Conference on Computational Complexity (CCC), 2014, pp. 209–216.

[ZJANTV$^+$20] Z. Ji, A. Natarajan, T. Vidick, J. Wright, and H. Yuen, $MIP^* = RE$, preprint (arXiv:2001.04383) (2020).

[MJMNCP$^+$11] M. Junge, M. Navascués, C. Palazuelos, D. Perez-Garcia, V.B. Scholz, and R.F. Werner, *Connes' embedding problem and Tsirelson's problem*, Journal of Mathematical Physics **52** (2011), 012102, 12pp.

[SJKAM17] S.-J. Kim and A. Mehta, *Chromatic numbers and a Lovász type inequality for non-commutative graphs*, preprint (arXiv:1709.05595) (2017).

[SJKVPCS18] S.-J. Kim, V.I. Paulsen, and C. Schafhauser, *A synchronous game for binary constraint systems*, Journal of Mathematical Physics **59** (2018), 032201.

[EK94] E. Kirchberg, *Discrete groups with Kazhdan's property T and factorization property are residually finite*, Mathematische Annalen **299** (1994), 35–63.

[EKMR14] E. Kirchberg and M. Rørdam, *Central sequence C^*-algebras and tensorial absorption of the Jiang-Su algebra*, Journal für die reine und angewandte Mathematik (Crelles Journal) **2014** (2014), no. 695, 175–214.

[DLBTJW13] D. Leung, B. Toner, and J. Watrous, *Coherent state exchange in multi-prover quantum interactive proof systems*, Chicago Journal of Theoretical Computer Science (2013), 1–18.

[RLVP] R. Levene and V.I. Paulsen, *Unique Games and Games Based on Groups*, Journal of Mathematical Physics.

[ZDLYLMMW+22] Z.-D. Li, Y.-L. Mao, M. Weilenmann, A. Tavakoli, H. Chen, L. Feng, S.-J. Yang, M.-O. Renou, D. Trillo, T.P. Le, N. Gisin, A. Acín, M. Navascués, Z. Wang, and J. Fan, *Testing real quantum theory in an optical quantum network*, Physical Review Letters **128** (2022), no. 4, 040402.

[MLLMVP+20] M. Lupini, L. Mančinska, V.I. Paulsen, D.E. Roberson, G. Scarpa, S. Severini, I.G. Todorov, and A. Winter, *Perfect strategies for non-signalling games*, Mathematical Physics, Analysis and Geometry **23** (2020), no. 7.

[LMVPITAW23] L. Mančinska, V.I. Paulsen, I.G. Todorov, and A. Winter, *Products of synchronous games*, Studia Mathematica **272** (2023), no. 3, 299–317.

[LMDR12] L. Mančinska and D.E. Roberson, *Graph homomorphisms for quantum players*, preprint (arXiv:1212.1724) (2012).

[LMDR16a] ______, *Quantum homomorphisms*, Journal of Combinatorial Theory, Series B **118** (2016), 228–267.

[LMDR16b] ______, *Oddities of quantum colorings*, Baltic Journal of Modern Computing **4** (2016), no. 4, 846–859.

[LMDR20] ______, *Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs*, 2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS), 2020, pp. 661-672.

[LMTV18] L. Mančinska and T. Vidick, *Unbounded entanglement in non-local games*, preprint(arxiv:1402.4145) (2018).

[NM90] N.D. Mermin, *Simple unified form for the major no-hidden variables theorems*, Physical Review Letters **65** (1990), no. 27, 3373–3376.

[MMMR20] M. Musat and M. Rørdam, *Non-closure of quantum correlation matrices and factorizable channels that require infinite dimensional ancilla (with an appendix by Narutaka Ozawa)*, Communications in Mathematical Physics **375** (2020), 1761–1776.

[MNSPAA08] M. Navascués, S. Pironio, and A. Acín, *A convergent hierarchy of semidefinite programs characterizing the set of quantum correlations*, New Journal of Physics **10** (2008), 073013.

[NO13] N. Ozawa, *About the Connes embedding conjecture–algebraic approaches*, Japanese Journal of Mathematics **8** (2013), 147–183.

[CPWS23] C. Paddock and W. Slofstra, *Satisfiability problems and algebras of boolean constraint system games*, preprint(arXiv:2310.07901) (2023).

[KPTV10] K.F. Pái and T. Vértesi, *Maximal violation of the I3322 inequality using infinite dimensional quantum systems*, Phys. Rev. A **82** (2010).

[CPTV16] C. Palazuelos and T. Vidick, *Survey on nonlocal games and operator space theory*, Journal of Mathematical Physics **57** (2016), 015220.

[VP16] V.I. Paulsen, *Entanglement and non-locality*, Online course notes (2016).

[VPMR21] V.I. Paulsen and M. Rahaman, *Bisynchronous games and factorizable maps*, Annales Henri Poincaré **22** (2021), 593–614.

[VPSSDS+16] V.I. Paulsen, S. Severini, D. Stahlke, I.G. Todorov, and A. Winter, *Estimating quantum chromatic numbers*, Journal of Functional Analysis **270** (2016), no. 6, 2188–2222.

[VPIT15] V.I. Paulsen and I.G. Todorov, *Quantum chromatic numbers via operator systems*, Quarterly Journal of Mathematics **66** (2015), no. 2, 677–692.

[RR98] R. Raz, *A parallel repetition theorem*, SIAM Journal on Computing **27** (1998), no. 3, 763–803.

[ORTV15] O. Regev and T. Vidick, *Quantum XOR games*, ACM Transactions on Computation Theory **7** (2015), no. 15, 1–43.

[MORDTMW$^+$21] M.-O. Renou, D. Trillo, M. Weilenmann, T.P. Le, A. Tavakoli, N. Gisin, A. Acíin, and M. Navascués, *Quantum theory based on real numbers can be experimentally falsified*, Nature **600** (2021), 625–629.

[DR13] D.E. Roberson, *Variations on a theme: graph homomorphisms*, Ph.D. Thesis (University of Waterloo), 2013.

[YSLT02] Y. Samoilenko and L. Turowska, *On bounded and unbounded idempotents whose sum is a multiple of the identity*, Methods of Functional Analysis and Topology **8** (2002), no. 1, 79–100.

[WS19a] W. Slofstra, *The set of quantum correlations is not closed*, Forum of Mathematics, Pi **7** (2019), e1.

[WS19b] ———, *Tsirelson's problem and an embedding theorem for groups arising from non-local games*, Journal of the American Mathematical Society **33** (2019), no. 1, 1–56.

[DS16] D. Stahlke, *Quantum zero-error source-channel coding and non-commutative graph theory*, IEEE Transactions on Information Theory **62** (2016), no. 1, 554–577.

[ITLT] I.G. Todorov and L. Turowska, *Quantum no-signalling correlations and non-local games*, preprint(arxiv:2009.07016).

[BT80] B.S. Tsirelson, *Quantum generalizations of Bell's inequality*, Letters in Mathematical Physics **4** (1980), 93–100.

[BT93] ———, *Some results and problems on quantum Bell-type inequalities*, Hadronic Journal Supplement **8** (1993), no. 4, 329–345.

[WvDPH03] W. van Dam and P. Hayden, *Universal entanglement transformations without communication*, Physical Review A **67** (2003), 060302.

[TV22] T. Vidick, *Almost synchronous quantum correlations*, Journal of Mathematical Physics **63** (2022), 022201.

[TVJW15] T. Vidick and J. Watrous, *Quantum Proofs*, Foundations and Trends in Theoretical Computer Science **11** (2015), 1-215.

[NW21] N. Weaver, *Quantum graphs as quantum relations*, Journal of Geometric Analysis **31** (2021), 9090–9112.

[HY16] H. Yuen, *A parallel repetition theorem for all entangled games*, International Colloquium of Automata, Languages, and Programming (ICALP 2016), 2016, pp. 77:1–77:13.

Institute for Quantum Computing and Department of Pure Mathematics, University of Waterloo, Waterloo, ON, Canada N2L 3G1

Email address: `vpaulsen@uwaterloo.ca`